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POISSON APPROXIMATIONS FOR 2-DIMENSIONAL PATTERNS*

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Abstract. Let $X = (X_{ij})_{n\times n}$ be a random matrix whose elements are independent Bernoulli random variables, taking the values 0 and 1 with probability q_{ij} and p_{ij} ($p_{ij} + q_{ij} = 1$) respectively. Upper and lower bounds for the probabilities of m non-overlapping occurrences of a square submatrix with all its elements being equal to 1, are obtained. Some Poisson convergence theorems are established for $n \to \infty$. Numerical results indicate that the proposed bounds perform very well, even for moderate and small values of n.

Key words and phrases: Random matrix, Bernoulli random variables, Poisson approximation, patterns, consecutive-k-out-of-n:F system.

1. Introduction

Let X_{ij}, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, n$ be a double sequence of independent Bernoulli random variables with success (failure) probabilities p_{ij} (q_{ij}) and denote by $X = (X_{ij})_{n\times n}$ the respective random matrix. A *success-block of size* k (k a positive integer) is a $k \times k$ square submatrix of X with all its elements being equal to 1 (successes). The purpose of this paper is to study the distribution of the number $N_{n,k}$ of occurrences of non-overlapping success blocks of size k. The term "non-overlapping" means that no two success submatrices have elements in common (we count from scratch every time a success block occurs).

It is obvious that the distribution of $N_{n,k}$ is a two-dimensional version of the so-called Binomial distribution of order k which has been extensively studied (see Feller (1968), Hirano (1986), Aki and Hirano (1989), Godbole (1991), Fu (1993), Fu and Koutras (1994) and references therein).

The distribution of $N_{n,k}$ is closely related to the reliability of a system which is called 2-*dimensional consecutive-k-out-of-n:F system*. This model was introduced by Salvia and Lasher (1990) as follows: A k^2/n^2:F system is a square grid of size

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n (containing n^2 components). The system fails if and only if there is at least one square grid of size k ($2 \le k \le n-1$) that contains all failed components. As Salvia and Lasher (1990) indicate, this model is useful in integrated-circuit design and pattern detection. It is clear that the reliability of a k^2/n^2:F system is equal to $P[N_{n,k} = 0]$. Generalizing Salvia and Lasher's model, we may introduce a *2-dimensional m-consecutive-k-out-of-n:F system*, assuming that the system fails if and only if at least m non-overlapping failure square grids of size k have occurred ($m \ge 1$).

The present paper is organized as follows: In Section 2 we approximate the probability $P[N_{n,k} = 0]$ by a lower and an upper bound which perform very well, especially if the failure probabilities q_{ij} are high (p_{ij} are low). Numerical comparisons to the "Chen-Stein method"-based bounds, given by Koutras *et al.* (1993), are also provided and some limit theorems for large n are established. In Section 3 we develop lower and upper bounds for the probabilities $P[N_{n,k} = x]$, $x \ge 1$ in the iid case, and use them to obtain a Poisson convergence theorem. Finally, in Section 4 we introduce some additional models and indicate how the analysis conducted in Sections 2 and 3 can be properly modified to cover these situations as well.

2. Probability of zero success block occurrences

The next theorem provides a lower and an upper bound for the probability of no occurrence of success block of size k, in an $n \times n$ matrix $X = (X_{ij})$ consisting of independent Bernoulli random variables with

$$P[X_{ij} = 1] = p_{ij}, \qquad P[X_{ij} = 0] = q_{ij}, \qquad p_{ij} + q_{ij} = 1.$$

THEOREM 2.1. *Let $\gamma_{ij} = \gamma_{ij}(X)$, $k \le i \le n$, $k \le j \le n$ be the following binary random variables*

$$\gamma_{ij} = \prod_{\mu=i-k+1}^{i} \prod_{\nu=j-k+1}^{j} X_{\mu\nu}.$$

Then

$$\prod_{i=k}^{n}\prod_{j=k}^{n}\{1 - P[\gamma_{ij} = 1]\} \le P[N_{n,k} = 0]$$
$$\le \prod_{i=k}^{n}\prod_{j=k}^{n}\{1 - q_{i-k,j-k+1}q_{i,j-k}P[\gamma_{ij} = 1]\}.$$

Convention. $q_{ij} = P[X_{ij} = 0] = 1$ if at least one of i, j is equal to zero.

