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The Equivalence of Perfect Fluid Space-Times and Viscous Magnetohydrodynamic Space-Times in General Relativity

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Abstract

The work of a previous article [1] is extended to show that space-times which are the exact solutions of the field equations for a perfect fluid also may be exact solutions of the field equations for a viscous magnetohydrodynamic fluid. Conditions are found for this equivalence to exist and viscous magnetohydrodynamic solutions are found for a number of known perfect fluid space-times.

§(1): *Introduction*

It has been shown [1] that, under certain circumstances, it is possible for the stress-energy tensor of a perfect fluid to have identical components to those of the stress-energy tensor of a magnetohydrodynamic fluid with heat conduction. This equality implies that the field equations of a perfect fluid, viz.,

$$G_{\mu\nu} = H_{\mu\nu} \equiv (\rho + p)\, v_\mu v_\nu + p g_{\mu\nu} \tag{1}$$

and the field equations for a magnetohydrodynamic fluid with heat conduction, viz.,

$$G_{\mu\nu} = K_{\mu\nu} \equiv E_{\mu\nu} + (\bar{\rho} + \bar{p})\, u_\mu u_\nu + \bar{p} g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \tag{2}$$

may give rise to space-time solutions which are identical, i.e., the same space-time metric, in the same coordinate system, may satisfy both sets of field equations, so that an exact perfect fluid solution may be also an exact solution of the magnetohydrodynamic field equations, implying that the space-time may be interpreted physically as corresponding to either type of matter content. Such a duality of interpretation also exists between Einstein-Maxwell solutions and viscous fluid solutions [2].

In [1], it was shown that the equality of the stress-energy tensors $H_{\mu\nu}$ and $K_{\mu\nu}$ necessarily implies that the electromagnetic field contained in $K_{\mu\nu}$ is null and that the densities and pressures in the two distributions are identical, i.e., $\rho = \bar{\rho}$ and $p = \bar{p}$. As a result the 4-velocity, u_μ, and the heat conduction vector, q_μ, can be determined to within a sign, so that the problem of finding the exact magnetohydrodynamic solution corresponding to a known perfect fluid solution is well-defined and can be solved with comparative ease.

In this paper we generalize the problem considered in [1] by including viscous terms in the field equations (1), i.e., we seek the conditions under which a space-time solution of the perfect fluid field equations (1) is also an exact solution of the field equations for a viscous magnetohydrodynamic fluid, viz.,

$$G_{\mu\nu} = M_{\mu\nu} \equiv E_{\mu\nu} + (\bar{\rho} + \bar{p}^*) u_\mu u_\nu + \bar{p}^* g_{\mu\nu} - 2\eta\sigma_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \tag{3}$$

where $\bar{\rho}$ is the density, $\bar{p}^* = \bar{p} - \xi\Theta$ is the kinetic pressure, $\bar{p}$ is the thermodynamic pressure, Θ is the expansion of the velocity congruence u_μ, $\sigma_{\mu\nu}$ is the shear tensor, q_μ is the heat conduction vector, $\xi(\geqslant 0)$ is the bulk viscosity coefficient, $\eta(\geqslant 0)$ is the shear viscosity coefficient, and $E_{\mu\nu}$ is the electromagnetic stress-energy tensor given by

$$E_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \tfrac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \tag{4}$$

where $F_{\mu\nu}$ is the Maxwell tensor.

In Section 2 we find the general conditions that must hold if the stress-energy tensors $H_{\mu\nu}$ and $K_{\mu\nu}$ are identical. In Section 3 we discuss the canonical tetrad form of the field equations and in Sections 4 and 5 we apply these results to the cases of nonnull and null, respectively, electromagnetic fields. In Section 6 we discuss viscous fluids without electromagnetic fields. A number of examples illustrating the results are given.

§(2): *The General Equations*

Equating $H_{\mu\nu}$ defined by equation (1) with $M_{\mu\nu}$ defined by equation (3) we obtain

$$(\rho + p) v_\mu v_\nu + p g_{\mu\nu} = E_{\mu\nu} + (\bar{\rho} + \bar{p}^*) u_\mu u_\nu + \bar{p}^* g_{\mu\nu} - 2\eta\sigma_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \tag{5}$$

and contracting this equation leads to

$$\rho - \bar{\rho} = 3(p - \bar{p}^*) \tag{6}$$

Introducing the notation

$$\rho + p = M, \qquad \bar{\rho} + \bar{p}^* = N \tag{7}$$

$$v_\mu u^\mu = -\alpha, \qquad v_\mu q^\mu = \beta Q \tag{8}$$

where $\alpha \geqslant 1$, $Q^2 = q_\mu q^\mu$, and $M > 0, N > 0$ from energy considerations, equation (5) can be written in the form

$$E_{\mu\nu} = M v_\mu v_\nu - N u_\mu u_\nu + \tfrac{1}{4}(M - N) g_{\mu\nu} + 2\eta\sigma_{\mu\nu} - q_\mu u_\nu - q_\nu u_\mu \tag{9}$$

Now $E_{\mu\nu}$ can be written in the form

$$E_{\mu\nu} = (\tfrac{1}{2} g_{\mu\nu} + u_\mu u_\nu)(E_\alpha E^\alpha + B_\alpha B^\alpha) - (E_\mu E_\nu + B_\mu B_\nu) - (u_\mu S_\nu + u_\nu S_\mu) \tag{10}$$

where E_μ, B_μ, and S_μ are, respectively, the electric field, the magnetic field, and the Poynting vector as measured by a comoving observer and defined by

$$E_\mu = F_{\mu\nu} u^\nu, \qquad B_\mu = \tfrac{1}{2} \eta_{\mu\nu\alpha\beta} u^\nu F^{\alpha\beta}, \qquad S_\mu = \eta_{\nu\mu\alpha\beta} u^\nu E^\alpha B^\beta$$

These quantities satisfy $E_\mu u^\mu = B_\mu u^\mu = S_\mu u^\mu = E_\mu S^\mu = B_\mu S^\mu = 0$. (Note that the definition of S_μ given here differs in sign from that used in [1]; the present definition corresponds to the usual three-dimensional definition $\mathbf{S} = \mathbf{E} \wedge \mathbf{B}$). We shall assume that the 4-velocity u^μ appearing in equation (10) is identical with that appearing in equations (5) and (9). Contracting each of equations (9) and (10) with u^ν and comparing the results we obtain

$$q_\mu - S_\mu = M\alpha(v_\mu - \alpha u_\mu) \tag{11}$$

and

$$E^2 + B^2 = 2M\alpha^2 - \tfrac{1}{2}(M + 3N) \tag{12}$$

where $E^2 = E_\alpha E^\alpha$ and $B^2 = B_\alpha B^\alpha$.

Applying the Rainich condition $E_{\mu\alpha} E^{\nu\alpha} = \frac{1}{4} \delta_\mu{}^\nu E_{\alpha\beta} E^{\alpha\beta}$ to the expression (9) leads to

$$\begin{aligned}
&- \tfrac{1}{2} M(M + N) v_\mu v^\nu + [Q^2 - \tfrac{1}{2} N(M + N)] u_\mu u^\nu + M(N\alpha - \beta Q)(v_\mu u^\nu + v^\nu u_\mu) \\
&\quad - \tfrac{1}{2}(M + N)(u_\mu q^\nu + u^\nu q_\mu) + M\alpha(v_\mu q^\nu + v^\nu q_\mu) - q_\mu q^\nu \\
&\quad + 2M\eta(\sigma^{\nu\alpha} v_\alpha v_\mu + \sigma_{\mu\alpha} v^\nu v^\alpha) - 2\eta(\sigma_{\mu\alpha} q^\alpha u^\nu + \sigma^{\nu\alpha} q_\alpha u_\mu) + (M - N)\eta\sigma_\mu{}^\nu \\
&\quad + 4\eta^2 \sigma_{\mu\alpha} \sigma^{\nu\alpha} = \delta_\mu{}^\nu [\tfrac{1}{2} N(M + N) + \tfrac{1}{2} M(M - N)\alpha^2 + 2M\alpha\beta Q - Q^2]
\end{aligned} \tag{13}$$

together with

$$\begin{aligned}
&\tfrac{1}{2}(M + N)(M - 3N) + 2M(2N - M)\alpha^2 - 4M\alpha\beta Q \\
&\qquad + 2Q^2 + 8\eta^2\sigma^2 + 4M\eta\sigma^{\alpha\beta} v_\alpha v_\beta = 0
\end{aligned} \tag{14}$$

where $\sigma^2 = \frac{1}{2}\sigma_{\alpha\beta}\sigma^{\alpha\beta}$. Contracting equation (13) with u^ν leads to

$$-M[\tfrac{1}{2}(M-N)\alpha + \beta Q]\ (v_\mu - \alpha u_\mu) + [M\alpha^2 - \tfrac{1}{2}(M+N)]\ q_\mu = 2\eta\sigma_{\mu\nu}S^\nu \quad (15)$$

Equations (10), (11), and (15) play a large part in the subsequent discussion, but no further useful information can be obtained from these equations without making some simplifying assumptions, such as the assumption $\sigma_{\mu\nu} = 0$, which was the basis of [1]. Before proceeding we need to consider in some detail the electromagnetic field and the canonical form of its stress-energy tensor.

