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Classification of Singular Space–Times

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§(1): *Introduction*

The scheme we wish to propose has been discussed in detail in "Singular Space Times," by G. F. R. Ellis and B. G. Schmidt [*Gen. Rel. Grav.*, 8, 915 (1977)]. It is also discussed in [9]. Ellis and Schmidt's paper gives the basic ideas of the classification; it shows briefly the difference between the singularity types differentiated by the proposed classification, namely *quasi-regular singularities* (as in a cone, for example), *nonscalar singularities* (as can occur in a plane wave), and *scalar singularities* (as in the Schwarzschild and Robertson–Walker solutions).

This classification enables certain theorems to be proved regarding generic behavior, existence, and stability; these theorems are discussed in the paper by C. J. S. Clarke (and see also [9]).

§(2): *Definitions*

Given a C^k space–time (M, g) in general relativity,[1] the existence of incomplete inextendible geodesics or other curves indicates that the space–time has an edge [1,2]. One can then attach a set ∂M to M representing the set of *boundary*

[1] C^k is the differentiability of the curvature tensor (differentiability requirements are discussed briefly at the end of this section). In general we assume that $\Lambda = 0$ (this makes no essential difference to our discussion).

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points[2] of the space-time, by the *b-boundary* [3] or similar constructions. Each curve $\gamma(v)$ in M that is incomplete when v is generalized affine parameter ends at a point $q \in \partial M$; and at least one such curve ends at each point $q \in \partial M$. We let F_q denote the family of incomplete curves in M ending at q. More precisely, F_q is the maximal set of rectifiable curves in M such that for each curve $\gamma(v) \in F_q$, v is a generalized affine parameter and $\exists v_+$: $\gamma(v) \subset M$ for $v \in [0, v_+)$, and $\gamma(v_+) = q$).[3] Then for each $q \in \partial M$, F_q is a nonempty set.

The existence of such boundary points poses a problem for any space-time theory; for if a timelike curve ends at a point $q \in \partial M$, then a particle moving on this world line finds that its possible future suddenly comes to an end. If it starts at such a point, it begins with no previous history. Clearly we need to understand how such beginnings and ends occur.

The different ways the space-time can go wrong on ∂M can be partially understood by using the classification scheme shown in Figure 1 (developed, as a result of useful discussions with J. Ehlers, from related schemes in [2], [5], and [6]).

In the first place, the space-time may simply not have been extended far enough. We will call the point $q \in \partial M$ a C^r *regular boundary* point ($r \geqslant 0-$) if there is an extension [2, 7] of the space-time (M, g) into a larger space-time (M', g') such that the Riemann tensor of (M, g) is defined and is C^r, and q is an interior point of M'. Thus in this case there is no barrier to extending the space-time further; the singularity is "removable." We will call $q \in \partial M$ a C^r *singular boundary point* ($r \geqslant 0-$) if it is not a C^r regular boundary point; so in this case it is not possible to extend (M, g) through q in a C^r way.

If q is a singular boundary point of (M, g), it may sometimes be that this is because the space-time curvature prevents one from making an extension. We will call $q \in \partial M$ a C^k *(or C^{k-}) curvature* singularity ($k \geqslant 0$) if there is a curve $\gamma(v) \in F_q$ such that when an orthonormal tetrad $\{\mathbf{E}_a(v)\}$ parallel along $\gamma(v)$ is used as a basis, at least one curvature tensor component $R_{abcd;e_1 \ldots e_k}(v)$ does not behave in a C^0 (or C^{0-}) way on $[0, v_+]$. Clearly this is a singular boundary point, for the curvature tensor could not be badly behaved in a parallel frame on $[0, v_+]$ if an extension $\bar{\gamma}$ through $\gamma(v_+)$ were possible, for then $\gamma(v_+)$ would just be a regular point on the extended curve $\bar{\gamma}$. Thus this condition (called a "p.p. curvature singularity" in [2]) is sufficient to show that the local properties of the space-time are badly behaved as one approaches q.

On the other hand, there may be boundary points $q \in \partial M$ at which no extension is possible even though the total geometry is perfectly well-behaved as one approaches q; this occurs, for example, at the vertex of a cone. We will call

[2] ∂M is the set of boundary points at a finite distance from points $r \in M$. It will not represent points "at infinity" such as are contained in the conformal boundary of Geroch, Kronheimer, and Penrose [4].