PROOF. For the lower bound, note that

$$P[N_{n,k} = 0] = P\left[\prod_{i=k}^{n}\prod_{j=k}^{n}(1 - \gamma_{ij}) = 1\right]$$

and make use of the following inequality which is valid for any sequence of associated Bernoulli variables Y_r, $r = 1, 2, \ldots, N$ (see Barlow and Proschan (1981))

$$\prod_{r=1}^{N} P[Y_r = 1] \le P\left[\prod_{r=1}^{N} Y_r = 1\right]. \tag{2.1}$$

To establish the upper bound we work as follows: For every given (i, j) with $k \le i, j \le n$ we define $r = 1 + (i - k) + (j - k)(n - k + 1)$ (a 1-1 transformation performing a column-by-column traversing of the matrix X) and denote by A_r the event $\{\gamma_{ij} = 1\}$. It is clear that

$$P[N_{n,k} = 0] = P\left[\bigcap_{r=1}^{N} A'_r\right], \qquad N = (n - k + 1)^2$$

(A'_r is the complementary event of A_r) and employing the chain rule we deduce

$$\begin{aligned} P[N_{n,k} = 0] &= (1 - P[A_1])(1 - P[A_2 \mid A'_1]) \cdots \left(1 - P\left[A_N \middle| \bigcap_{l=1}^{N-1} A'_l\right]\right) \\ &= \prod_{r=1}^{N} (1 - \alpha_r) \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \alpha_1 &= P[A_1] = P[\gamma_{kk} = 1] = q_{o1} q_{ko} P[\gamma_{kk} = 1], \\ \alpha_r &= P\left[A_r \middle| \bigcap_{l=1}^{r-1} A'_l\right], \qquad r > 1. \end{aligned} \tag{2.3}$$

For given $r > 1$ (which corresponds to a specific (i, j)), we introduce the index set

$$I_r = \{1, 2, \ldots, r - 1\} \cap \{1 + (i_1 - k) + (j_1 - k)(n - k + 1) \mid (i_1, j_1) \in J_r\}$$

where

$$\begin{aligned} J_r = \{(i_1, j_1) \mid \max[(i - k + 1), k] \le i_1 \le \min[(i + k - 1), n], \\ \max[(j - k + 1), k] \le j_1 \le j\} \end{aligned}$$

and the event $B_r : X_{i-k,j-k+1} = 0$ and $X_{i,j-k} = 0$. Since
(i) $B_r(\cap_{l=1}^{r-1} A'_l) = B_r(\cap_{l=1, l \notin I_r}^{r-1} A'_l) = C_r$,
(ii) A_r and C_r are independent
we conclude that

$$P\left[A_r \middle| B_r \left(\bigcap_{l=1}^{r-1} A'_l\right)\right] = P[A_r \mid C_r] = P[A_r] \tag{2.4}$$

and therefore,

$$\alpha_r \geq P\left[A_r \middle| B_r\left(\bigcap_{l=1}^{r-1} A_l'\right)\right] P\left[B_r \middle| \bigcap_{l=1}^{r-1} A_l'\right] = P[A_r] P\left[B_r \middle| \bigcap_{l=1}^{r-1} A_l'\right]. \tag{2.5}$$

Now, it is quite intuitive, that the information that some of the variables associated with A_l ($1 \leq l \leq r-1$) are equal to 0 (i.e. the occurrence of the condition $\cap_{l=1}^{r-1} A_l'$) can only increase the probability of the event B_r (or else leave the probability unchanged if $X_{i-k,j-k+1}$ and $X_{i,j-k}$ are not used in $\cap_{l=1}^{r-1} A_l'$).

Hence, intuitively,

$$P\left[B_r \middle| \bigcap_{l=1}^{r-1} A_l'\right] \geq P[B_r]. \tag{2.6}$$

For a formal proof of this, we form the index set I_r^* which consists of all $l \in \{1, 2, \ldots, r-1\}$ such that A_l makes use of the variables $X_{i-k,j-k+1}$ or $X_{i,j-k}$. Carrying out the same analysis as the one used in the proof of (2.4) we obtain

$$\begin{aligned} P\left[B_r\left(\bigcap_{l=1}^{r-1} A_l'\right)\right] &= P\left[B_r\left(\bigcap_{\substack{l=1 \\ l\notin I_r^*}}^{r-1} A_l'\right)\right] \\ &= P[B_r] P\left[\bigcap_{\substack{l=1 \\ l\notin I_r^*}}^{r-1} A_l'\right] \geq P[B_r] P\left[\bigcap_{l=1}^{r-1} A_l'\right] \end{aligned}$$

which is obviously equivalent to (2.6). The upper bound of the theorem is now immediately deduced from (2.2), (2.3), (2.5) and (2.6). □