§(3): *The Canonical Form of* $E_{\mu\nu}$

Introducing the notation that Latin suffixes refer to tetrad components, we choose a tetrad frame $e^i{}_\mu$ and its inverse $e_i{}^\mu$ such that the tetrad components of the Maxwell tensor are

$$F_{ij} = e_i{}^\mu e_j{}^\nu F_{\mu\nu}$$

Assuming first that the electromagnetic field is nonnull, it is possible to choose a local tetrad frame such that the only nonzero tetrad components of the Maxwell tensor are $F_{0'1'}$ and $F_{2'3'}$ [3], where the primes denote tetrad suffixes. We have the freedom to make a rotation in the $(2', 3')$ plane and a hyperbolic rotation in the $(0', 1')$ plane without introducing other nonzero components of F_{ij}. This freedom can be used to eliminate $u_{1'}$ and $u_{3'}$ from the tetrad components, u_i, of the 4-velocity, but, in general, we cannot eliminate $u_{2'}$, so that we have $u_i = (u_{0'}, 0, u_{2'}, 0)$. The tetrad components of E_μ, B_μ, and S_μ are then

$$E_i = (0, F_{0'1'}u_{0'}, 0, -F_{2'3'}u_{2'}) \quad (16)$$

$$B_i = (0, F_{2'3'}u_{0'}, 0, F_{0'1'}u_{2'}) \quad (17)$$

$$S_i = (A^2 u_{0'}u_{2'}{}^2, 0, A^2 u_{0'}{}^2 u_{2'}, 0) \quad (18)$$

and $E_{\mu\nu}$ has tetrad components

$$E_{ij} = \mathrm{diag}(\tfrac{1}{2}A^2, -\tfrac{1}{2}A^2, \tfrac{1}{2}A^2, \tfrac{1}{2}A^2) \quad (19)$$

where

$$A^2 = F_{0'1'}{}^2 + F_{2'3'}{}^2 \quad (20)$$

The expression (19) is the familiar canonical form for the stress-energy tensor of a nonnull electromagnetic field; the point of the above discussion is that the tetrad components of u_i have, in general, one nonzero spacelike component which is zero if and only if $S_\mu = 0$, i.e., if and only if E_μ and B_μ are parallel, in which case u_i has the comoving form $u_i = (-1, 0, 0, 0)$.

Consider now the consequences of transforming away $u_{2'}$. If we put $u_{0'} =$

$-\cosh\phi$ and $u_{2'} = \sinh\phi$, then a Lorentz transformation of the local tretrad frame of the form

$$x^{0'} = \bar{x}^{0'} \cosh\phi + \bar{x}^{2'} \sinh\phi$$
$$x^{2'} = \bar{x}^{0'} \sinh\phi + \bar{x}^{2'} \cosh\phi \tag{21}$$

transforms u_i into $\bar{u}_i = (-1, 0, 0, 0)$. The nonzero tetrad components of the Maxwell tensor are now

$$\bar{F}_{0'1'} = F_{0'1'} \cosh\phi, \qquad \bar{F}_{2'1'} = F_{0'1'} \sinh\phi$$
$$\bar{F}_{2'3'} = F_{2'3'} \cosh\phi, \qquad \bar{F}_{0'3'} = F_{2'3'} \sinh\phi \tag{22}$$

and the tetrad components of the other electromagnetic quantities are

$$\bar{E}_i = (0, -\bar{F}_{0'1'}, 0, -\bar{F}_{0'3'})$$
$$\bar{B}_i = (0, -\bar{F}_{2'3'}, 0, -\bar{F}_{1'2'}) \tag{23}$$
$$\bar{S}_i = (0, 0, A^2 \sinh\phi \cosh\phi, 0)$$

and

$$\bar{E}_{ij} = \begin{bmatrix} \tfrac{1}{2}A^2 \cosh 2\phi & 0 & \tfrac{1}{2}A^2 \sinh 2\phi & 0 \\ 0 & -\tfrac{1}{2}A^2 & 0 & 0 \\ \tfrac{1}{2}A^2 \sinh 2\phi & 0 & \tfrac{1}{2}A^2 \cosh 2\phi & 0 \\ 0 & 0 & 0 & \tfrac{1}{2}A^2 \end{bmatrix} \tag{24}$$

Now, from equation (20), we can write $F_{0'1'} = A\cos\gamma$, $F_{2'3'} = A\sin\gamma$, where γ is a parametric function, and we put $A\cosh\phi = C$ and $A\sinh\phi = D$. Dropping the bars, equations (22), (23), and (24) become

$$F_{0'1'} = C\cos\gamma, \qquad F_{2'1'} = D\cos\gamma$$
$$F_{2'3'} = C\sin\gamma, \qquad F_{0'3'} = D\sin\gamma \tag{25}$$

$$E_i = (0, -C\cos\gamma, 0, -D\sin\gamma)$$
$$B_i = (0, -C\sin\gamma, 0, D\cos\gamma) \tag{26}$$
$$S_i = (0, 0, CD, 0)$$

and

$$E_{ij} = \begin{bmatrix} \tfrac{1}{2}(C^2 + D^2) & 0 & CD & 0 \\ 0 & -\tfrac{1}{2}(C^2 - D^2) & 0 & 0 \\ CD & 0 & \tfrac{1}{2}(C^2 + D^2) & 0 \\ 0 & 0 & 0 & \tfrac{1}{2}(C^2 - D^2) \end{bmatrix} \tag{27}$$

When $u_{2'} = 0$, i.e., $\sinh \phi = 0$, then $D = 0$, S_μ and the form (27) reverts to the form (19). Hence, when $S_\mu \neq 0$, we can use either the standard canonical form (19) for the stress-energy tensor of a nonnull electromagnetic field, which corresponds to u_μ having the local tetrad form $u_i = (-\cosh \phi, 0, \sinh \phi, 0)$, or we can use the form (27) for which $u_i = (-1, 0, 0, 0)$, When $S_\mu = 0$ both forms are identical and correspond to $u_i = (-1, 0, 0, 0)$.

For null electromagnetic fields, E_μ, B_μ, S_μ, and u_μ are mutually orthogonal with $E^2 = B^2$ and $S_\alpha S^\alpha \equiv S^2 = E^4$. We can choose a local tetrad frame such that

$$\begin{aligned} E_i &= E(0,0,0,1) \equiv E e_i \\ B_i &= \pm E(0,1,0,0) \equiv E b_i \\ S_i &= \pm E^2 (0,0,1,0) \equiv E^2 s_i \\ u_i &= (-1,0,0,0) \end{aligned} \tag{28}$$

where e_i, b_i, and s_i are the unit vectors so defined. Then E_{ij} has the standard canonical form for null electromagnetic fields, viz.,

$$E_{ij} = \begin{bmatrix} E^2 & 0 & \pm E^2 & 0 \\ 0 & 0 & 0 & 0 \\ \pm E^2 & 0 & E^2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

and the nonzero local components of the Maxwell tensor are

$$F_{0'3'} = \pm F_{2'3'} = -E \tag{30}$$

The ambiguous sign in expressions (28) to (30) can be eliminated by a rotation through π radians about the $x^{3'}$ axis, but we shall find it convenient to maintain the dual sign. Note that the form (28) to (30) can be obtained from the forms (25) to (27) by putting $\pm C = D = E$ and $\gamma = -\pi/2$. Thus the expressions (25) to (27) constitute a set of basic canonical forms for electromagnetic fields, from which the appropriate standard forms for nonnull or null fields may be obtained as explained above.

In the subsequent sections we shall apply the results of this section to find examples of known fluid space-times that also satisfy the field equations (3). Such solutions not only must satisfy the algebraic conditions of Section 2, and others to be given later, but also the Maxwell tensor must satisfy the Maxwell equations

$$F_{[\mu\nu;\sigma]} = 0, \qquad F^{\mu\nu}{}_{;\nu} = J^\mu \tag{31}$$

and the 4-current J^μ must be consistent with the expression [4]

$$(J^\mu - \epsilon u^\mu)(1 + \zeta^2 B^2) = \lambda E^\mu + \lambda \zeta^2 E_\alpha B^\alpha B^\mu + \lambda \zeta S^\mu \tag{32}$$

in which J^μ is expressed as the sum of a convection current and a conduction current, where ϵ is the charge density, λ is the conductivity, and $\lambda\zeta$ is the transverse conductivity. We also shall require the heat conduction vector to satisfy the equation of state

$$q_\mu = -\kappa h_\mu{}^\nu (T_{,\nu} + Ta_\nu) \tag{33}$$

where $\kappa (\geqslant 0)$ is the thermal conductivity, T is the temperature, $a_\nu = u_{\nu;\alpha}u^\alpha$ is the acceleration vector, and $h_\mu{}^\nu = \delta_\mu{}^\nu + u_\mu u^\nu$ is the projection tensor.

§(4): *Nonnull Electromagnetic Field*

We consider the two cases $S^\mu \neq 0$ and $S^\mu = 0$, and give examples of perfect fluid space-times which admit the viscous magnetohydrodynamic fluid interpretation.