[3] Strictly, we refer to curves in $\bar{M} \equiv M \cup \partial M$ contained in M except for an endpoint on ∂M and such that their restriction to M is rectifiable.

a singular boundary point $q \in M$ a C^k *(or* C^{k-}*) quasi-regular singularity* ($k \geq 0$) (called a "locally extendible singularity" in [5], [6]) if it is not a C^k (or C^{k-}) curvature singularity. Therefore in this case the curvature tensor components $R_{abcd;e_1 \ldots e_k}(v)$ measured in a parallel frame $\{\mathbf{E}_a(v)\}$ behave in a C^0 (or C^{0-}) way on all curves $\gamma(v) \in F_q$, and the space is locally well-behaved near q, even though q is an irremovable singularity.

A curvature singularity necessarily occurs if some physical quantity (e.g., the density or pressure of a fluid) or some curvature tensor invariant (e.g., $R_{abcd}R^{abcd}$) is badly behaved as one approaches q. We will call a point $q \in \partial M$ a C^k *(or* C^{k-}*) scalar singularity* ($k \geq 0$) if there is a curve $\gamma(v) \in F_q$ on which some C^k curvature scalar field (i.e., polynomial scalar field constructed from the tensors g_{ab}, η^{abcd}, $R_{abcd;e_1^1 \ldots e_k}$ on M) does not behave in a C^0 (or C^{0-}) way on $[0, v_+]$. In this case (called a "s.p. curvature singularity" in [2]), irrespective of the choice of C^k orthonormal tetrad used as a basis along $\gamma(v)$, at least one curvature tensor component $R_{abcd;e_1 \ldots e_k}(v)$ does not behave in a C^0 (or C^{0-}) way on $[0, v_0]$; therefore such points are necessarily C^k-curvature singularities.

On the other hand, there may be singular points for which no scalar invariant field causes any obstacle to extension, but it is still the curvature that causes problems. We shall call a curvature singularity $q \in \partial M$ a C^k *(or* C^{k-}*) nonscalar singularity* ($k \geq 0$) if it is not a C^k (or C^{k-}) scalar singularity. This situation can arise in the following way: suppose that along every curve approaching q, there is some C^k-orthonormal tetrad $\{\mathbf{y}_i(v)\}$ such that all curvature tensor components $R_{abcd;e_1 \ldots e_k}(v)$ are well-behaved on $\gamma(v)$ as $v \to v_+$ when this tetrad is used as a basis; then clearly no scalar singularity occurs at q. A nonscalar singularity can occur if on some curve $\gamma(v) \in F_q$ this basis is related to a parallel propagated basis $\{\mathbf{E}_a(v)\}$ on $\gamma(v)$ by a Lorentz transformation $\Lambda_i{}^a(v)$ which is badly behaved as $v \to v_+$, i.e., if $\mathbf{y}_i(v) = \Lambda_i{}^a(v)\,\mathbf{E}_a(v)$ where at least one of the functions $\Lambda_i{}^a(v)$ fails to be C^k (or C^{k-}) on $[0, v_+]$. In this case, the curvature tensor is "well-behaved" as one approaches q in that its components are perfectly regular when $\{\mathbf{y}_i\}$ is used as a basis; but the curvature measured in a parallel propagated frame can be badly behaved. (When a nonscalar singularity arises in this way, it is the same as the "intermediate singularities" of [5], [6].)

Each of these boundary behaviors is rather different in character. While examples of each of them are known, little is known about their general properties; and unfortunately the singularity theorems of Hawking and Penrose [8, 2], which prove the existence of boundary points under a wide variety of circumstances, do not specify which kind of behavior is likely to occur.

We do not claim that quasi-regular or nonscalar singularities discussed are likely to occur in physically realistic situations; rather we claim that only when we understand which singularities can occur in general space-times or in space-times with the field equations satisfied for particular matter content can we hope to discuss fruitfully their occurrence, equations of motion, and so on.