Introducing the notation:

$$\begin{aligned} a_{ij} &= P[\gamma_{ij} = 1] = E[\gamma_{ij}] = \prod_{\mu=i-k+1}^{i} \prod_{\nu=j-k+1}^{j} p_{\mu\nu} \quad \text{and} \\ b_{ij} &= q_{i-k,j-k+1} q_{i,j-k} a_{ij}, \end{aligned}$$

it follows that the lower and upper bounds given in Theorem 2.1 can be written as

$$\begin{aligned} LB_n &= LB_n((p_{ij})_{n\times n}) = \prod_{i=k}^{n} \prod_{j=k}^{n} (1 - a_{ij}), \\ UB_n &= UB_n((p_{ij})_{n\times n}) = \prod_{i=k}^{n} \prod_{j=k}^{n} (1 - b_{ij}). \end{aligned}$$

It is clear that, when the failure probabilities q_{ij} are high (i.e. the successes in the double Bernoulli sequence are sparse), then UB_n is expected to be close to LB_n; thus, a good approximation to the reliability $P[N_{n,k}=0]$ is yielded.

Theorem 2.1 is directly applicable to a k^2/n^2:F system with independent but not necessarily identical components. Denoting by $p_{ij}=P[X_{ij}=0]$ the failure probability of component (i,j), we may state the following:

(i) $P[N_{n,k}=0]$ is the exact reliability R_n of the system.

(ii) LB_n is a lower bound developed from minimal cut sets, see Barlow and Proschan (1981).

(iii) UB_n is an upper bound based on minimal cut sets too. The advantage of UB_n over the so called minimal path set upper bound, Barlow and Proschan (1981) (whose computation for this specific system is tedious), is that the former performs very well with systems containing high reliability components, a fact that is not valid in general for the latter.

(iv) When component reliabilities are close to 1, the lower and upper bounds (LB_n and UB_n) provide very good approximations for the system reliability R_n.

Koutras *et al.* (1993) studied the k^2/n^2:F system using the Chen-Stein method. Since the Chen-Stein's lower bound LB_s was found to be, in most of the cases, worse than the minimal cut bound LB_n hence they proposed to use LB_n for approximating the reliability R_n from below, and to use the Chen-Stein's upper bound UB_s for approximating the reliability R_n from above. Our extensive numerical experimentation indicated that UB_n is better than UB_s in most cases. Tables 1 and 2 (next page) present the values of LB_s, LB_n, UB_n, UB_s for several values of n and k, and component reliabilities

$$\text{I.}\quad q_{ij}=\begin{cases}0.70 & \text{if } i+j \text{ odd}\\ 0.75 & \text{if } i+j \text{ even,}\end{cases}\qquad \text{II.}\quad q_{ij}=\begin{cases}0.5 & \text{if } |i-j|\le 1\\ 1-\dfrac{1}{|i-j|} & \text{if } |i-j|>1,\end{cases}$$

respectively.

It is worth mentioning that Theorem 2.1 could also be stated for non square matrices X and nonsquare success blocks. More specifically we have the next

THEOREM 2.2. *Let $X=(X_{ij})_{n_1\times n_2}$ be a random matrix of binary variables X_{ij} with $E[X_{ij}]=p_{ij}=1-q_{ij}$, and denote by $N(n_1,n_2;k_1,k_2)$ the number of occurrences of success blocks of size $k_1\times k_2$. If*

$$\gamma_{ij}=\gamma_{ij}(X)=\prod_{\mu=i-k_1+1}^{i}\prod_{\nu=j-k_2+1}^{j}X_{\mu\nu}\qquad k_1\le i\le n_1,\quad k_2\le j\le n_2$$

then

$$\prod_{i=k_1}^{n_1}\prod_{j=k_2}^{n_2}\{1-P[\gamma_{ij}=1]\}\le P[N(n_1,n_2;k_1,k_2)=0]$$

$$\le\prod_{i=k_1}^{n_1}\prod_{j=k_2}^{n_2}\{1-q_{i-k_1,j-k_2+1}q_{i,j-k_2}P[\gamma_{ij}=1]\}.$$