Case 1. $S^\mu \neq 0$. We shall use the canonical form given by equations (16)-(20) with $v_i = (v_{0'}, v_{1'}, v_{2'}, v_{3'})$. From the tetrad versions of equations (5)-(8), (11) and equation (12) we obtain

$$q_i = [A^2 u_{0'} u_{2'}{}^2 + M\alpha(v_{0'} - \alpha u_{0'}), M\alpha v_{1'}, A^2 u_{0'}{}^2 u_{2'} + M\alpha(v_{2'} - \alpha u_{2'}), M\alpha v_{3'}] \tag{34}$$

$$\beta Q = M\alpha(\alpha^2 - 1) + A^2 u_{0'}(v_{0'} - \alpha u_{0'}) \tag{35}$$

$$v_{0'}u_{0'} - v_{2'}u_{2'} = \alpha \tag{36}$$

$$A^2(u_{0'}{}^2 + u_{2'}{}^2) = 2M\alpha^2 - \tfrac{1}{2}(M + 3N) \tag{37}$$

$$\begin{aligned}
2\eta\sigma_{0'0'} &= -Mv_{0'}{}^2 + M\alpha(u_{0'}v_{0'} + u_{2'}v_{2'}) - \tfrac{1}{2}(M+N)u_{2'}{}^2 \\
2\eta\sigma_{0'1'} &= -Mv_{1'}(v_{0'} - \alpha u_{0'}) \\
2\eta\sigma_{0'2'} &= -Mv_{0'}v_{2'} + M\alpha(u_{0'}v_{2'} + u_{2'}v_{0'}) - \tfrac{1}{2}(M+N)u_{0'}u_{2'} \\
2\eta\sigma_{0'3'} &= -Mv_{3'}(v_{0'} - \alpha u_{0'}) \\
2\eta\sigma_{1'1'} &= -\tfrac{1}{2}A^2 - Mv_{1'}{}^2 - \tfrac{1}{4}(M-N) \\
2\eta\sigma_{1'2'} &= -Mv_{1'}(v_{2'} - \alpha u_{2'}) \\
2\eta\sigma_{1'3'} &= -Mv_{1'}v_{3'} \\
2\eta\sigma_{2'2'} &= -Mv_{2'}{}^2 + M\alpha(u_{0'}v_{0'} + u_{2'}v_{2'}) - \tfrac{1}{2}(M+N)u_{0'}{}^2 \\
2\eta\sigma_{2'3'} &= -Mv_{3'}(v_{2'} - \alpha u_{2'}) \\
2\eta\sigma_{3'3'} &= \tfrac{1}{2}A^2 - Mv_{3'}{}^2 - \tfrac{1}{4}(M-N)
\end{aligned} \tag{38}$$

The components of $\sigma_{\mu\nu}$ calculated from u_μ and its derivatives must satisfy these expressions. Unfortunately, at this stage the expressions are too complex to enable us to find an example. Part of the difficulty is due to the fact that v_i has four nonzero components, whereas, in their standard coordinate systems, most known perfect fluid solutions correspond to a comoving coordinate system. Accordingly, we shall look for a solution with $v_i = (-1, 0, 0, 0)$. From equation (36) this implies that $u_i = (-\alpha, 0, \pm(\alpha^2 - 1)^{1/2}, 0)$, and equations (34) and (35) become

$$q_i = [(M - A^2)\,\alpha(\alpha^2 - 1), 0, -(M - A^2)\,\alpha^2 u_{2'}, 0] \tag{39}$$

$$\beta Q = (M - A^2)\,\alpha(\alpha^2 - 1) \tag{40}$$

From these two expressions we find that

$$\beta^2 = (\alpha^2 - 1) \tag{41}$$

so that

$$u_i = (-\alpha, 0, \beta, 0) \tag{42}$$

$$q_i = Q(\beta, 0, -\alpha, 0) \tag{43}$$

$$Q = (M - A^2)\,\alpha\beta \tag{44}$$

and the expressions (38) simplify to

$$\begin{aligned} 2\eta\sigma_{0'0'} &= \tfrac{1}{2}(M - N)\,\beta^2 \\ 2\eta\sigma_{0'2'} &= -\tfrac{1}{2}(M - N)\,\alpha\beta \\ 2\eta\sigma_{2'2'} &= \tfrac{1}{2}(M - N)\,\alpha^2 \\ 2\eta\sigma_{1'1'} &= -\tfrac{1}{2}A^2 - \tfrac{1}{4}(M - N) \\ 2\eta\sigma_{3'3'} &= \tfrac{1}{2}A^2 - \tfrac{1}{4}(M - N) \end{aligned} \tag{45}$$

Our problem may be stated as follows: given the space-time metric of a known perfect fluid solution, we want to find a velocity vector u_i, i.e., the functions α and β, such that the space-time is a viable exact solution of the field equation (3). However, as noted in [2], for viscous fluids the same stress-energy tensor components can result from different choices of the 4-velocity, so that there is no unique solution and we are left with the task of making a suitable choice for u_i which will lead to a viable solution. This is unlike the situation described in [1] in which, in the absence of viscosity, u_i was completely determined.

Example 1. Consider the Kasner solution with metric

$$ds^2 = -dt^2 + dx^2 + t^{2b}dy^2 + t^{2(1-b)}\,dz \tag{46}$$

which is a perfect fluid solution with $\rho = p = b(1-b)t^{-2}$, i.e., $M = 2b(1-b)t^{-2}$, and $v_i = (-1, 0, 0, 0)$. For the viscous, magnetohydrodynamic fluid solution we shall assume that the electromagnetic field consists only of the single nonzero component $F_{2'3'} = A$. The corresponding coordinate component is $F_{23} = At$ and, applying Maxwell's equations, we find that

$$A = A_0 t^{-1} \tag{47}$$

where A_0 is a constant, and $J^\mu = 0$. The electric field, magnetic field, and Poynting vector are

$$\begin{aligned} E_\mu &= (0, 0, 0, -A_0 \beta t^{-b}) \\ B_\mu &= (0, -A_0 \alpha t^{-1}, 0, 0) \\ S_\mu &= (-A_0{}^2 \alpha \beta^2 t^{-2}, 0, A_0{}^2 \alpha^2 \beta t^{b-2}, 0) \end{aligned} \tag{48}$$

We shall look for a solution in which α and β are constants. Calculating the shear tensor components from $u_\mu = (-\alpha, 0, \beta t^b, 0)$ we find that the only nonzero components are

$$\begin{aligned} \sigma_{0'0'} &= \tfrac{1}{3}(3b-1)\,\alpha\beta^2 t^{-1} \\ \sigma_{0'2'} &= -\tfrac{1}{3}(3b-1)\,\alpha^2 \beta t^{-1} \\ \sigma_{2'2'} &= \tfrac{1}{3}(3b-1)\,\alpha^3 t^{-1} \\ \sigma_{1'1'} &= -\tfrac{1}{3}\alpha t^{-1} \\ \sigma_{3'3'} &= -\tfrac{1}{3}(3b-2)\,\alpha t^{-1} \end{aligned}$$

which are entirely consistent with the expressions (45) with

$$\begin{aligned} M - N &= \tfrac{4}{3}\eta(3b-1)\,\alpha t^{-1} \\ A^2 &= 2\eta(1-b)\,\alpha t^{-1} \end{aligned}$$

The field equations (3) yield

$$\begin{aligned} b(1-b)t^{-2} &= \tfrac{1}{2}A_0{}^2 t^{-2} + \bar{\rho}\alpha^2 + \bar{p}^*\beta^2 - \tfrac{2}{3}\eta(3b-1)\,\alpha\beta^2 t^{-1} - 2Q\alpha\beta \\ 0 &= -(\bar{\rho} + \bar{p}^*)\,\alpha\beta + \tfrac{2}{3}\eta(3b-1)\,\alpha^2\beta t^{-1} + Q(\alpha^2 + \beta^2) \\ b(1-b)t^{-2} &= -\tfrac{1}{2}A_0{}^2 t^{-2} + \bar{p}^* + \tfrac{2}{3}\eta\alpha t^{-1} \\ b(1-b)t^{-2} &= \tfrac{1}{2}A_0{}^2 t^{-2} + \bar{\rho}\beta^2 + \bar{p}^*\alpha^2 - \tfrac{2}{3}\eta(3b-1)\,\alpha^3 t^{-1} - 2Q\alpha\beta \\ b(1-b)t^{-2} &= \tfrac{1}{2}A_0{}^2 t^{-2} + \bar{p}^* + \tfrac{2}{3}\eta(3b-2)\,\alpha t^{-1} \end{aligned} \tag{49}$$

and the solution is

$$\bar{\rho} = b(1-b)(1-2b)(1+2\beta^2)\,[(1-2b)+(1-b)\beta^2]^{-1}\,t^{-2} \tag{50}$$

$$\bar{p}^* = \tfrac{1}{3}b(1-b)\,[3(1-2b)+2(2-3b)\beta^2]\,[(1-2b)+(1-b)\beta^2]^{-1}\,t^{-2} \tag{51}$$

$$\eta = b(1-b)\,[(1-2b)+(1-b)\beta^2]^{-1}\,\alpha^{-1}\beta^2 t^{-1} \tag{52}$$

$$Q = 2\alpha\beta(1+2\beta^2)^{-1}\,\bar{\rho} \tag{53}$$

$$A_0{}^2 = 2b(1-b)^2\,[(1-2b)+(1-b)\beta^2]^{-1}\,\beta^2 \tag{54}$$

The conditions $\bar{\rho} - \bar{p}^* \geqslant 0$, $\bar{\rho} > 0$, $\bar{p}^* \geqslant 0$, $\eta > 0$, and $A_0{}^2 > 0$ are satisfied if $b \leqslant \frac{1}{3}$. Note that the viscous fluid alone satisfies the strong energy condition but not the dominant energy condition [5]. However, the latter condition is obviously satisfied by the total stress-energy tensor, since the perfect fluid stress-energy tensor does so, and by the electromagnetic field alone.

Equation (33) takes the form

$$Q = \kappa\beta(\dot{T} + bt^{-1}T)$$

assuming that $T = T(t)$. If we further assume that

$$T = T_0 t^{-m} \tag{55}$$

where T_0 is a constant and $m \geqslant 0$, then we find that

$$\kappa = 2b(1-b)(1-2b)(b-m)^{-1}\,[(1-2b)+(1-b)\beta^2]^{-1}\,\alpha t^{m-1} \tag{56}$$

which will be positive if $b > m$.

