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On the Interdependency of the Gauss-Codazzi-Ricci Equations of Local Isometric Embedding

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Abstract

The Gauss-Codazzi-Ricci equations governing the local isometric embedding of Riemannian spaces $V_n \subset V_N$ ($N = n + p, p > 0$) are interrelated by the Bianchi identities in V_n and V_N. This leads to redundancies which permit great simplification in the embedding problem, i.e., allows a neglect of part of the equations. By transcription, to the case of semi-Riemannian spaces, of a result of R. Blum we obtain a number of theorems and corollaries expressing for $V_n \subset V_N$ this interdependency of the Gauss-Codazzi-Ricci equations. They form a generalization of previous results and are felt to be useful for the study of the geometrical properties of space-time and its three-dimensional space sections.

§(1): *Introduction*

Recently, in this journal, Y. K. Gupta and P. Goel [1] have given a class-2 analog and a class-p generalization (suggested by Barnes) of T. Y. Thomas's theorem [2] concerning an interrelation of the equations that govern the local isometric embedding of a Riemannian space V_n in a flat space E_N ($N = n + p$, $p > 0$). In this note, I would like to communicate a more general result essentially due to R. Blum [3]-[5], which was established a long time ago but seems to be virtually unknown to relativists and differential geometers.[1]

Consider a Riemannian space V_n, locally, as a subspace of a V_N. If y^A ($A = 1, 2, \ldots, N$) are coordinates of a chart in the atlas covering V_N and x^α ($\alpha = 1,$

[1] For example, in the book of Kobayashi and Nomizu [6] in which the problem of local isometric embedding is treated, Blum's result is not mentioned.

2, . . . , n) coordinates in V_n the local isometric embedding of $V_n \subset V_N$ is described by

$$y^A = y^A(x^\alpha) \tag{1.1}$$

$$g_{\alpha\beta} = \bar{g}_{AB} y^A{}_{,\alpha} y^B{}_{,\beta} \tag{1.2}$$

where $g_{\alpha\beta}$ and $\bar{g}_{AB}$ are the components of the metric tensors of V_n and V_N, respectively. If, at any point of $V_n \subset V_N$ an N-leg $\{y^A{}_{,\alpha} | n^{jA}\}$ is affixed with $y^A{}_{,\alpha}$ tangent to V_n and n^{jA} ($j = 1, 2, \ldots, p = N - n$) normal to V_n a kind of generalized Frenet equations can be derived [7][2]:

$$y^A{}_{;\alpha;\beta} = \sum_j e_j b^j_{\alpha\beta} n^{jA} - \bar{\Gamma}_{BC}{}^A y^B{}_{;\alpha} y^C{}_{;\beta} \tag{1.3a}$$

$$n^{jA}{}_{;\alpha} = -b^j_{\sigma\alpha} g^{\sigma\kappa} y^A{}_{;\kappa} + \sum_k e_k s^{kj}_\alpha n^{kA} - \bar{\Gamma}_{CD}{}^A y^C{}_{;\alpha} n^{jD} \tag{1.3b}$$

where

$$\bar{g}_{AB} n^{jA} n^{kB} = e^j \delta^{jk} \tag{1.4a}$$

$$\bar{g}_{AB} n^{jA} y^B{}_{;\alpha} = 0 \tag{1.4b}$$

with sign factors $(e_j)^2 = (e^j)^2 = 1$.

The $p = N - n$ symmetric bilinear forms $b^j_{\alpha\beta} = b^j_{\beta\alpha}$ and $\binom{p}{2}$ vectors $s^{ij}_\alpha = -s^{ji}_\alpha$ defined by (1.3a) and (1.3b) are called second fundamental forms and torsion vectors of $V_n \subset V_{n+p}$ [8]. The integrability conditions of equations (1.3a) and (1.3b) form a mixed system of algebraical and differential equations, the Gauss-Codazzi-Ricci equations[3]:

$$0 = G_{\alpha\beta\gamma\delta} := R_{\alpha\beta\gamma\delta} - \sum_j e_j b^j_{\alpha[\gamma} b^j_{\delta]\beta} - \bar{R}_{ABCD} y^A{}_{;\alpha} y^B{}_{;\beta} y^C{}_{;\gamma} y^D{}_{;\delta} \tag{1.5}$$

$$0 = C^j_{\alpha\beta\gamma} := b^j_{\alpha[\beta;\gamma]} - \sum_k e_k b^k_{\alpha[\beta} s^{kj}_{\gamma]} - \bar{R}_{ABCD} y^A{}_{,\alpha} n^{jB} y^C{}_{;\beta} y^D{}_{;\gamma} \tag{1.6}$$

$$0 = K^{ij}_{\alpha\beta} := s^{ij}_{[\alpha;\beta]} + \sum_k e_k s^{ki}_{[\alpha} s^{kj}_{\beta]} + g^{\mu\nu} b^i_{\mu[\alpha} b^j_{\beta]\nu} + \bar{R}_{ABCD} n^{iA} n^{jB} y^C{}_{;\alpha} y^D{}_{;\beta} \tag{1.7}$$

[2]We use the summation convention for indices A and α. A comma denotes partial derivatives, a semicolon covariant derivatives with respect to $g_{\alpha\beta}$, while double verticals stand for covariant derivatives with respect to $\bar{g}_{AB}$. $\bar{\Gamma}_{AB}{}^C$ and $\bar{R}^A{}_{BCD}$ are the Christoffel symbol and Riemannian curvature tensor of V_N taken on the subspace V_n.

[3]The antisymmetrization bracket is defined by $A_{[\alpha;\beta]} = A_{\alpha;\beta} - A_{\beta;\alpha}$ while $A_{\langle\alpha\beta\gamma\rangle} = A_{\alpha\beta\gamma} + A_{\gamma\alpha\beta} + A_{\beta\gamma\alpha}$.

With Blum [3, 4] we call $G_{\alpha\beta\gamma\delta}$, $C^j_{\alpha\beta\gamma}$, and $K^{ij}_{\alpha\beta}$ the Gauss, Codazzi, and Ricci tensors, respectively.

The Gauss-Codazzi-Ricci equations are linked by the Bianchi identities in V_n:

$$R_{\alpha\beta\langle\gamma\delta;\epsilon\rangle} = 0 \tag{1.8}$$

and in V_N:

$$\bar{R}_{AB\langle CD\|E\rangle} = 0 \tag{1.9}$$

Equations (1.8) and (1.9) lead to a redundancy permitting, under certain conditions on the $b^j_{\alpha\beta}$, a great deal of reduction of the Gauss-Codazzi-Ricci equations. In many cases, the problem of local isometric embedding then is transformed into a purely algebraical one. In Section 2, Blum's result concerning the interdependency of the Gauss-Codazzi-Ricci equations is described and a sketch of the proof for the case of semi-Riemannian spaces given. In Section 3, applications of the general theorem are derived, one of which contains, as a special subcase, the results of Gupta and Goel [1] and of Barnes (see footnote in [1]).

§(2): *Theorem of Blum*

By repeated use of (1.5), (1.6), and (1.7) a straightforward calculation leads to the following expressions:

$$\begin{aligned}(\bar{R}_{ABCD}y^A{}_{;\alpha}y^B{}_{;\beta}y^C{}_{;\gamma}y^D{}_{;\delta})_{;\epsilon} = {} & \bar{R}_{ABCD\|E}y^A{}_{;\alpha}y^B{}_{;\beta}y^C{}_{;\gamma}y^D{}_{;\delta}y^E{}_{;\epsilon} \\ & - \sum_j e_j b^j_{\alpha\epsilon}\left[-C^j_{\beta\gamma\delta} + b^j_{\beta[\gamma;\delta]} - \sum_k e_k b^k_{\beta[\gamma}s^{kj}_{\delta]}\right] \\ & + \sum_j e_j b^j_{\beta\epsilon}\left[-C^j_{\alpha\gamma\delta} + b^j_{\alpha[\gamma;\delta]} - \sum_k e_k b^k_{\alpha[\gamma}s^{kj}_{\delta]}\right] \\ & - \sum_j e_j b^j_{\gamma\epsilon}\left[-C^j_{\delta\alpha\beta} + b^j_{\delta[\alpha;\beta]} - \sum_k e_k b^k_{\delta[\alpha}s^{kj}_{\beta]}\right] \\ & + \sum_j e_j b^j_{\delta\epsilon}\left[-C^j_{\gamma\alpha\beta} + b^j_{\gamma[\alpha;\beta]} - \sum_k e_k b^k_{\gamma[\alpha}s^{kj}_{\beta]}\right]\end{aligned} \tag{2.1}$$