Table 1. Lower and upper bounds for the reliability of a 2-dimensional system with

$$q_{ij} = \begin{cases} 0.70 & \text{if } i+j \text{ odd} \\ 0.75 & \text{if } i+j \text{ even.} \end{cases}$$

n	k	LB_s	LB_n	UB_n	UB_s
3	2	0.9577	0.9777	0.9834	0.9978
5	2	0.8364	0.9137	0.9448	0.9915
5	3	0.9999	0.9999	0.9999	0.9999
5	4	1.0000	1.0000	1.0000	1.0000
10	2	0.3044	0.6332	0.7708	0.9637
10	3	0.9993	0.9994	0.9997	0.9996
10	4	1.0000	1.0000	1.0000	1.0000
20	2	−0.6514	0.1305	0.3292	0.9139
20	3	0.9965	0.9972	0.9984	0.9978
20	4	1.0000	1.0000	1.0000	1.0000
50	2	−0.9009	0.0000	0.0007	0.9009
50	3	0.9755	0.9801	0.9892	0.9848
50	4	1.0000	1.0000	1.0000	1.0000
50	5	1.0000	1.0000	1.0000	1.0000

Table 2. Lower and upper bounds for the reliability of a 2-dimensional system with

$$q_{ij} = \begin{cases} 0.50 & \text{if } |i-j| \le 1 \\ 1 - \dfrac{1}{|i-j|} & \text{if } |i+j| > 1. \end{cases}$$

n	k	LB_s	LB_n	UB_n	UB_s
3	2	0.1014	0.7725	0.8661	1.4562
5	2	−1.2723	0.4302	0.7156	2.1536
5	3	0.9718	0.9881	0.9942	1.0045
5	4	1.0000	1.0000	1.0000	1.0000
10	2	−2.6844	0.0872	0.4342	2.8706
10	3	0.9075	0.9611	0.9851	1.0147
10	4	0.9999	0.9999	1.0000	1.0000
20	2	−3.0461	0.0035	0.1570	3.0542
20	3	0.7841	0.9091	0.9672	1.0342
20	4	0.9996	0.9998	0.9999	0.9999
50	2	−3.0625	0.0000	0.0074	3.0625
50	3	0.4524	0.7695	0.9154	1.0868
50	4	0.9989	0.9993	0.9997	0.9997
50	5	1.0000	1.0000	1.0000	1.0000

Convention. $q_{ij} = P[X_{ij} = 0] = 1$ if at least one of i, j is equal to zero.

PROOF. It can be easily proved with exactly the same technique as the one used in the proof of Theorem 2.1. □

Another interesting point in the proof of Theorem 2.1 is that the upper bound UB_n depends on the traversing direction used when jumping from (i,j) to r. Performing the numbering in a different fashion, it is not difficult to verify that (for example) the following upper bounds for $P[N_{n,k} = 0]$ could also be obtained

$$UB_n^{(2)} = \prod_{i=k}^{n}\prod_{j=k}^{n}\{1 - q_{i+1,j-1}q_{i,j-k}P[\gamma_{ij} = 1]\},$$
$$UB_n^{(3)} = \prod_{i=k}^{n}\prod_{j=k}^{n}\{1 - q_{i,j-k}q_{i+1,j}P[\gamma_{ij} = 1]\}$$

(Convention: $q_{ij} = 1$ if $i = 0,\ k+1$ or $j = 0,\ k+1$). If q_{ij} are changing in a systematic fashion (for example, increase or decrease as i, j increase), a proper choice of the upper bound may improve slightly the upper approximating value.

The rest of the present paragraph is dedicated to the development of limit theorems for the probability $P[N_{n,k} = 0]$ as $n \to \infty$. To this purpose let us assume that the success (failure) probabilities are functions of n (the dimension of the matrix, or the number of system's components in reliability terms) i.e., $p_{ij} = p_{ij}(n)$, $q_{ij} = q_{ij}(n)$.

Introducing the notations

$$a_{ij}(n) = \prod_{\mu=i-k+1}^{i}\ \prod_{\nu=j-k+1}^{j} p_{\mu\nu}(n),$$
$$b_{ij}(n) = q_{i-k,j-k+1}(n)q_{i,j-k}(n)a_{ij}(n)$$

we have the next theorem.

