SHORT COMMUNICATION

A DIRECT APPROACH TO INDIRECT PROOFS

INTRODUCTION

I begin with observation, continue with generalization and end with speculation. The observation concerns the classical proof-by-contradiction that there are infinitely many primes. It is about the mismatch between the simple appearance of the proof, and the mystification and frustration students often feel when they first encounter it. The generalization seeks to extend this observation to other indirect proofs and suggests an alternative approach. The speculation is about cognitive processes that may partly account for this phenomenon.

In a way, I am continuing a theme begun in [2] and [3]: While most of the work in maths education seeks to improve the learning and communication of mathematics by supplementing or bypassing mathematical formalism, it is also important to consider at the same time how the formalism itself might be improved to become more communicative of the ideas behind it.

IS IT SIMPLE?

Recently I was discussing with a class of student teachers the classical theorem that there are infinitely many prime numbers. The proof of this theorem is traditionally hailed by mathematicians for its simplicity and 'elegance'. Indeed, it can't be denied that the proof is extremely simple *when looked at as a piece of text*. There are indications, however, that its *processing* in the student's mind on first introduction is anything but simple, and in fact gives rise to feelings of frustration and bewilderment. Let us first take a look at the proof. Here it is – eternal, shining, elegant.

> Suppose on the contrary that there is only a finite number of primes, say $p_1, p_2, \ldots, p_n$. Consider the number $M = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$. As is the case with all natural numbers greater than 1, M must have a prime factor, p. Now p must be different from $p_1, p_2, \ldots, p_n$, since these do not divide M (they each leave a remainder of 1). Thus p is a new prime number – contradicting the assumption that we have listed them all.

I had experienced the phenomenon many times before, that this proof, despite its apparent simplicity, leaves most students (young and old alike) mystified

Educational Studies in Mathematics **16** (1985) 321–325. 0013–1954/85.10

the first time they see it. (This is even more pronounced in an oral presentation because of its linearity and uni-directedness.) So I took great care to present it slowly and clearly, involving the students with intermediate steps and carefully explaining all terms used, etc. However, when I finished the proof with the triumphant announcement of the contradiction, I could clearly see that the students were not feeling triumphant at all. In fact their faces showed the same glazed look of bewilderment and blankness. Once more I was left wondering about the nature of this proof and its processing by the students. What is it in the proof that makes it so simple to me, yet so perplexing to the students?

AN ALTERNATIVE

One common approach to making the proof appear more natural, less mysterious, is through discovery learning (e.g., [5]). In spite of all the advantages of this approach, it does not solve my problem – it bypasses it. The question still remains, what is it in the proof itself that makes it hard to digest? This question is not primarily an educational question. It requires an inquiry into the nature of mathematical formalism and the human mind's ways of processing it. However, it also assumes practical value in cases where one cannot afford the time or the conditions necessary for the type of dialogue depicted in [5]. We certainly want students to be actively engaged in constructing and discovering as much of the mathematics they learn as possible, but we also want to find better ways of *presenting* and *communicating* the products of such mathematical activities. Before going on to speculate on the source of these difficulties, here is a presentation of the proof which I found to be much more illuminating and less mysterious and frustrating to students.

> We first introduce a construction (an algorithm) whereby given any list of primes, say $p_1, p_2, \ldots, p_n$, a *new* prime can be obtained. The construction and its proof are as before, namely produce the number $M = p_1 \cdot p_2 \cdot \ldots \cdot p_n + 1$ and pick any of its prime factors.

This part includes the 'meat' of the mathematical argument of our main theorem; what remains is almost trivial. Still, there is no negative assumption and no feeling of mystery about it. It is a good, old-fashioned construction that can be manipulated and played with by the students until they can feel they 'own' it; e.g., given 3, 7, 11, use the construction to obtain a new prime.

Back to the main theorem, we inquire: can the number of prime numbers be finite? Of course not, for our construction guarantees that no finite list of primes can exhaust all primes.

MENTAL REALITY OF MATHEMATICAL OBJECTS

What follows is purely speculative. It is based on reflections on my own thinking and my interaction with students. I hope readers will respond with their own experience and thoughts on these issues.

I think we have here a fundamental psychological issue involving mathematical thinking in relation to *proof by contradiction*. As I have elaborated elsewhere [3], most non-trivial proofs pivot around an *act of construction* – a construction of a new mathematical object (a number, a function, a point, a line, a set, a partition of a set, etc).