and

$$(\bar{R}_{ABCD}y^A{}_{;\alpha}n^{jB}y^C{}_{;\beta}y^D{}_{;\gamma})_{;\delta} = \bar{R}_{ABCD\|E}y^A{}_{;\alpha}n^{jB}y^C{}_{;\beta}y^D{}_{;\gamma}y^E{}_{;\epsilon}$$

$$+ \sum_k e_k \bar{R}_{ABCD}n^{jB}n^{kC}y^A{}_{;\alpha}y^D{}_{;[\gamma}b^k_{\beta]\delta}$$

$$+ \sum_k e_k b^k_{\alpha\delta}\Big[K^{kj}_{\beta\gamma} - s^{kj}_{[\beta;\gamma]} - \sum_q s^{qk}_{[\beta}s^{qj}_{\gamma]}$$

$$- g^{\mu\nu}b^k_{\mu[\beta}b^j_{\gamma]\nu}\Big] + b^j{}_{\sigma\delta}g^{\sigma\nu}\Big[G_{\alpha\nu\beta\gamma} - R_{\alpha\nu\beta\gamma}$$

$$+ \sum_k e_k b^k_{\alpha[\beta}b^k_{\gamma]\nu}\Big] + \sum_k e_k s^{kj}_{\delta}\Big[-C^k_{\alpha\beta\gamma} + b^k_{\alpha[\beta;\gamma]}$$

$$- \sum_q s^{qk}_{[\gamma}b^q_{\beta]\alpha}\Big] \tag{2.2}$$

Application of (1.8), (1.9) and the Ricci identity for $b^j_{\alpha\beta;[\gamma;\delta]}$ together with (2.1), (2.2) gives the following system of derived Gauss and Codazzi tensors:

$$G_{\alpha\beta\langle\gamma\delta;\delta\rangle} = \sum_j e_j(C^j_{\alpha\langle\gamma\delta}b^j_{\epsilon\rangle\beta} - C^j_{\beta\langle\gamma\delta}b^j_{\epsilon\rangle\alpha}) \tag{2.3}$$

and

$$C^j_{\alpha\langle\beta\gamma;\delta\rangle} = G^\sigma{}_{\alpha\langle\beta\gamma}b^j_{\epsilon\rangle\sigma} - \sum_k e_k[b^k_{\alpha\langle\beta}K^{kj}_{\gamma\delta\rangle} - C^k_{\alpha\langle\beta\gamma}s^{kj}_{\delta\rangle}] \tag{2.4}$$

For $e_1 = e_2 = \cdots = e_p = 1$, i.e., an ordinary Riemannian space equations (2.3), (2.4) are due to Blum [5].

If the Gauss equation (1.5) is satisfied by a set of $b^j_{\alpha\beta}$, equation (2.3) is a linear system for the $p \cdot n(n^2 - 1)/3$ unknowns $C^j_{\alpha\beta\gamma}$:

$$0 = \sum_{j=1}^{p} C^j_{\kappa\mu\nu}M^{j\kappa\mu\nu}_{\alpha\beta\gamma\delta\epsilon} \tag{2.5}$$

where

$$M^{j\kappa\mu\nu}_{\alpha\beta\gamma\delta\epsilon} := \tfrac{1}{2}e^j\delta^{[\mu}_{\langle\gamma}\delta^{\nu]}_{\delta}b^j{}_{\epsilon\rangle[\beta}\delta^\kappa_{\alpha]}$$

(no summation on j) is a matrix with $\frac{1}{2}\binom{n+1}{2}\binom{n}{3}$ rows and $\frac{1}{3}pn(n^2 - 1)$ columns.