THEOREM 2.3. *If*

$$\lim_{n\to\infty}\sum_{l=1}^{\infty}\frac{1}{l}\sum_{i=k}^{n}\sum_{j=k}^{n}[a_{ij}(n)]^l = \lim_{n\to\infty}\sum_{l=1}^{\infty}\frac{1}{l}\sum_{i=k}^{n}\sum_{j=k}^{n}[b_{ij}(n)]^l = \lambda,$$

then

$$\lim_{n\to\infty} P[N_{n,k} = 0] = e^{-\lambda}.$$

PROOF. Observe that both quantities

$$-\log LB_n = \sum_{i=k}^{n}\sum_{j=k}^{n}[-\log(1 - a_{ij}(n))] = \sum_{l=1}^{\infty}\frac{1}{l}\sum_{i=k}^{n}\sum_{j=k}^{n}[a_{ij}(n)]^l$$

and

$$-\log UB_n = \sum_{l=1}^{\infty} \frac{1}{l} \sum_{i=k}^{n} \sum_{j=k}^{n} [b_{ij}(n)]^l$$

tend to λ as $n \to \infty$. Therefore, in virtue of Theorem 2.1 we have

$$\lim_{n\to\infty} \{\log P[N_{n,k} = 0]\} = -\lambda$$

and the proof is completed. □

In the iid case with $q_n = q_{ij}(n)$, $p_n = p_{ij}(n)$, $i \le i, j \le n$, we obtain the following upper and lower bounds

$$LB_n = (1 - p_n^{k^2})^{(n-k+1)^2}, \qquad UB_n = (1 - q_n^2 p_n^{k^2})^{(n-k+1)^2} \tag{2.7}$$

and Theorem 2.3 yields the next corollary.

COROLLARY 2.1. *If* $\lim_{n\to\infty} n^2 p_n^{k^2} = \lambda$, *then* $\lim_{n\to\infty} P[N_{nk} = 0] = e^{-\lambda}$.

Similar results are also valid when $k = k_n$ depends on n. More specifically we have the following corollary.

COROLLARY 2.2. *If* k_n *is unbounded and satisfies* (a) $\lim_{n\to\infty}(n - k_n) = \infty$, (b) $\lim_{n\to\infty}(n - k_n)^2 p_n^{k_n^2} = \lambda$, *and* (c) $\lim_{n\to\infty} q_n = 1$, *then*

$$\lim_{n\to\infty} P[N_{n,k} = 0] = e^{-\lambda}.$$

PROOF. Note that (a), (b) and (c) imply

$$\lim_{n\to\infty} (n - k_n + 1)^2 p_n^{k_n^2} = \lim_{n\to\infty} (n - k_n + 1)^2 q_n^2 p_n^{k_n^2} = \lambda$$

and apply Theorem 2.3 or make direct use of (2.7). □

There are many cases of p_n and k_n (unbounded) which satisfy the conditions (a), (b) and (c), for example $k_n^2 \sim \log n / \log(\log n)$ and $p_n \sim [\lambda/(n - k_n)^2]^{1/k_n^2}$.

COROLLARY 2.3. *If the sequence* k_n *is bounded from above, and*

$$\lim_{n\to\infty} n^2 p_n^{k_n^2} = \lambda \tag{2.8}$$

then $\lim_{n\to\infty} P[N_{n,k} = 0] = e^{-\lambda}$.

PROOF. It is not difficult to verify that when k_n is bounded, condition (2.8) implies the validity of conditions (a), (b) and (c) of Corollary 2.2; hence, the result. □

We mention that Corollaries 2.1 and 2.2 (under slightly different assumptions) were also proved (in reliability terminology) by Koutras *et al.* (1993).

3. Poisson convergence

Employing the bounds developed in the previous section, in this section we are going to derive the limiting distribution of $N_{n,k}$ for the iid case

$$q_n = q_{i,j}(n), \quad p_n = p_{ij}(n) \qquad \text{for all } 1 \le i, j \le n.$$

The method to be used is an extension of Fu's (1993) technique in the 2-dimensional space which needs no computation of moments of any order. It is worth mentioning that for the approximation of $P[N_{n,k} = m]$, $m > 0$ (in the non-overlapping case) the Chen-Stein method (see Arratia *et al.* (1989), Barbour and Holst (1989), Barbour *et al.* (1992)) is not applicable unless certain couplings are employed. In this case, the establishment of bounds for the total variation distance between the auxiliary variables is rather tedious.

For proving the main theorem of this section, the next two lemmas will be needed.