Two final remarks: First, it may (in the case of scalar and nonscalar curvature singularities) be useful to refine the classification to discuss whether the singularity is a *matter* singularity (if it is the Ricci tensor that causes the problem) and whether the relevant components are unbounded (a *divergent* singularity) or bounded (an *oscillatory* singularity). We shall sometimes use such further subdivisions in this paper when we consider curvature singularities. Second, our classification depends on the differentiability assumed. We assume that the Hausdorff manifold M is C^∞, that (cf. [10]) the metric components $g_{\mu\nu}$ are continuous with locally bounded weak derivatives, and that the curvature tensor components $R^\lambda{}_{\mu\nu\sigma}$ are C^k (or C^{k-}) functions. Then we call (M, g) a C^k (or C^{k-}) space-time ($k \geqslant 0$). Here, C^{k-} ($k \geqslant 1$) means that the ($k-1$)st derivatives obey Lipschitz conditions, while C^{0-} means locally bounded.

We will usually be interested in C^0 space-times (when there exist coordinates in which $g_{\mu\nu}$ are C^2 functions [10]) or C^{0-} space-times (when there are coordinates in which $g_{\mu\nu}$ are C^{2-} functions [10]), corresponding to continuous or locally bounded Riemann tensors. The second case admits shock waves, but the first does not.

§(3): *Singular Plane Waves*

The quasi-regular singularities are represented by cone singularities and the Taub-NUT singularity [11]. Scalar singularities are represented by the $R = 0$ singularity in Friedmann universes and the $r = 0$ singularity in the Schwarzschild solution. These are quite familiar; we concentrate here on an example of a non-scalar singularity, as these are less familiar.

The plane wave exact solutions of Einstein's empty space field equations can be given in the form [12]

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + 2du\,dv + 2H(x,y,u)\,du^2 \\ H(x,y,u) &= \tfrac{1}{2}A(u)\,\{(x^2 - y^2)\cos\theta(u) - 2xy\sin\theta(u)\} \end{aligned} \tag{3.1}$$

where $-\infty < x, y, v < \infty$ and $A(u)$, $\theta(u)$ are arbitrary C^1 functions on some open interval $I \subset R$. Using the orthonormal frame $\{\mathbf{E}_a\}$ as a basis, where

$$\mathbf{E}_1 \equiv \frac{\partial}{\partial x}, \quad \mathbf{E}_2 \equiv \frac{\partial}{\partial y}, \quad \mathbf{E}_3 \equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u} + (1-H)\frac{\partial}{\partial v}\right), \quad \mathbf{E}_4 \equiv \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial u} - (1+H)\frac{\partial}{\partial v}\right)$$

we see that the curvature tensor components take the form (characteristic of a type N Weyl tensor)

$$\begin{aligned} E_{11} = -E_{22} = -H_{12} = -H_{21} &\equiv \alpha(u), \qquad R_{ab} = 0 \\ E_{12} = E_{21} = H_{11} = -H_{22} &\equiv \beta(u), \qquad E_{ij} = H_{ij} = 0 \text{ otherwise} \end{aligned} \tag{3.2}$$

where E_{ab} and H_{ab} are the "electric" and "magnetic" parts of the Weyl tensor defined by $E_{ac} \equiv C_{a4c4}$, $H_{ab} \equiv \frac{1}{2}\eta_{a4}{}^{ef}C_{ef4b}$, and

$$\alpha(u) = A(u)\cos\theta(u), \qquad \beta(u) = A(u)\sin\theta(u) \tag{3.3}$$

The tetrad $\{\mathbf{E}_a\}$ is parallel propagated along the timelike geodesic $\{x = y = 0, u = v = s/\sqrt{2}\}$ which has $\mathbf{E}_4$ as the tangent vector (note that $H = 0$ on this curve). This geodesic is equivalent to every other timelike geodesic because of the invariance of the space under a five-parameter group of isometries [12]; the components (3.2), (3.3) of the tensor E_{ab} are therefore just the physical components of the tidal force felt by any observer freely falling in this space-time.