Hence, we have shown that the space-time with metric (46), which is an exact solution of the field equations for a perfect fluid, is also an exact solution of the field equations (3) for a viscous magnetohydrodynamic fluid with $u_\mu = (-\alpha, 0, \beta t^b, 0)$, $q_\mu = Q(\beta, 0, -\alpha t^b, 0)$, where α, β are arbitrary real constants satisfying $\alpha^2 - \beta^2 = 1$, and the other physical quantities given by equations (48) and (50)–(56). All energy conditions and all positivity conditions are satisfied if $\frac{1}{3} \geqslant b > m \geqslant 0$, and all the physical quantities are infinite at $t = 0$ and approach zero as $t \to \infty$. The expansion and shear of the fluid velocity vector are given by

$$\theta = \alpha t^{-1}, \qquad \sigma^2 = (b^2 - b + \tfrac{1}{3})\,\alpha^2 t^{-2} \tag{57}$$

and the vorticity and acceleration are zero.

Case 2. $S^\mu = 0$. We again use the canonical form given by equations (16)–(20), but in this case $u_i = (-1, 0, 0, 0)$ so that $v_{0'} = -\alpha$ and $q_{0'} = 0$. Equation (11) becomes

$$q_\mu = M\alpha(v_\mu - \alpha u_\mu) \tag{58}$$

Contracting with v^μ gives

$$\beta Q = M\alpha(\alpha^2 - 1) \tag{59}$$

and contracting with q^μ gives

$$Q^2 = M\alpha\beta Q$$

i.e.,

$$Q = M\alpha\beta \tag{60}$$

Comparing equations (58) and (60), we again obtain the relation (41).

We shall present two examples of this case, each resulting from slightly different simplifying assumption. We first make the assumption $v_{2'} = v_{3'} = 0$ and then transform v_i into $\bar{v}_i = (-1, 0, 0, 0)$ by a Lorentz transformation in the $(x^{0'}, x^{1'})$ plane under which u_i becomes $\bar{u}_i = (-\alpha, \beta, 0, 0)$. Dropping the bars, the tetrad form of equation (58) is

$$q_i = Q(\beta, -\alpha, 0, 0) \tag{61}$$

and the nonzero shear tensor components are given by

$$\begin{aligned} 2\eta\sigma_{0'0'} &= (M + N - 2M\alpha^2)\,\beta^2 \\ 2\eta\sigma_{0'1'} &= -(M + N - 2M\alpha^2)\,\alpha\beta \\ 2\nu\sigma_{1'1'} &= (M + N - 2M\alpha^2)\,\alpha^2 \\ 2\eta\sigma_{2'2'} &= 2\eta\sigma_{3'3'} = M\alpha^2 - \tfrac{1}{2}(M + N) \end{aligned} \tag{62}$$

Example 2. Consider the Einstein universe with metric

$$ds^2 = -dt^2 + R_0{}^2(1 - r^2)^{-1}\, dr^2 + R_0{}^2 r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \tag{63}$$

This is a perfect fluid solution with $\rho = 2R_0{}^{-2}$, $p = 0$, (i.e., $M = 2R_0{}^{-2}$), and cosmological constant $\Lambda = R_0{}^{-2}$. [In this context the cosmological term $\Lambda g_{\mu\nu}$ should be added to the left-hand sides of equations (1) and (3).] For the viscous magnetohydrodynamic solution we assume that the electromagnetic field consists only of a magnetic field, so that $F_{2'3'} = A$ and $F_{23} = AR_0{}^2 r^2 \sin\theta$ is the only nonzero Maxwell tensor component. Assuming that $A = A(r)$, the Maxwell equations yield $A = A_0 r^{-2}$, where A_0 is a constant, so that $F_{23} = A_0 R_0{}^2 \sin\theta$ and $J^\mu = 0$. The 4-velocity is $u_\mu = [-\alpha, \beta R_0(1 - r^2)^{-1/2}, 0, 0]$, so that the magnetic field components are

$$B_\mu = A_0 r^{-2} \operatorname{cosec}\theta\, [\beta, -\alpha R_0 (1 - r^2)^{-1/2}, 0, 0] \tag{64}$$

The tetrad form of the field equation (3) yields

$$\begin{aligned} 2R_0{}^{-2} &= \tfrac{1}{2}A_0{}^2 r^{-4} + \bar{\rho}\alpha^2 + \bar{p}^*\beta^2 - \tfrac{4}{3}\eta\beta^2 X - 2Q\alpha\beta \\ 0 &= -\tfrac{1}{2}A_0{}^2 r^{-4} + \bar{\rho}\beta^2 + \bar{p}^*\alpha^2 - \tfrac{4}{3}\eta\alpha^2 X - 2Q\alpha\beta \\ 0 &= \tfrac{1}{2}A_0{}^2 r^{-4} + \bar{p}^* + \tfrac{2}{3}\eta X \\ 0 &= -(\bar{\rho} + \bar{p}^*)\,\alpha\beta + \tfrac{4}{3}\eta\alpha\beta X + Q(\alpha^2 + \beta^2) \end{aligned} \tag{65}$$

where, assuming that α and β are functions of r only,

$$X = (\beta' - \beta r^{-1})(1 - r^2)^{1/2} R_0^{-1} \tag{66}$$

the prime denoting differentiation with respect to r. Equations (65) yield

$$\begin{aligned} \bar{\rho} &= 2R_0^{-2}\alpha^2 - \tfrac{1}{2}A_0^2 r^{-4} \\ \bar{p}^* &= \tfrac{2}{3}R_0^{-2}\beta^2 - \tfrac{1}{6}A_0^2 r^{-4} \\ Q &= 2R_0^{-2}\alpha\beta \\ \eta X &= -R_0^{-2}\beta^2 - \tfrac{1}{2}A_0^2 r^{-4} \end{aligned} \tag{67}$$

Let us choose β so that $\bar{p}^* = 0$, i.e., $\bar{p}^* = p$, so that, from equation (6), we have $\bar{\rho} = \rho$ and $M = N$. Then

$$\beta = \tfrac{1}{2}A_0 R_0 r^{-2} \tag{68}$$

and equations (67) become

$$\bar{\rho} = 2R_0^{-2}, \qquad \bar{p}^* = 0, \qquad \Lambda = R_0^{-2} \tag{69}$$

$$Q = A_0 R_0^{-1} r^{-2}(1 + \tfrac{1}{4}A_0^2 R_0^2 r^{-4})^{1/2} \tag{70}$$

$$\eta = \tfrac{1}{2}A_0 r^{-1}(1 - r^2)^{-1/2} \tag{71}$$

Assuming that $T = T(r)$, the equation of state (33) takes the form

$$Q = \kappa R_0^{-1}(1 - r^2)^{1/2} (\alpha T)'$$

and choosing T to be of the form

$$T = T_0 \alpha^{-1} r^m \tag{72}$$

where T_0 is a constant, we find that

$$\kappa = A_0 r^{-m-1}(1 - r^2)^{-1/2} T_0^{-1} m^{-1} \alpha \tag{73}$$

so that we must have $m > 0$ for $\kappa > 0$.

Here, we have shown that the static, spherically symmetric Einstein universe is an exact solution of the field equations for a viscous magnetohydrodynamic fluid. There are many possible choices of velocity vector which will allow this space-time to satisfy the equations (3). In our particular choice the density, pressure, and cosmological constant are identical to that of the perfect fluid solution, but there is a tilting 4-velocity given by

$$u_\mu = [-(1 + \tfrac{1}{4}A_0^2 R_0^2 R^{-4})^{1/2}, \tfrac{1}{2}A_0^2 R_0^2 r^{-2}(1 - r^2)^{-1/2}, 0, 0] \tag{74}$$

The heat conduction vector is

$$q_\mu = Q[\tfrac{1}{2}A_0 R_0 r^2, -(1 + \tfrac{1}{4}A_0^2 R_0^2 r^{-4})^{1/2} R_0(1 - r^2)^{-1/2}, 0, 0] \tag{75}$$

and Q and other physical quantities given by equations (64) and (70)–(73). The fluid velocity vector has neither vorticity nor expansion, but has shear given by

$$\sigma^2 = \tfrac{3}{4} A_0{}^2 r^{-6} (1 - r^2) \tag{76}$$

and the acceleration vector is

$$a_\mu = \tfrac{1}{4} A_0{}^2 R_0{}^2 r^{-5} [A_0 r^{-2} (1 - r^2)^{1/2} \alpha^{-1}, -2, 0, 0] \tag{77}$$

Note that, although the dominant energy condition is everywhere satisfied for the total field, the requirement that the condition should hold for each of the electromagnetic and viscous fluid fields is satisfied only in the region of space-time given by $r^4 > \frac{1}{2} A_0{}^2 R_0{}^2$, which can be very large if the electromagnetic field, i.e., A_0, is very small. This is similar to the situation found in [1], in which some of the examples of magnetohydrodynamic solutions were not valid in the entire domain of validity of the associated perfect fluid solution.

For another example of this case we look for a solution in which both the perfect fluid and the viscous magnetohydrodynamic fluid have the same 4-velocity, i.e.,

$$u_i = v_i = (-1, 0, 0, 0) \tag{78}$$

so that $\alpha = 1$, $\beta = 0$, and, from equation (58), $q_\mu = 0$. Equation (37) becomes

$$A^2 = E^2 + B^2 = \tfrac{3}{2}(M - N) \tag{79}$$

so that $M > N$, and the only nonzero components of equation (38) are

$$\begin{aligned} 2\eta\sigma_{1'1'} &= -(M - N) \\ 2\eta\sigma_{2'2'} = 2\eta\sigma_{3'3'} &= \tfrac{1}{2}(M - N) \end{aligned} \tag{80}$$

To illustrate this situation we need a space-time for which the comoving velocity vector has a nonzero shear, since otherwise $M = N$ and the solution degenerates into the original perfect fluid solution.