LEMMA 3.1. *Let S be any subset of $\{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$ and denote by $N_{n,k}(S)$ the number of occurrences of a success block of size k within S. Then*

$$P[N_{n,k}(S) = 0] \ge (1 - p_n^{k^2})^{|S|}$$

where $|S|$ denotes the cardinality of S.

PROOF. If $A = \{(i,j) : k \le i \le n, k \le j \le n\}$ it is obvious that

$$P[N_{n,k}(S) = 0] \ge P[\gamma_{ij} = 0 \text{ for all } (i,j) \in S \cap A] \ge P\left[\prod_{(i,j)\in S\cap A} (1 - \gamma_{ij}) = 1\right]$$

and by virtue of (2.1) we conclude that

$$P[N_{n,k}(S) = 0] \ge \prod_{(i,j)\in S\cap A} (1 - P[\gamma_{ij} = 1]) = (1 - p_n^{k^2})^{|S\cap A|} \ge (1 - p_n^{k^2})^{|S|}. \quad \square$$

LEMMA 3.2. *Let $N(n^*, n; k)$, $n^* \le n$ be the number of occurrences of a success block of size $k \le n$ in a binary random matrix $X = (X_{ij})_{n^* \times n}$. Then*

$$P[N(n^*, n; k) = 0] \le (1 - q_n^2 p_n^{k^2})^{(n^*-k+1)(n-k+1)}$$

for all $n^* = 1, 2, \ldots, n$.

PROOF. If $n^* \ge k$, the result is a direct consequence of Theorem 2.1. For $n^* < k$ the inequality becomes trivial since its LHS equals 1, while the RHS is greater than or equal to 1. $\square$

The next theorem states that, roughly speaking, if the success probabilities p_n are small, then the random variable $N_{n,k}$ follows asymptotically $(n \to \infty)$ the Poisson distribution. More specifically, we have the following

THEOREM 3.1. *If the success probabilities p_n satisfy the condition*

$$n^2 p_n^{k^2} \to \lambda, \quad \text{as} \quad n \to \infty \tag{3.1}$$

then

$$\lim_{n\to\infty} P[N_{n,k} = m] = e^{-\lambda}\frac{\lambda^m}{m!} \tag{3.2}$$

for every $m = 0, 1, 2, \ldots$.

PROOF. The special case $m = 0$ has already been treated in Corollary 2.1. Let $m \geq 1$ and assume that $n \geq m(k+1)$. For every pair (i,j), $k+1 \leq i, j \leq n$ we introduce the index set

$$I_{ij} = \{(\mu,\nu) : i-k \leq \mu \leq i, j-k \leq \nu \leq j\} \tag{3.3}$$

and the binary variable

$$\delta_{ij} = \gamma_{ij}(1 - X_{i,j-k})(1 - X_{i-k,j})(1 - X_{i-k,j-k}).$$

Also, for every $\boldsymbol{j} = (j_1, j_2, \ldots, j_m)$ with $k+1 \leq j_r \leq n$, $r = 1, 2, \ldots, m$ and every $\boldsymbol{i} = (i_1, i_2, \ldots, i_m)$ satisfying the constraints

$$i_1 \geq k+1, \quad i_r - i_{r-1} \geq k+1 \qquad r = 2, \ldots, m \tag{3.4}$$

we define

$$I_*(\boldsymbol{i},\boldsymbol{j}) = \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} - \bigcup_{r=1}^{m} I_{i_r, j_r}$$

and denote by $A_*(\boldsymbol{i},\boldsymbol{j})$ the event

$$\{\delta_{i_r,j_r} = 1 \text{ for all } r = 1, 2, \ldots, m \text{ and } N_{n,k}(I_*(\boldsymbol{i},\boldsymbol{j})) = 0\}.$$

The definition of $A_*(\boldsymbol{i},\boldsymbol{j})$ implies that

$$A_*(\boldsymbol{i},\boldsymbol{j}) \subseteq \{N_{n,k} = m\} \qquad \text{for every } \boldsymbol{i},\boldsymbol{j},$$

and since $A_*(\cdot,\cdot)$ are disjoint, we conclude that

$$P[N_{n,k} = m] \geq P\left[\bigcup_{\boldsymbol{i}}\bigcup_{\boldsymbol{j}} A_*(\boldsymbol{i},\boldsymbol{j})\right] = \sum_{\boldsymbol{i}}\sum_{\boldsymbol{j}} P[A_*(\boldsymbol{i},\boldsymbol{j})]. \tag{3.5}$$