We may further introduce the orthonormal tetrad field $\{\mathbf{y}_i\}$, where

$$\mathbf{y}_1 = \cos\tfrac{1}{2}\theta(u)\,\mathbf{E}_1 + \sin\tfrac{1}{2}\theta(u)\,\mathbf{E}_2$$
$$\mathbf{y}_2 = \cos\tfrac{1}{2}\theta(u)\,\mathbf{E}_2 - \sin\tfrac{1}{2}\theta(u)\,\mathbf{E}_1$$
$$\mathbf{y}_3 = \cosh\xi(u)\,\mathbf{E}_3 + \sinh\xi(u)\,\mathbf{E}_4$$
$$\mathbf{y}_4 = \cosh\xi(u)\,\mathbf{E}_4 + \sinh\xi(u)\,\mathbf{E}_3$$

Here $\theta(u)$ is given by (3.1), and $\xi(u)$ is an arbitrary C^1 function of u. With this tetrad as a basis, the curvature tensor again has the form (3.2), but now

$$\alpha(u) = A(u)\exp 2\xi(u), \qquad \beta(u) = 0 \tag{3.4}$$

These space-times are geodesically complete if $A(u)$ and $\theta(u)$ are C^1 functions defined for $-\infty < u < \infty$, i.e., if $I \equiv R$ (see [12]); but they are geodesically incomplete with C^0 or C^{0-} nonscalar singularities if either $A(u)$ or $\theta(u)$ are badly behaved at some finite value, say at $u = u_+$. Then the curvature tensor components in a parallel frame [given by (3.2), (3.3)] are badly behaved, so a curvature singularity occurs as $u \to u_+$; however, (3.4) shows the existence of tetrads $\{\mathbf{y}_i\}$ for which the curvature is perfectly regular. For example, we could take $\xi(u) = -\frac{1}{2}\log(|A(u)|)$ when $A \neq 0$ and $\xi(u) = 0$ when $A = 0$; then $\alpha = +1$ if $A > 0$, $\alpha = -1$ if $A < 0$, and $\alpha = 0$ if $A = 0$. However, this tetrad field is discontinuous when $A \to 0$, so it is preferable to take, for example, $\xi = -\frac{1}{2}\log(1 + A^2)$; then in the frame $\{\mathbf{y}_i\}$ one has

$$\alpha(u) = A(u)\cdot\{1 + A^2(u)\}^{-1}, \qquad \beta(u) = 0 \tag{3.5}$$

so the curvature tensor components are continuous and bounded by $\pm\frac{1}{2}$ at all values of u for which $A(u)$ is defined. Calculating any C^0 curvature scalar with this tetrad as a basis shows that they are bounded for $u \in [0, u_+]$; and they can at worst be C^{0-} on this interval.

A particularly interesting case is the C^0 nonscalar singularity when $A = cu^{-2}$, $\theta = 2K\log u$ (c, K constants, $c > 0$), for these are homogeneous space-times invariant under a transitive six-dimensional group of isometries [12]. An observer falling into the singularity on a geodesic (e.g., that generated by $\mathbf{E}_4$) will feel unbounded tidal forces in a finite time; however, an observer moving arbitrarily near the singularity[4] on an integral curve of y_4 will feel perfectly finite tidal forces. (However, he may feel very large inertial forces). This example is in contrast to the situation in positive definite spaces, when a homogeneous space is necessarily complete and therefore singularity-free.

§(4): *The Effect of Curvature Singularities*

Of primary importance is the fact (Section 2) that a singularity can lead to an observer's future suddenly coming to an end. In the case of a curvature singularity this may be accompanied by the observer and his apparatus being torn to pieces before reaching the singularity; a graphic description of this process for an observer falling into the (conformal scalar) singularity at $r = 0$ in the Schwarzschild solution is given in Section 32.6 of [13]. In the principle this is the additional danger that threatens on any curve running into a curvature singularity. (Clearly it does not happen at a quasi-regular singularity.) Accordingly we can attempt to divide curvature singularities into *weak curvature singularities*—ones such that some object falling into the singularity can arrive intact at the singularity—and *strong curvature singularities*, where this is not possible.

To decide what happens to an object falling on a geodesic $\gamma(\tau)$ into a curvature singularity at $\tau = \tau_+$, one approach is to use the same classical methods (based on the geodesic deviation equation) as in the calculations for gravitational wave detectors (see, e.g., Section 37 of [13]). Suppose the object is a metal bar with amplitude $b_n(\tau)$ for the nth normal mode, where τ is proper time measured along its world line. Then this amplitude satisfies the question

$$\frac{d^2B_n}{d\tau^2} + \frac{2}{\tau_n}\frac{dB_n}{d\tau} + \omega_n{}^2 B_n = R_n(\tau) \tag{4.1}$$