Example 3. Consider the type-II cosmology with metric [4]

$$ds^2 = -dt^2 + k^{-2} t^{2a} dx^2 + t^{4a-2} (dy + x\, dz)^2 + t^{2a} dz^2 \tag{81}$$

where $k^2 = 2(1 - a)(2a - 1)$ and $\frac{1}{2} < a \leqslant \frac{3}{4}$. This is a perfect fluid solution with density and pressure given by

$$\rho = \tfrac{1}{2}(3a - 1)(4a - 1) t^{-2}, \qquad p = \tfrac{1}{2}(3a - 1)(3 - 4a) t^{-2} \tag{82}$$

so that $M = (3a - 1) t^{-2}$. Note that $p = 0$ when $a = \frac{3}{4}$, and $p \to \rho$ as $a \to \frac{1}{2}$.

The nonzero components of the tetrad and its inverse that diagonalize the Einstein tensor are

$$\begin{aligned} &e^{0'}{}_0 = 1, \quad e^{1'}{}_1 = k^{-1} t^a, \quad e^{2'}{}_2 = t^{2a-1}, \quad e^{2'}{}_3 = x t^{2a-1}, \quad e^{3'}{}_3 = t^a \\ &e_{0'}{}^0 = 1, \quad e_{1'}{}^1 = k t^{-a}, \quad e_{2'}{}^2 = t^{1-2a}, \quad e_{3'}{}^2 = -x t^{-a}, \quad e_{3'}{}^3 = t^{-a} \end{aligned} \tag{83}$$

and the nonzero local components of the shear tensor computed from u_μ are

$$\sigma_{2'2'} = -\tfrac{2}{3}(1-a)\,t^{-1}, \qquad \sigma_{1'1'} = \sigma_{3'3'} = \tfrac{1}{3}(1-a)\,t^{-1} \tag{84}$$

These values are in accordance with the expressions (80), and, since $a < 1$, also satisfy the requirement $M > N$, but the labeling of the variables has been permuted $1' \to 2' \to 3' \to 1'$. Accordingly, the nonzero local Maxwell tensor components are

$$F_{0'2'} = A\cos\gamma, \qquad F_{3'1'} = A\sin\gamma \tag{85}$$

and, using the tetrad (87), the coordinate components are

$$\begin{aligned} F_{02} &= At^{2a-1}\cos\gamma \\ F_{03} &= Axt^{2a-1}\cos\gamma \\ F_{31} &= Ak^{-1}t^{2a}\sin\gamma \end{aligned} \tag{86}$$

The electric and magnetic field vectors are thus

$$\begin{aligned} E_\mu &= -At^{2a-1}\cos\gamma\,(0,0,1,x) \\ B_\mu &= -At^{2a-1}\sin\gamma\,(0,0,1,x) \end{aligned} \tag{87}$$

We find that Maxwell's equations are satisfied by

$$A = A_0 t^{-m} \tag{88}$$

where A_0 is an arbitrary constant and $m > 0$, and by γ = const given by

$$\tan\gamma = k(m-2a)^{-1} \tag{89}$$

The only nonzero component of J^μ is

$$J^2 = -[k\sin\gamma + (m-2a)\cos\gamma]\,A_0 t^{-m-2a} \tag{90}$$

and this is consistent with equation (32) if $\zeta = 0$ and

$$\lambda = [k^2 + (m-2a)^2]\,(m-2a)^{-1}\,t^{-1} \tag{91}$$

which shows that $m \geqslant 2a$ for $\lambda > 0$. Note that the value $m = 2a$, which corresponds to $\cos\gamma = 0$, i.e., zero electric field, yields the case of infinite conductivity.

The tetrad form of the field equations (3) gives

$$\begin{aligned} \tfrac{1}{2}(3a-1)(4a-1)\,t^{-2} &= \tfrac{1}{2}A_0{}^2 t^{-2m} + \bar{\rho} \\ \tfrac{1}{2}(3a-1)(3-4a)\,t^{-2} &= -\tfrac{1}{2}A_0{}^2 t^{-2m} + \bar{p}^* + \tfrac{4}{3}\eta(1-a)\,t^{-2} \\ \tfrac{1}{2}(3a-1)(3-4a)\,t^{-2} &= \tfrac{1}{2}A_0{}^2 t^{-2m} + \bar{p}^* - \tfrac{2}{3}\eta(1-a)\,t^{-2} \end{aligned} \tag{92}$$

from which we obtain

$$\bar{\rho} = \tfrac{1}{2}(3a-1)(4a-1)\,t^{-2} - \tfrac{1}{2}A_0{}^2 t^{-2m} \tag{93}$$

$$\bar{p}^* = \tfrac{1}{2}(3a-1)(3-4a)\,t^{-2} - \tfrac{1}{6}A_0{}^2 t^{-2m} \tag{94}$$

$$\eta = \tfrac{1}{2}A_0{}^2(1-a)^{-1}\,t^{1-2m} \tag{95}$$

Hence $\eta > 0$ always and $\bar{p}$ will be positive after a certain time t which will be small if A_0 is sufficiently small. The same will be true of $\bar{p}^*$ except in the case of $a = \frac{3}{4}$, i.e., $p = 0$, when $\bar{p}^*$ is always negative. However, since $\bar{p}^* = \bar{p} - \xi\Theta$ and ξ and Θ are both positive, it is not unexpected that $\bar{p}^*$ can be negative. If, for example, we take $a = \frac{5}{8}$, which is the case of radiating matter, $\rho = 3p$, and consequently $\bar{\rho} = 3\bar{p}^*$, we find that $\bar{p}$ and $\bar{p}^*$ are positive and the electromagnetic and viscous fluid fields each satisfy the dominant energy condition after a time t_0 given by

$$t_0^{\,2m-2} = \tfrac{16}{7} A_0^{\,2} \tag{96}$$

Since $m \geqslant 2a > 1$, the power of t_0 is positive and so t_0 will be very small when A_0 is very small.

Thus the Bianchi type-II perfect fluid model with metric (81) is also an exact solution of the field equations (3) with the identical comoving 4-velocity of the perfect fluid solution, with no heat conduction, and with values of the various associated physical quantities given by equations (87)-(91) and (93)-(95). Unlike the perfect fluid solution, this viscous magnetohydrodynamic solution is not valid for all values of t, but is valid after a certain finite time which depends on the magnitude of the electromagnetic field. All physical quantities are finite at that time and tend to zero as $t \to \infty$. The kinematical quantities are, of course, identical to those of the perfect fluid solution, viz.,

$$\Theta = (4a - 1)\, t^{-1}, \qquad \sigma^2 = \tfrac{1}{3}(1 - a)^2\, t^{-2} \tag{97}$$

and $a^\mu = 0$.

§(5): *Null Electromagnetic Field*

Using the canonical form given by equations (28)-(30), we have $u_i = (-1, 0, 0, 0)$ and

$$v_i = (-\alpha, v_{1'}, v_{2'}, v_{3'}) \tag{98}$$

The tetrad versions of equations (5), (8), (11) and equation (12) yield

$$q_i = (0, M\alpha v_{1'}, \pm E^2 + M\alpha v_{2'}, M\alpha v_{3'}) \tag{99}$$

$$\beta Q = M\alpha(\alpha^2 - 1) \pm E^2 v_{2'} \tag{100}$$

$$Q^2 = M^2\alpha^2(\alpha^2 - 1) \pm 2E^2 M\alpha v_{2'} + E^4 \tag{101}$$

$$\left.\begin{aligned}
2\eta\sigma_{1'1'} &= -Mv_{1'}^{\,2} - \tfrac{1}{4}(M - N)\\
2\eta\sigma_{1'2'} &= -Mv_{1'}v_{2'}\\
2\eta\sigma_{1'3'} &= -Mv_{1'}v_{3'}\\
2\eta\sigma_{2'2'} &= M(\alpha^2 - v_{2'}^{\,2}) - \tfrac{1}{2}(M + N)\\
2\eta\sigma_{2'3'} &= -Mv_{2'}v_{3'}\\
2\eta\sigma_{3'3'} &= -Mv_{3'}^{\,2} - \tfrac{1}{4}(M - N)
\end{aligned}\right\} \tag{102}$$

As in the case of nonnull fields, we shall first look for a solution in which $v_{1'} = v_{3'} = 0$, $v_{2'} = \pm(\alpha^2 - 1)^{1/2}$. In this case $q_i = (0, 0, \pm E^2 + M\alpha v_{2'}, 0)$, i.e., q_i is parallel to S_i. Contracting equation (11) with q^μ gives

$$Q^2 - M\alpha\beta Q = E^2(E^2 \pm M\alpha v_{2'}) \tag{103}$$

and equations (100), (101), and (103) yield $v_{2'}{}^2 = \beta^2$, so that equation (41) again holds. Choosing $v_{2'} = -\beta$, we have $v_i = (-\alpha, 0, -\beta, 0)$. We now transform v_i into $\bar{v}_i = (-1, 0, 0, 0)$ by a Lorentz transformation in the $(x^{0'}, x^{2'})$ plane under which u_i becomes $\bar{u}_i = (-\alpha, 0, \beta, 0)$ and E_{ij} retains the form (29) with E^2 replaced by $E^2(\alpha \mp \beta)^2$, S_i becomes $\bar{S}_i = \pm E^2(-\beta, 0, \alpha, 0)$, and q_i becomes $\bar{q}_i = (M\alpha\beta \mp E^2)(\beta, 0, -\alpha, 0)$. Dropping the bars, equations (102) are replaced by

$$\begin{aligned} 2\eta\sigma_{0'0'} &= \tfrac{1}{2}(M-N)\beta^2, \qquad 2\eta\sigma_{0'2'} = -\tfrac{1}{2}(M-N)\alpha\beta \\ 2\eta\sigma_{2'2'} &= \tfrac{1}{2}(M-N)\alpha^2, \qquad 2\eta\sigma_{1'1'} = 2\eta\sigma_{3'3'} = -\tfrac{1}{4}(M-N) \end{aligned} \tag{104}$$