If the Gauss and Codazzi equations (1.5) and (1.6) are satisfied by a set of $b^j_{\alpha\beta}$ and $s^{ij}_{\alpha\beta}$ equation (2.4) is a linear system for the $\binom{p}{2} \cdot \binom{n}{2}$ unknowns $K^{ij}_{\alpha\beta}$:

$$0 = \sum_{k=1}^{p} K^{kj}_{\mu\nu}N^{k\mu\nu}_{\alpha\beta\gamma\delta} \tag{2.6}$$

$(j = 1, 2, \ldots, p)$
where

$$N^{k\mu\nu}_{\alpha\beta\gamma\delta} := \tfrac{1}{2} e^k b^k_{\alpha\langle\beta} \delta^{[\mu}_{\gamma} \delta^{\nu]}_{\delta\rangle}$$

(no summation over k) is a matrix with $\frac{1}{2}p\binom{n+1}{2}\binom{n-1}{2}$ rows and $\binom{p}{2} \cdot \binom{n}{2}$ columns.

By counting independent components of all tensors occurring, Blum arrived at the following result:

> *Theorem:* If the Gauss equation (1.5) is satisfied by a set of $b^j_{\alpha\beta}$ for which the ranks of matrices M and N of (2.5) and (2.6) have maximum value, then (1) for $0 \leqslant p = N - n \leqslant \frac{1}{8}n(n-2)$ all Codazzi and Ricci equations are consequences of the Gauss equations; (2) for $\frac{1}{8}n(n-2) < p \leqslant \binom{n}{2}$ a system of $\frac{1}{3}n(n^2-1)[p - \frac{1}{8}n(n-2)]$ equations of (1.6) are independent. The remainder of the Codazzi equations and all Ricci equations are a consequence of the independent system and of the Gauss equations.

For the case of M, N having less than maximal value we refer to Blum's paper [5].

In the case of space-time V_4, in which we are most interested, the dimensions of M and N are formidable. For class p $(1 \leqslant p \leqslant 6)$ M is a $20 \times 20\,p$ matrix while N is a $15p \times 3p(p-1)$ matrix. Thus, in the form given above, Blum's result is of no direct calculational use. It seems desirable to replace the assumptions concerning the ranks of M and N by conditions on $b^j_{\alpha\beta}$, directly (which themselves hopefully may be connected to properties of curvature invariants). A step in this direction is taken in the next section.

§(3): *Applications*

As a first consequence of the system (2.3), (2.4) more suited to direct application the following theorem is proved:

> *Theorem 1.* Let rank $r \geqslant 4$ of one of the p second fundamental forms $b^j_{\alpha\beta}$ of $V_n \subset V_N$, e.g., rank $b^1_{\alpha\beta} \geqslant 4$. Then, the Codazzi equation $C^1_{\alpha\beta\gamma} = 0$ and Ricci equations $K^{1j'}_{\alpha\beta} = 0$ $(j' = 2, 3, \ldots, p)$ follow from the Gauss equation, the remaining Codazzi equations $C^{j'}_{\alpha\beta\gamma} = 0$ $(j' = 2, 3, \ldots, p)$, and remaining Ricci equations $K^{i'j'}_{\alpha\beta} = 0$.