where τ_n is a damping constant for the mode, ω_n is an angular frequency, and $R_n(\tau)$ is the curvature-induced driving term. (It is $R_{j0k0}(\tau)|_\gamma$ in an appropriate parallel propagated orthonormal frame,[5] multiplied by a suitable moment-of-inertia factor describing the properties and orientation of the bar: see Box 37.4 and Example 37.10 of [13].) Define $\overline{\omega}_n \equiv \omega_n(1 - \tau_n{}^{-2}\omega_n{}^{-2})^{1/2}$. Then the general solution to (4.1) is

$$B_n(\tau) = x_n(\tau) B_n\bigg|_0 + y_n(\tau)\frac{dB_n}{d\tau}\bigg|_0 + z_n(\tau) \tag{4.2}$$

[4] Moving at a point $p \in M$ in any open neighborhood $v \subset \overline{M}$ of the singularity $q \in \partial M$.
[5] We assume for the moment that the bar falls so that the principal axes are parallel propagated along the geodesic γ.

where the complementary functions x_n, y_n are given by

$$x_n(\tau) = \exp\{-\tau/\tau_n\}\left\{\cos(\bar{\omega}_n) + \frac{1}{\omega_n \tau_n}\sin(\bar{\omega}_n\tau)\right\} \tag{4.3a}$$

$$y_n(\tau) = \exp\{-\tau/\tau_n\}\left\{\frac{1}{\bar{\omega}_n}\sin(\bar{\omega}_n\tau)\right\} \tag{4.3b}$$

and the particular integral $z_n(\tau)$ is

$$z_n(\tau) = \int_0^\tau G_n(\tau - u)R_n(u)\,du, \qquad G_n(\xi) = \frac{1}{\bar{\omega}_n}\sin(\bar{\omega}_n\xi)\exp\{-\xi/\tau_n\} \tag{4.4}$$

The crucial question is whether these amplitudes remain finite as the bar falls into the singularity; more realistically, do they remain below the elastic limit for the bar in question? Since x_n and y_n are well-behaved, as a first step we can simply consider whether the integral (4.4) is finite for all n. If it is finite, one may think that a strong enough bar will not be destroyed before $v \to v_+$. Consequently we may proceed as follows: we can mentally let a bar fall down the geodesic γ. If z_n is finite as $\tau \to \tau_+$ for all n, the bar does not disintegrate (unless we have bad luck with the convergence of the Fourier series for the total displacement), and the curvature singularity is weak. If not, we need to try a stronger bar; and if we are convinced that no "real" bar could survive, it is a strong curvature singularity. Thus a Mark I detector is just a selection of metal bars; the advanced Mark II detector would have strain gauges added (just like the gravitational wave detectors, but with different sensitivity!) to give some advanced warning of the imminent disintegration of the bar.

To make the discussion specific, let the bar fall on a geodesic in a plane wave space-time; then $F_n(\tau)$ is an arbitrarily disposable function of τ (by appropriate choice of the functions $A(u)$, $\theta(u)$ in (3.1)). Suppose $F_n(u) = \sin u^{-1}$, a C^0 (but not C^{0-}) curvature singularity. Now the bar will respond to this with finite oscillations in each mode as $\sin u^{-1}$ sweeps through the resonant frequency of that mode; but the oscillations in each mode will die away as the frequency increases (c.f. Box 37.4 of [13]) beyond the resonant frequency. Clearly any reasonably strong bar will survive this treatment. Similarly any C^0 (but not C^{0-}) curvature singularity will be a weak curvature singularity. Now consider infinite oscillations, for example $F_n(u) = u^{-1}\sin u^{-1}$. Again the forcing term sweeps through the frequency band; and even though the amplitude is divergent, once it has passed the resonant frequency the forcing term will have a negligible effect on a given mode. Hence it is conceivable that some classical bars could survive such treatment if their microscopic structure was suitable (the question would be how τ_n depended on n). Now one should note that oscillations are not necessary for the argument to apply: even for a monotonically divergent forcing term $F_n(u)$, the finite response time of the bar could allow it to survive a C^{0-}

singularity if the force increases so fast that the bar does not have time to respond to it. An interesting example is the Schwarzschild solution. The discussion in Section 32.6 of [13] does not consider the response time, but is based on the fact that the forcing term $F_n \alpha r^{-3}$, where r is the Schwarzschild coordinate. For an object freely falling into the singularity at $r = 0$, proper time τ along the geodesic world line goes as $r^{3/2}$, so the forcing term $F_n \alpha \tau^{-2}$. This is just sufficient to make z_n diverge (roughly, we integrate F_n twice, obtaining a logarithmic divergence). Thus the picture presented in [13] is correct, but only just so: if the object could fall in rather faster, it could escape destruction before hitting the singularity!