On the other hand, employing Lemma 3.1, we obtain

$$\begin{aligned} P[A_*(\boldsymbol{i},\boldsymbol{j})] &= P[\delta_{i_r,j_r}=1, r=1,2,\dots,m]P[N_{n,k}(I_*(\boldsymbol{i},\boldsymbol{j}))=0] \\ &\geq (p_n^{k^2} q_n^3)^m (1-p_n^{k^2})^{n^2-m(k+1)^2}. \end{aligned} \tag{3.6}$$

Employing (3.5), (3.6) and taking into account the fact that the number of possible selections of the vector $\boldsymbol{j}$ is $(n-k)^m$ and of the vector $\boldsymbol{i}$ is $\binom{n-mk}{m}$ (because of constraints (3.4)), we deduce

$$P[N_{n,k}=0] \geq (n-k)^m \binom{n-mk}{m} q_n^{3m} p_n^{mk^2} (1-p_n^{k^2})^{n^2-m(k+1)^2}. \tag{3.7}$$

In order to obtain an upper bound for the probability $P[N_{n,k}=0]$ we proceed as follows. Let

$$I_{ij}^* = \{(\mu,\nu) : i-k+1 \leq \mu \leq i, j-k+1 \leq \nu \leq j\}, \qquad k \leq i,j \leq n$$

and denote by $\boldsymbol{i}=(i_1,i_2,\dots,i_m)$ an m-tuple of integers such that $k \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n$. For every choice of $\boldsymbol{i}$, we select $\boldsymbol{j}=(j_1,j_2,\dots,j_m)$ so that $k \leq j_r \leq n$ and

$$I_{i_r,j_r}^* \cap I_{i_s,j_s}^* = \phi \quad \text{for all } s=1,2,\dots,r-1, \; r=2,3,\dots,m. \tag{3.8}$$

Notice that, no matter what $\boldsymbol{i}$ is, there always exist proper choices of $\boldsymbol{j}$; for example, $j_r=rk$, $r=1,2,\dots,m$, always provides a valid selection.

Introducing

$$I^*(\boldsymbol{i},\boldsymbol{j}) = \{1,2,\dots,n\} \times \{1,2,\dots,n\} - \bigcup_{r=1}^{m} I_{i_r,j_r}^*$$

and the event

$$A^*(\boldsymbol{i},\boldsymbol{j}) = \{\gamma_{i_r,j_r}=1 \text{ for all } r=1,2,\dots,m \text{ and } N_{n,k}(I^*(\boldsymbol{i},\boldsymbol{j}))=0\}$$

we may state that

$$\begin{aligned} P[N_{n,k}=m] &\leq \sum_{\boldsymbol{i}} \sum_{\boldsymbol{j}} P[A^*(\boldsymbol{i},\boldsymbol{j})], \\ P[A^*(\boldsymbol{i},\boldsymbol{j})] &= (p_n^{k^2})^m P[N_{n,k}(I^*(\boldsymbol{i},\boldsymbol{j}))=0]. \end{aligned} \tag{3.9}$$

Next, observe that the set

$$I^{**} = \{1,2,\dots,n\} - \bigcup_{r=1}^{m} \{i_r-k+1,\dots,i_r\}$$

can be expressed as a union of disjoint sets of consecutive integers, i.e.

$$I^{**} = \bigcup_{s=1}^{l} \{i_s^*, i_s^* + 1, \dots, i_s^* + n_s^* - 1\},$$

where

$$\sum_{s=1}^{l} n_s^* \geq n - mk, \qquad 1 \leq l \leq m+1. \tag{3.10}$$

Defining

$$I_s^{**} = \{i_s^*, i_s^* + 1, \dots, i_s^* + n_s^* - 1\} \times \{1, 2, \dots, n\}$$

we obtain

$$P[N_{n,k}(I^*(\boldsymbol{i}, \boldsymbol{j})) = 0] \leq \prod_{s=1}^{l} P[N_{n,k}(I_s^{**}) = 0]$$

which, by using Lemma 3.2 and (3.10), yields

$$P[N_{n,k}(I^*(\boldsymbol{i}, \boldsymbol{j})) = 0] \leq (1 - q_n^2 p_n^{k^2})^{(n-k+1)[n-m(2k-1)-k+1]}. \tag{3.11}$$

Finally, (3.9) and (3.11) give the following upper bound for the probability $P[N_{n,k} = m]$.

$$P[N_{n,k} = m] \leq \binom{m+n-k}{m} (n-k+1)^m \cdot p_n^{mk^2} (1 - q_n^2 p_n^{k^2})^{(n-k+1)[n-m(2k-1)-k+1]}. \tag{3.12}$$

Under the condition (3.1), the limiting expression (3.2) is now an immediate consequence of the lower and upper bounds given by equations (3.7) and (3.12) respectively. □

The next corollary describes the limiting ($n \to \infty$) behaviour of a 2-dimensional m-consecutive-k-out-of-n system with iid components.