Example 4. Consider the Einstein-de Sitter universe with metric

$$ds^2 = -dt^2 + t^{4/3}(dx^2 + dy^2 + dz^2) \tag{105}$$

which is a dust model with $\rho = \frac{4}{3}t^{-2}$. For the viscous magnetohydrodynamic solution we have, from equation (30) and the Lorentz transformation of the last paragraph,

$$F_{0'3'} = \pm F_{2'3'} = -(\alpha \mp \beta)E = -A \tag{106}$$

where A is introduced for convenience. Then $F_{03} = -At^{2/3}$, $F_{23} = \mp At^{4/3}$ and, assuming that $A = A(t)$, Maxwell's equations give

$$A = A_0 t^{-4/3} \tag{107}$$

where A_0 is a constant, and $J^\mu = 0$, so that

$$F_{03} = -A_0 t^{2/3}, \qquad F_{23} = \mp A_0 \tag{108}$$

The 4-velocity is $u_\mu = (-\alpha, 0, \beta t^{2/3}, 0)$ and, if α, β are functions of t only, the tetrad components of the shear tensor calculated from u_μ are

$$\begin{gathered} \sigma_{0'0'} = \tfrac{2}{3}\beta^2\dot{\alpha}, \qquad \sigma_{0'2'} = \tfrac{2}{3}\alpha\beta\dot{\alpha}, \qquad \sigma_{2'2'} = \tfrac{2}{3}\alpha^2\dot{\alpha} \\ \sigma_{1'1'} = \sigma_{3'3'} = -\tfrac{1}{3}\dot{\alpha} \end{gathered} \tag{109}$$

which are in accord with the expressions (104) if

$$\eta\dot{\alpha} = \tfrac{3}{8}(M - N) \tag{110}$$

Note that this shows that α cannot be a constant, since this would imply that $M = N$, the shear is zero, and the matter distribution degenerates to the perfect fluid case.

Using the expressions (104), the tetrad components of the field equations

(3) are

$$\tfrac{4}{3}t^{-2} = A_0{}^2 t^{-8/3} + \bar{\rho}\alpha^2 + \bar{p}^*\beta^2 - \tfrac{1}{2}(M-N)\beta^2 - 2Q\alpha\beta$$
$$0 = \pm A_0{}^2 t^{-8/3} - (\bar{\rho} + \bar{p}^*)\alpha\beta + \tfrac{1}{2}(M-N)\alpha\beta + Q(\alpha^2 + \beta^2)$$
$$0 = A_0{}^2 t^{-8/3} + \bar{\rho}\beta^2 + \bar{p}^*\alpha^2 - \tfrac{1}{2}(M-N)\alpha^2 - 2Q\alpha\beta$$
$$0 = \bar{p}^* + \tfrac{1}{4}(M-N)$$

the solution of which is

$$\begin{aligned} \bar{\rho} &= \tfrac{4}{3}\alpha^2 t^{-2} - A_0{}^2(\alpha \pm \beta)^2\, t^{-8/3} \\ \bar{p}^* &= \tfrac{4}{9}\beta^2 t^{-2} - \tfrac{1}{3}A_0{}^2(\alpha \pm \beta)^2\, t^{-8/3} \\ Q &= \tfrac{4}{3}\alpha\beta t^{-2} \mp A_0{}^2(\alpha \pm \beta)^2\, t^{-8/3} \\ \eta\dot{\alpha} &= -\tfrac{2}{3}\beta^2 t^{-2} + \tfrac{1}{2}A_0{}^2(\alpha \pm \beta)^2\, t^{-8/3} \end{aligned} \tag{111}$$

The velocity components α and β must be chosen such that the usual energy conditions hold. A suitable choice is

$$\alpha = (1 + ht^{-1/3})(1 + 2ht^{-1/3})^{-1/2}, \qquad \beta = \mp ht^{-1/3}(1 + 2ht^{-1/3})^{-1/2} \tag{112}$$

where h is an arbitrary positive constant and the dual sign in the expression for β corresponds to the dual sign in the $E_{0'2'}$ term. Equations (111) then become

$$\bar{\rho} = [\tfrac{4}{3}(1 + ht^{-1/3})^2\, t^{-2} - A_0{}^2 t^{-8/3}]\,(1 + 2ht^{-1/3})^{-1} \tag{113}$$

$$\bar{p}^* = \tfrac{1}{3}(\tfrac{4}{3}h^2 - A_0{}^2)\, t^{-8/3}(1 + 2ht^{-1/3})^{-1} \tag{114}$$

$$Q = \mp[4h(1 + ht^{-1/3}) + A_0{}^2 t^{-1/3}]\, t^{-7/3}(1 + 2ht^{-1/3})^{-1} \tag{115}$$

$$\eta = \tfrac{1}{2}(4 - 3A_0{}^2 h^{-2})\, t^{-1}(1 + 2ht^{-1/3})^{1/2} \tag{116}$$

from which we see that βQ is always positive, and $\bar{\rho}$, $\bar{p}^*$, and η are also always positive provided that we choose

$$4h^2 > 3A_0{}^2 \tag{117}$$

From equation (106), the magnitude of the electric and magnetic fields is

$$|E| = |A|\,(\alpha \pm \beta) = |A_0|\, t^{-4/3}(1 + 2ht^{-1/3})^{-1/2} \tag{118}$$

In order to find the thermal conductivity we shall assume that T is of the form

$$T = T_0\alpha^{-1} t^{-m} \tag{119}$$

where T_0 is a constant and $m > 0$. Equation (33) then leads to

$$\kappa = [\tfrac{4}{3}(1 + ht^{-1/3}) + h^{-1}A_0{}^2 t^{-1/3}]\,(1 + ht^{-1/3})^2\,(1 + 2ht^{-1/3})^{-1} \times [(\tfrac{1}{3} - m) + (\tfrac{2}{3} - m)ht^{-1/3}]^{-1}\, t^{m-1} \tag{120}$$

which is always positive if $m \leqslant \frac{1}{3}$. T will be finite at $t = 0$ if $m \leqslant \frac{1}{6}$ and infinite at $t = 0$ if $m > \frac{1}{6}$.

By diagonalizing the stress-energy tensor of the viscous fluid alone we find that, although the total stress-energy tensor always satisfies the dominant energy condition, the viscous fluid stress-energy tensor satisfies this condition only after a time t_0 given by

$$t_0 = (\tfrac{3}{2})^{3/2} A_0{}^3$$

which is very small if A_0 is small.

Hence, the Einstein–de Sitter universe is an exact solution of the viscous magnetohydrodynamic field equations, with a null electromagnetic field given by equations (108) and (118). The 4-velocity is given by

$$u_\mu = (1 + 2ht^{-1/3})^{-1/2} \,[-(1 + ht^{-1/3}), 0, \mp ht^{1/3}, 0] \tag{121}$$

and the heat conduction vector by

$$q_\mu = Q(1 + 2ht^{-1/3})^{-1/2} \,[\mp ht^{-1/3}, 0, -(1 + ht^{-1/3})\, t^{2/3}, 0] \tag{122}$$

where h is a positive constant satisfying equation (117). The other physical quantities are given by equations (113)–(116) and (118)–(120). The fluid velocity vector has expansion, shear, and acceleration given by

$$\Theta = (2 + 6ht^{-1/3} + \tfrac{11}{3} h^2 t^{-2/3})\, t^{-1} (1 + 2ht^{-1/3})^{-3/2}$$
$$\sigma^2 = \tfrac{1}{27} h^4 t^{-10/3} (1 + 2ht^{-1/3})^{-3}$$
$$a_\mu = \mp \tfrac{1}{3} ht^{-1/3} (1 + 3ht^{-1/3})\,(1 + 2ht^{-1/3})^{-3/2}\,(\beta, 0, -\alpha, 0)$$

If $\frac{1}{3} \leqslant m < \frac{1}{6}$, all physical quantities are infinite at $t = 0$ and tend to zero as $t \to \infty$.

For our next example we attempt to find a solution corresponding to Example 3 in which the perfect fluid and the viscous fluid have the same comoving 4-velocity, i.e., $u_i = v_i = (-1, 0, 0, 0)$, so that $\alpha = 1$ and, from equation (100), $\beta = 0$. Equations (101) become

$$2\eta\sigma_{2'2'} = \tfrac{1}{2}(M - N), \qquad 2\eta\sigma_{1'1'} = 2\eta\sigma_{3'3'} = -\tfrac{1}{4}(M - N) \tag{123}$$

and equation (12) is

$$E^2 = \tfrac{3}{4}(M - N) \tag{124}$$

so that $M > N$, i.e., $\sigma_{2'2'} > 0$. Equations (11) and (28) show that

$$q_i = S_i = \pm \tfrac{3}{4}(M - N)\,(0, 0, 1, 0) \tag{125}$$

The tetrad components of the field equation (3) give no further information.

Hence, we require a perfect fluid solution with a shearing locally comoving 4-velocity such that the $\sigma_{2'2'}$ tetrad component of the shear is positive, while $\sigma_{1'1'} = \sigma_{3'3'}$ are negative. Unfortunately, we have been unable to find such a solution; all of our attempts have resulted in models for which $\sigma_{2'2'}$ is negative if the energy conditions are to hold for the viscous fluid. However, we have no reason to assume that such a solution does not exist.