At its best, mathematical learning of a proof is based on the learner's construction of a corresponding *mental* entity, an image perhaps, that can then be manipulated in the mind in place of the mathematical object or its symbol on the paper (see also [1]). This is also what gives us the Platonic sense that we are working on a mathematical 'reality', manipulating real objects. (Our mind-entities are very real to us.)

In indirect proofs, however, something strange happens to the 'reality' of these objects. We begin the proof with a declaration that we are about to enter a false, impossible world, and all our subsequent efforts are directed towards 'destroying' this world, proving it is indeed false and impossible. We are thus involved in an act of *mathematical destruction*, not construction. Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered.

What have we really proved in the end? What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created? I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick – a sorcery – that has been played on us. Take the number M for example. Does it really exist (at least in the same sense that other mathematical objects 'exist'), or has it been irrevocably destroyed by the eventual contradiction? Can we still make any *positive* use of the *construction* we carried out in the proof? It has often been said that a proof does not merely establish the truth of a theorem – it can also give insight into the reasons why the theorem is true. What insight do we get from a proof-by-contradiction? Yu. I. Manin said 'A good proof is one that makes you wiser.' What wisdom can be derived from a contradiction?

CONSTRUCTIVE PROOFS

Actually, there is a way to alleviate the frustration, as I hope the above example demonstrates. It is based on the observation that in many indirect proofs, the main construction is independent of the negative assumption. You can therefore *separate out the construction from the negative assumption*, making it a positive act preceding the main (negative) argument. The rest of the proof then often follows instantly 'at a glance'. Besides relieving frustration, it also increases the *mathematical* benefits students can derive from the proof. Often the construction involved in such proofs is of independent interest, but students are not likely to ponder the proof long enough (after they have been convinced that a contradiction indeed results) to see it for themselves. Lastly, there is an additional learning benefit, as the construction (discussed in a positive direction before the main proof-by-contradiction) offers an opportunity for *activity* and exercise which is lacking in the original version.

Readers are invited to try out this 'constructivist' approach on more proofs. A good one to try is Cantor's 'diagonal' proof that the set of real numbers is uncountable [4].

THE NEGATIVE STRETCH OF A PROOF

Thinking of these issues, the following image comes to my mind. Of course, I cannot claim any factual basis for it.

The moment the negative assumption is declared, along with the intention of falsifying it by means of a future contradiction, a cognitive strain is set up in the mind of the learner, perhaps because of the difficulty of living in a false world, still operating as if it were real. This cognitive strain grows (linearly?) with the time spent living in this world, i.e. with the distance between the negative assumption and the terminal contradiction. Perhaps the feeling of frustration and incomprehensibility is proportional to the length of the 'negative stretch' of the proof? With this image, it is easy to see why the constructivist approach exemplified above has a relieving effect, since it greatly reduces the length of the negative stretch of the proof.

CONCLUSION

Summing up the benefits I see in the above 'constructive approach':

- Most of the proof is done in positive mode, enabling students to create and manipulate mental objects in the usual fashion.

- A constructive procedure is learned, which is often of independent interest, but is seldom extracted by students on their own from indirect proofs.
- The constructive procedure provides activities and practice which were not available when operating in negative mode. These activities are helpful in the process of creating the necessary mental objects to be manipulated in the proof proper. They help learners feel they 'own' the concepts involved.
- The extraction of a major part of the proof (often the one containing most of the technical work) out of the main proof, reduces the 'negative stretch' to a minimum. The resulting main proof can then often be taken in 'at a glance'. The frustration of trying to live and function in an impossible world is correspondingly reduced. In fact, it is often possible to view the whole main proof from the outside, with no need to enter and 'live' in the false world at all. There is no 'cognitive strain' here since we do not really enter this impossible world, but dispense with it, as it were, in one glance from the outside.
- The resulting proof is more structured (in the sense of [2]), resulting in a more comprehensible presentation.

ACKNOWLEDGEMENT

I am grateful to David Pimm for his helpful remarks on an earlier version of this communication.

REFERENCES

1. Van Arp, J.: 1984, 'Images of learning mathematics', *Mathematics Teaching* **108**, 14–17.
2. Leron, U.: 1983, 'Structuring mathematical proofs', *Amer. Math. Monthly* **90**, 174–185.
3. Leron, U.: 'Structuring and heuristic, for the learning of mathematics' (forthcoming).
4. Leron, U. and Moran, G.: 1983, 'The diagonal method', *The Mathematics Teacher* **76**, 674–676.
5. Nordgreen, J. E.: 1983, 'A plausible dialog', *Mathematics Teaching* **102**, 54–57.

Department of Science Education,
Technion-Israel Institute of Technology,
Haifa,
Israel 32000

URI LERON