One can ask whether there are other ways the bar could escape destruction. One possibility is that a very thin bar would not be destroyed if it could maintain its direction relative to the principal axes of the tidal force, so that there were no tidal tension or compression forces along the bar. The trouble here is that divergent forces would have to be exerted to keep the bar from rotating out of such a preferred direction. A second possibility would be to utilize rotation, so that before the bar had had time to respond to extending tidal forces (when aligned along one set of principal axes of tidal forces) it would have rotated and already been subjected to compressing tidal forces. That is, rotation of the bar relative to a parallel frame could effectively introduce oscillations in the term F_n. However, it seems one could not prevent destruction this way because the resulting centrifugal forces would then have to diverge as $v \to v_+$! While these methods do not avoid destruction, they are interesting because they indicate that one cannot ultimately discuss the effects of tidal forces near a singularity without considering inertial forces as well (c.f. the discussion in [14] and [15]). Again, in certain situations (see, e.g., the plane wave discussion), the inertial effects caused by nongeodesic motion near a singularity may become important.

§(5): *Conclusion*

We have presented a classification of singularities in general relativity that we believe usefully separates classes of singularities with very different behaviors. Many points need clarifying, for example, the question of the existence and stability of the various kinds of singularity in astronomically relevant situations and the relation of this classification to the questions raised by Taub (see his comment later in this section).

References

1. Geroch, R. P. (1968). "What is a Singularity in General Relativity?" *Ann. Phys.* (*N. Y.*), **48**, 526.

2. Hawking, S. W., and Ellis, G. F. R. (1973). *The Large-Scale Structure of Space-Time*, Cambridge University Press, Cambridge.
3. Schmidt, B. G. (1971). "A New Definition of Singular Points in General Relativity," *Gen. Rel. Grav.*, **1**, 269; 1973. "The Local Completeness of Space-Time," *Commun. Math. Phys.*, **29**, 49.
4. Geroch, R., Kronheimer, E., and Penrose, R. (1972). "Ideal Points of Space-Time," *Proc. R. Soc. London Ser. A*, **327**, 545.
5. Clarke, C. J. S. (1975). "Singularities in Globally Hyperbolic Space-Times," *Commun. Math. Phys.*, **41**, 65.
6. Ellis, G. F. R., and King, A. R. (1974). "Was the Big-Bang a Whimper?" *Commun. Math. Phys.*, **38**, 119.
7. Clarke, C. J. S. (1975). "The Classification of Singularities," *Gen. Rel. Grav.*, **6**, 35.
8. Hawking, S. W., and Penrose, R. (1970). "The Singularities of Gravitational Collapse and Cosmology," *Proc. R. Soc. London Ser. A*, **314**, 529.
9. Clarke, C. J. S., and Schmidt, B. G. (1977). "Singularities, the State of the Art," *Gen. Rel. Grav.*, **8**, 129–137.
10. Clarke, C. J. S. (1976). "On the Differentiability of Space-Time," preprint MPI-PAE/Astro 80, Max Planck Institute for Physics and Astrophysics, Munich.
11. Misner, C. W. (1967). "Taub-NUT Space as a Counterexample to Almost Anyting," in *Relativity Theory and Astrophysics. I: Relativity and Cosmology*, J. Ehlers, Lectures in Applied Mathematics, vol. 8, American Mathematical Society, Providence, Rhode Island.
12. Ehlers, J., and Kundt, W. (1962). "Exact Solutions of the Gravitational Field Equations," in *Gravitation*, edited by L. Witten, Wiley, New York, p. 49.
13. Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, Freeman, San Francisco.
14. Fishbone, L. G. (1973). "Relativistic Roche Problem," *Ap. J.*, **185**, 43.
15. Faulkner, J., and Flannery, B. P. (1978). "Tidal Fields in General Relativity: D'Alembert's Principle and the Test Rigid Rod," *Astrophys. J.*, **220**, 1125.