Our final example of the null electromagnetic case is obtained by considering the possibility $q_\mu = 0$. In this case β is not defined by equation (8), so we define β by equation (41). The $(0', 1')$ and $(0', 3')$ components of the field equations show that $v_{1'} = v_{3'} = 0$, so that $v_{2'}{}^2 = \beta^2$. We chose $v_{2'} = -\beta$ and make the same Lorentz transformation used previously to obtain $v_i = (-1, 0, 0, 0)$, $u_i = (-\alpha, 0, \beta, 0)$, and

$$E^2 = \pm M\alpha\beta \tag{126}$$

From equations (12) and (106) we find that

$$\alpha^2 = \frac{(M+3N)^2}{8M(3N-M)}, \qquad \beta^2 = \frac{3(M-N)^2}{8M(3N-M)} \tag{127}$$

so that $M < 3N$. The tetrad components of the shear tensor are again given by equation (104).

Example 5. Consider the zero-curvature FRW model with metric

$$ds^2 = -dt^2 + t(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2) \tag{128}$$

This model, which describes radiating matter, i.e., $\rho = 3p = \frac{3}{4}t^{-2}$, was the subject of one of the examples of [1], in which it was shown to be an exact solution of the field equation (3) with $\sigma_{\mu\nu} = 0$ and $q_\mu \neq 0$. Here we shall show that it is also an exact solution of the field equations (3) with $\sigma_{\mu\nu} \neq 0$ and $q_\mu = 0$. Note that we could use cartesian coordinates here, as in Example 4, but we shall work in spherical polar coordinates so that we can compare the results with those found in [1].

In order to obtain a shear tensor corresponding to equation (104) we must choose the spacelike component to be in the direction of the coordinate r, so we must permute the spacelike coordinate labels so that (x^1, x^2, x^3) correspond to (ϕ, r, θ). The tetrad components of the Maxwell tensor are given by equation (106) and the coordinate components are $F_{03} = -Art^{1/2}$, $F_{23} = \mp Art$. Maxwell's equations lead to

$$A = A_0 r^{-1} t^{-1} \operatorname{cosec}\theta \tag{129}$$

where A_0 is a constant, and $J^\mu = 0$, so that

$$F_{03} = -A_0 t^{-1/2} \operatorname{cosec}\theta, \qquad F_{23} = \mp A_0 \operatorname{cosec}\theta \tag{130}$$

The 4-velocity is $u_\mu = (-\alpha, 0, \beta t^{1/2}, 0)$ and the tetrad components of the resulting shear tensor are

$$\begin{gathered}\sigma_{0'0'} = \tfrac{1}{3}\beta^2 X, \qquad \sigma_{0'2'} = -\tfrac{1}{3}\alpha\beta X, \qquad \sigma_{2'2'} = \tfrac{1}{3}\alpha^2 X \\ \sigma_{1'1'} = \sigma_{3'3'} = -\tfrac{1}{6}X\end{gathered} \tag{131}$$

where

$$X = \dot\alpha + (\beta' - \beta r^{-1})\, t^{-1/2} \tag{132}$$

The expressions (104) and (131) are in accord if

$$\eta X = \tfrac{3}{4}(M - N) \tag{133}$$

Since $\rho = 3p$, it follows that $\bar{\rho} = 3\bar{p}^*$ and the field equation (3) become

$$\begin{aligned} \tfrac{3}{4}t^{-2} &= A_0{}^2 t^{-2} r^{-2} \operatorname{cosec}^2\theta + \bar{\rho}(\alpha^2 + \tfrac{1}{3}\beta^2) - \tfrac{4}{3}\eta\beta^2 X \\ \tfrac{1}{4}t^{-2} &= A_0{}^2 t^{-2} r^{-2} \operatorname{cosec}^2\theta + \bar{\rho}(\beta^2 + \tfrac{1}{3}\alpha^2) - \tfrac{4}{3}\eta\alpha^2 X \\ 0 &= \pm A_0{}^2 t^{-2} r^{-2} \operatorname{cosec}^2\theta - \tfrac{4}{3}\bar{\rho}\alpha\beta + \tfrac{4}{3}\eta\alpha\beta X \\ \tfrac{1}{4}t^{-2} &= \tfrac{1}{3}\bar{\rho} + \tfrac{2}{3}\eta X \end{aligned} \tag{134}$$

which yield

$$\pm\alpha\beta(\alpha \mp \beta)^2 = A_0{}^2 r^{-2} \operatorname{cosec}^2\theta \tag{135}$$

showing that the upper sign applies when $\beta > 0$ and the lower sign when $\beta < 0$. The solution of equation (135) is

$$\begin{aligned} \alpha &= \tfrac{1}{2}\left[1 + (1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)^{1/2}\right](1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)^{-1/4} \\ \beta &= \pm\tfrac{1}{2}\left[1 - (1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)^{1/2}\right](1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)^{-1/4} \end{aligned} \tag{136}$$

where we must have

$$r^2 \sin^2\theta > 4A_0{}^2 \tag{137}$$

i.e., the region of validity is that region of space-time outside the infinite cylinder with axis along the Cartesian z axis and radius $2|A_0|$. This is precisely the same region in which the solution of [1] is valid.

From equations (134) and (136) we obtain

$$\bar{\rho} = 3\bar{p}^* = \tfrac{1}{4}\left[1 + 2(1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta\right]^{1/2}]\, t^{-2} \tag{138}$$

$$\eta X = \tfrac{1}{4}\left[1 - (1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)^{1/2}\right] \tag{139}$$

Both expressions are always positive and the second shows that we must have $X > 0$ for $\eta > 0$. Now, from equations (132) and (136), we find that

$$X = \pm\tfrac{1}{4}r^{-1}t^{-1/2}(1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)^{-5/4}\left[(2 - 3A_0 r^{-2}\operatorname{cosec}^2\theta)\cdot(1 - 4A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)^{1/2} - (2 - A_0{}^2 r^{-2}\operatorname{cosec}^2\theta)\right] \tag{140}$$

Taking into account the restriction (137), it is easily shown that the quantity within the square brackets in equation (140) is always negative so that we must take the lower sign to ensure that $\eta > 0$. Hence, this example is unlike Example 4 in that both signs are not possible. The $u_{2'}$ tetrad component must be negative, i.e., in the inward direction, as is the case with the example discussed in [1].

The electromagnetic stress-energy tensor clearly satisfies the dominant energy condition. Diagonalizing the stress-energy tensor of the viscous fluid alone shows that this tensor satisfies this condition in the region of validity given by equation (137).

Thus we have shown that the radiation-filled zero curvature FRW cosmology with metric (128) is an exact solution of the field equations for a viscous magnetohydrodynamic fluid with no heat conduction. Putting $x = 4A_0{}^2 r^{-2} \operatorname{cosec}^2 \theta$ for brevity, the 4-velocity is

$$u_\mu = \{-\tfrac{1}{2}[1 + (1-x)^{1/2}](1-x)^{-1/4}, 0, -\tfrac{1}{2}[1-(1-x)^{1/2}](1-x)^{-1/4} t^{1/2}, 0\} \tag{141}$$

and the shear conductivity is

$$\eta = [(2-x) - (2-3x)(1-x)^{1/2}]^{-1}\,[1-(1-x)^{1/2}](1-x)^{5/4} r t^{1/2} \tag{142}$$

The density and pressure are given by equation (138) and the electric and magnetic fields are

$$E_\mu = E(0,0,0,1), \qquad \beta_\mu = -E(0,1,0,0) \tag{143}$$

where

$$E = \tfrac{1}{2}(3)^{1/2} A_0 t^{-1} r^{-1} \operatorname{cosec}\theta\,(1-x)^{-1/4} \tag{144}$$

The fluid velocity vector has expansion, shear, and acceleration given by

$$\Theta = \tfrac{3}{2}[1+(1-x)^{1/2}](1-x)^{-1/4} t^{-1} - \tfrac{1}{2}[(2-\tfrac{5}{2}x) - (2-\tfrac{3}{2}x)(1-x)^{1/2}](1-x)^{-5/4} t^{-1/2}$$

$$\sigma^2 = \tfrac{1}{192}[(2-\tfrac{1}{4}x) + (2-\tfrac{3}{4}x)(1-x)^{1/2}]^2 (1-x)^{-5/2} r^{-2} t^{-1}$$

$$a_\mu = -\tfrac{1}{8}[xr^{-1} + 4(1-x)t^{-1/2}](1-x)^{-3/2}\{[1-(1-x)^{1/2}]t^{-1/2}, 0, x, 0\}$$

§(6): *Viscous Fluid Only*

Suppose that the electromagnetic field in the field equation (3) is zero, so that we seek perfect fluid space-times which also satisfy the field equations for a viscous fluid. Putting $E_{\mu\nu} = 0$ in equation (9) we have

$$2\eta\sigma_{\mu\nu} = -Mv_\mu v_\nu + Nu_\mu u_\nu - \tfrac{1}{4}(M-N)g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu \tag{145}$$

and equation (11) becomes

$$q_\mu = M\alpha(v_\mu - \alpha u_\mu) \tag{146}$$

so that

$$Q = M\alpha\beta \tag{147}$$

and equation (41) holds. Equation (12) gives

$$\alpha^2 = \frac{M+3N}{4M}, \qquad \beta^2 = \frac{3(N-M)}{4M} \tag{148}$$

which shows that $N > M$.