COROLLARY 3.1. *Let R_n be the reliability of a 2-dimensional m-consecutive-k-out-of-n system with component reliabilities $q_n = 1 - p_n$. If $n^2 p_n^{k^2} \to \lambda$ as $n \to \infty$ then*

$$\lim_{n \to \infty} R_n = \sum_{x=0}^{m-1} e^{-\lambda} \frac{\lambda^x}{x!}.$$

PROOF. Due to the definition of the system, we have

$$R_n = P[N_{n,k} < m] = \sum_{x=0}^{m-1} P[N_{n,k} = x]$$

and employing Theorem 3.1, we get the result immediately. □

4. Additional models

In this paragraph we introduce some additional models, closely related to the one studied so far, and indicate how the analysis conducted in paragraphs 2 and 3 could be adapted to the new cases.

(a) Assume that besides the success blocks of size k obtained through $k \times k$ submatrices of X, we are also interested in success blocks over subsets of the form

$$\{(\mu, \nu) : i - k + 1 \leq \mu \leq i, \nu = n - k + j + 1, \dots, n, 1, \dots, j\} \qquad k \leq i \leq n, \quad 1 \leq j < k.$$

In other words, this means that the rows of X are wrapped around so that the first column of X becomes next to the last one. This formulation would be useful for the study of a two-dimensional consecutive-k-out-of-n reliability system whose components are placed on the side surface of a cylinder. Let $N_{n,k}^c$ denote the number of success blocks (regular and extended). Introducing the notation

$$X_{i,-j} = X_{i,n-j}, \qquad p_{i,-j} = p_{i,n-j}, \qquad q_{i,-j} = q_{i,n-j}, \qquad 1 \leq i \leq n \text{ and } 0 \leq j < k$$

we can extend the definition of the binary variables γ_{ij} over all $k \leq i \leq n$, $1 \leq j \leq n$ and employing the same reasoning as in Theorem 2.1 we can prove that

$$\prod_{i=k}^{n} \prod_{j=1}^{n} \{1 - P[\gamma_{ij} = 1]\} \leq P[N_{n,k}^c = 0] \leq \prod_{i=k}^{n} \prod_{j=1}^{n} \{1 - q_{i-k,j-k+1} q_{i,j-k} P[\gamma_{ij} = 1]\}.$$

With this in mind, it is not difficult to state and prove results similar to the ones given in Theorems 2.2, 2.3 and Corollaries 2.1, 2.2, 2.3.

(b) Assume that the "wrapping" procedure described in (a) is performed in both rows and columns of the matrix X and consider the total number of success blocks. In reliability context this setup fits to a 2-dimensional consecutive-k-out-of-n system whose components are placed on the surface of a sphere.

The extension of Theorems 2.1, 2.2, 2.3 is rather straightforward (all summations and products with respect to i and j are now performed over $1, 2, \dots, n$ instead of $k, k+1, \dots, n$). Corollaries 2.1 and 2.3 remain unchanged while conditions (a), (b) and (c) of Corollary 2.2 become (i) $\lim_{n\to\infty} n^2 p_n^{k_n^2} = \lambda$, (ii) $\lim_{n\to\infty} q_n = 1$.

(c) Consider a 3-dimensional random matrix $X = (X_{ijr})_{n\times n\times n}$ whose entries X_{ijr} are Bernoulli variables with success (failure) probabilities p_{ijr} (q_{ijr}). Let $N_{n,k}$ denote the number of success cubes of size k (3-dimensional $k \times k \times k$ submatrices of X with all its elements equal to 1). The probability $P[N_{n,k} = 0]$ is closely related to a 3-dimensional consecutive-k-out-of-n system which, as mentioned by Salvia and Lasher (1990), is applicable in medical diagnostics. Again, the adaptation of the results of the previous paragraphs to this setup is rather straightforward.

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