Proof. By assumption (2.3) reduced to

$$0 = C^1_{\alpha\langle\gamma\delta} b^1_{\epsilon\rangle\beta} - C^1_{\beta\langle\gamma\delta} b^1_{\epsilon\rangle\alpha} \tag{3.1}$$

For rank $b^1_{\alpha\beta} \geqslant 4$, by a tedious calculation involving contractions with the inverse of a 4×4 submatrix of $b^1_{\alpha\beta}$ with nonvanishing determinant, from (3.1) we derive

$$C^1_{\alpha\beta\gamma} = 0 \tag{3.2}$$

By the assumptions of Theorem 1 and by (3.2), equation (2.4) reduces to

$$0 = b^1_{\alpha\langle\beta} K^{1j'}_{\gamma\delta\rangle} \quad (j' = 2, 3, \ldots, p) \tag{3.3}$$

while, for $j = 1$, equation (2.4) shrinks to

$$0 = \sum_k e_k b^k_{\alpha\langle\beta} K^{k1}_{\gamma\delta\rangle} \tag{3.4}$$

Now, if rank $b^1_{\alpha\beta} \geqslant 3$, (3.3) leads to

$$K^{1j}_{\alpha\beta} = 0 \tag{3.5}$$

Equation (3.4) then is satisfied identically. For a local isometric embedding into flat space, of class one, the result of Thomas [2] follows from Theorem 1; for arbitrary class the generalization suggested by A. Barnes [1] obtains.

Equation (3.5) may be derived under a different assumption.

Theorem 2. Let rank $r \geqslant 3$ of one of the p second fundamental forms of $V_n \subset V_N$, e.g., rank $b^1_{\alpha\beta} \geqslant 3$. Then the Ricci equations $K^{1j'}_{\alpha\beta} = 0$ ($j' = 2, 3, \ldots, p$) follow from the Gauss equation (1.7), the Codazzi equations (1.6), and the remaining Ricci equations.

Proof. The only equation of the system (2.3) and (2.4) not yet satisfied by the assumption of Theorem 2 is (3.3). However, for rank $b^1_{\alpha\beta} \geqslant 3$, equation (3.3) implies $K^{1j'}_{\alpha\beta} = 0$. For spaces with embedding class $p = 2$ the following conclusion from Theorems 1 and 2 may be drawn:

Corollary 1. Let rank $r \geqslant 3$ of one of the two fundamental forms, e.g., rank $b^1_{\alpha\beta} \geqslant 3$. Then the Ricci equations follow from the Gauss equation and the Codazzi equations.

Corollary 2. Let rank $r \geqslant 4$ of one of the two fundamental forms, e.g., $b^1_{\alpha\beta} \geqslant 4$. Then, in addition to the Ricci equations, the Codazzi equation $C^1_{\alpha\beta\gamma} = 0$ follows from the Gauss equation and the remaining Codazzi equation $C^2_{\alpha\beta\gamma} = 0$.

For ordinary Riemannian spaces embedded in Euclidean space $V_n \subset E_{n+2}$ both corollaries were proven by Verbizkii [9]. Gupta and Goel derived Corollary 2 [1].

If, for class two, none of the fundamental forms reaches rank 3, the following result may be obtained:

Corollary 3. Let $n \geqslant 3$ and rank $b^1_{\alpha\beta} \leqslant 2$, rank $b^2_{\alpha\beta} \leqslant 2$. If $b^1_{\alpha\beta} + \lambda b^2_{\alpha\beta}$ is a regular pencil or a singular pencil of rank 3, then the Ricci equations follow from the Gauss and Codazzi equations.

Proof. By assumption, equations (2.3) and (2.4) reduce to

$$\begin{aligned} 0 &= b^1_{\alpha\langle\beta} K^{12}_{\gamma\delta\rangle} \\ 0 &= b^2_{\alpha\langle\beta} K^{12}_{\gamma\delta\rangle} \end{aligned} \tag{3.6}$$

(3.6) admits nontrivial solutions only if rank $b^1_{\alpha\beta} \leqslant 2$, rank $b^2_{\alpha\beta} \leqslant 2$. A detailed analysis then shows that, in this case, $b^1_{\alpha\beta} + \lambda b^2_{\alpha\beta}$ forms a singular pencil of rank $\leqslant 2$.

It is hoped that the results presented here may be useful (1) for the study of the geometrical properties of space-times V_4 regarded, locally, as subspaces of a flat space or a space of constant curvature of higher dimension; (2) for the study of the geometry of time- or spacelike sections of space-time itself ($V_3 \subset V_4$).

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