We choose the local tetrad frame in which the perfect fluid 4-velocity has the comoving form $v_i = (-1, 0, 0, 0)$. Then $u_i = (-\alpha, u_{1'}, u_{2'}, u_{3'})$, $q_i = (\beta Q, -M\alpha^2 u_{1'}, -M\alpha^2 u_{2'}, -M\alpha^2 u_{3'})$, and the tetrad components of the shear tensor are

$$\begin{aligned} 2\eta\sigma_{0'0'} &= -\tfrac{1}{2}(N-M)\beta^2 \\ 2\eta\sigma_{0'a'} &= \tfrac{1}{2}(N-M)\alpha u_{a'} \\ 2\eta\sigma_{a'a'} &= \tfrac{1}{4}(N-M) - \tfrac{1}{2}(M+N)u_{a'}{}^2 \\ 2\eta\sigma_{a'b'} &= -\tfrac{1}{2}(M+N)u_{a'}u_{b'} \end{aligned} \tag{149}$$

where a, b take the values 1, 2, 3, and there is no summation over the repeated suffix.

In seeking a solution, we first note that if we put $A_0 = 0$ in Example 4, we have a viscous fluid solution which satisfies the necessary conditions, and so provides an example of the present case. However, if we put $A_0 = 0$ in the other examples we do not obtain viscous fluid models since the solutions degenerate into the original perfect fluid models. We shall give another example of a solution containing viscous fluid only by considering the special case in which $u_{2'} = u_{3'} = 0$, so that $u_i = (-\alpha, \beta, 0, 0)$ and $q_i = Q(\beta, -\alpha, 0, 0)$. The expressions (149) reduce to

$$\begin{aligned} &2\eta\sigma_{0'0'} = -\tfrac{1}{2}(N-M)\beta^2, \qquad 2\eta\sigma_{0'1'} = \tfrac{1}{2}(N-M)\alpha\beta \\ &2\eta\sigma_{1'1'} = -\tfrac{1}{2}(N-M)\alpha^2, \qquad 2\eta\sigma_{2'2'} = 2\eta\sigma_{3'3'} = \tfrac{1}{4}(N-M) \end{aligned} \tag{150}$$

Example 6. Consider the rotating dust solution with metric [6, Section 19.2]

$$ds^2 = -(dt - z\,d\phi)^2 + dr^2 + r(d\phi^2 + dz^2) \tag{151}$$

for which $v^\mu = (1, 0, 0, 0)$ and $\rho = r^{-2}$. The nonzero components of the tetrad and its inverse that diagonalize the Einstein tensor are

$$\begin{aligned} &e^{0'}{}_0 = 1, \quad e^{0'}{}_2 = -z, \quad e^{1'}{}_1 = 1, \quad e^{2'}{}_2 = r^{1/2}, \quad e^{3'}{}_3 = r^{1/2} \\ &e_{0'}{}^0 = 1, \quad e_{2'}{}^0 = zr^{-1/2}, \quad e_{1'}{}^1 = 1, \quad e_{2'}{}^2 = r^{-1/2}, \quad e_{3'}{}^3 = r^{-1/2} \end{aligned} \tag{152}$$

where the coordinates are labeled $(t, r, \phi, z) = (x^0, x^1, x^2, x^3)$.

For the viscous fluid solution we choose as the local 4-velocity $u_i = (-\alpha, \beta, 0, 0)$, where α and β are constants. The coordinate components are

$$u^\mu = (\alpha, \beta, 0, 0), \qquad u_\mu = (-\alpha, \beta, \alpha z, 0) \tag{153}$$

and the tetrad components of the shear tensor calculated from u_μ are

$$\sigma_{0'0'} = -\tfrac{1}{3}\beta^3 r^{-1}, \quad \sigma_{0'1'} = \tfrac{1}{3}\alpha\beta^2 r^{-1}, \quad \sigma_{1'1'} = -\tfrac{1}{3}\alpha^2\beta r^{-1}, \quad \sigma_{2'2'} = \sigma_{3'3'} = \tfrac{1}{6}\beta r^{-1} \tag{154}$$

which are in accord with the expressions (150) if

$$3(N - M) = 4\eta\beta r^{-1} \tag{155}$$

Since $N > M$ it follows that we must have $\beta > 0$.

The tetrad components of the field equations are

$$\begin{aligned} r^{-2} &= \bar{\rho}\alpha^2 + \bar{p}^*\beta^2 + \tfrac{2}{3}\eta\beta^3 r^{-1} - 2Q\alpha\beta \\ 0 &= -(\bar{\rho} + \bar{p}^*)\alpha\beta - \tfrac{2}{3}\eta\alpha\beta^2 r^{-1} + Q(\alpha^2 + \beta^2) \\ 0 &= \bar{\rho}\beta^2 + \bar{p}^*\alpha^2 + \tfrac{2}{3}\eta\alpha^2\beta r^{-1} - 2Q\alpha\beta \\ 0 &= \bar{p}^* - \tfrac{1}{3}\eta\beta r^{-1} \end{aligned} \tag{156}$$

which yield

$$\bar{\rho} = \alpha^2 r^{-2}, \qquad \bar{p}^* = \tfrac{1}{3}\beta^2 r^{-2} \tag{157}$$

$$\eta = \beta r^{-1}, \qquad Q = \alpha\beta r^{-2} \tag{158}$$

which are all positive. The coordinate components q_μ are

$$q_\mu = \alpha\beta r^{-2}(\beta, -\alpha, -\beta z, 0) \tag{159}$$

and, if $T = T(r)$, equation (33) leads to

$$\kappa T' = \beta r^{-2} \tag{160}$$

where the prime denotes differentiation. Thus T must be an increasing function of r in order that $\kappa > 0$. The dominant energy condition is satisfied by the original perfect fluid stress-energy tensor and so is satisfied by the viscous fluid stress-energy tensor.

Hence, we have shown that the rotating dust model (151) also satisfies the viscous fluid field equations with 4-velocity given by equation (153), and the various physical quantities by equations (157)-(160). The fluid velocity vector has no acceleration, but has nonzero expansion, shear, and vorticity given by

$$\Theta = \beta r^{-1}, \qquad \sigma = \tfrac{1}{6}(3)^{1/2}\,\beta r^{-1}, \qquad \omega = \tfrac{1}{2}\alpha$$

§(7): *Conclusion*

In general relativity theory an exact solution of the field equations consists of two parts, namely, the geometrical part, which is the space-time metric, and the physical part, which consists of the values of the density, pressure, 4-velocity, electromagnetic field, etc. What we have shown is that the geometrical part of an exact solution of the field equations for a perfect fluid can be identical to the geometrical part of an exact solution of the field equations for a viscous magnetohydrodynamic fluid. The physical parts of the two solutions are different but are related by the fact that the total contribution to the stress-energy tensor in each case will be the same. Thus, given the geometrical part of the solution, we do not necessarily have a unique result for the physical part of the solution. In fact, there may be many possibilities for the physical part of the solution, since, for the viscous magnetohydrodynamic case, there is often some freedom of choice in the velocity vector components resulting in a number of possible solutions associated with the same space-time metric, as shown by Example 5 of this article and Example 1 of [1]. Hence, there may exist many alternative physical interpretations of the space-time solutions which we normally regard as describing perfect fluid distributions.

These alternative physical solutions may be described mathematically in the following manner: Consider the Raychaudhuri equation [6, Section 6.2]

$$R_{\mu\nu}u^\mu u^\nu = a^\mu{}_{;\mu} + \omega_{\mu\nu}\omega^{\mu\nu} - \sigma_{\mu\nu}\sigma^{\mu\nu} - \Theta_{;\mu}u^\mu - \tfrac{1}{3}\Theta^2$$

which relates the Ricci tensor to a timelike unit vector field and its associated kinematical quantities. If $R_{\mu\nu}$ is the Ricci tensor of a perfect fluid space-time, then our work shows that, in general, there are many timelike unit vectors satisfying this equation. One of these is the 4-velocity of the perfect fluid solution, and the others are the 4-velocities of the various viscous and/or magnetohydrodynamic exact solutions which share the same space-time geometry.

Most of the examples that we have given are based on spatially homogeneous cosmological models. In such models there is a preferred observer moving along the unique timelike eigenvector, $v^\mu = (1, 0, 0, 0)$, of the Ricci tensor. In the usual perfect fluid interpretation of the model, the preferred observer views the universe to be filled with a perfect fluid which is comoving with him. In the alternative interpretations, the preferred observer views the universe to be filled with a viscous magnetohydrodynamic fluid which, in general, is not comoving with him but whose total stress-energy is the same as that of the perfect fluid. In the particular case of Example 3, the preferred observer actually sees either a perfect fluid, or a viscous magnetohydrodynamic fluid, comoving with himself.

Note that, as in [1], although the constituent physical parts of the perfect fluid solution inherit the symmetries of the spacetime, i.e., their Lie derivatives with respect to the Killing vectors vanish, the constituent physical parts of the viscous magnetohydrodynamic solution do not necessarily inherit these symmetries, although, of course, the total stress-energy tensor does. For example, the Einstein–de Sitter universe of Example 4 is homogeneous and isotropic, but the 4-velocity, and the electric and magnetic fields of the viscous magnetohydrodynamic solution are not isotropic.

Attempts have been made to consider the effects of viscosity and heat conduction in cosmological models by perturbations of standard perfect fluid models [7]. However, we have shown that perturbations are not necessary to introduce these effects; we can obtain exact solutions of the imperfect fluid field equations without any change in the metric of the perfect fluid solution. In a sense our methods can be thought of as a generating technique for obtaining new solutions from old in which only the physical part of the solution changes. Using this technique, solutions of a set of apparently unpleasant field equations can be found with remarkable ease.

The possible effect of these alternative exact viscous magnetohydrodynamic solutions in theoretical cosmology is currently under investigation.

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