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A Charged Generalization of Florides' Interior Schwarzschild Solution

Ø. Grøn[1]

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The new class of interior Schwarzschild solutions found by Florides is generalized to the charged case. A particular solution within this class is found, which represents an electromagnetic mass-model of a neutral spherically symmetric system. The pressure is isotropic, decreasing monotonously with increasing radius and vanishes at the surface of the matter distribution. The solution is regular everywhere inside a radius R, and is joined continuously to the exterior Schwarzschild solution at this radius.

1. INTRODUCTION

A new class of interior Schwarzschild solutions was presented by Florides [1] and investigated by Kofinti [2] recently. The solutions are static and spherically symmetric. This class of solutions is defined by the condition that the component $T^r{}_r$ of the energy-momentum tensor vanishes. It was shown by Florides that this implies a nonvanishing tangential stress, $T^\theta{}_\theta = T^\phi{}_\phi \neq 0$. Thus, the solution describes space-time in a medium which cannot be considered a static perfect fluid. If $R > 3m$, where R is the radius of the matter distribution and m is its Schwarzshild mass, the system may be interpreted as an Einstein cluster.

Charged generalizations of Florides' class of interior Schwarzschild solutions were recently given by Mehra [3] and Florides [4]. Mehra assumed that the components of the stress-energy tensor of the electromagnetic fields $E^1{}_1$ and $E^4{}_4$, are equal, and that the components of the stress-energy tensor of matter, $T^1{}_1$ and $T^4{}_4$, are 0 and $-\rho(r)$, respectively.

[1] Institute of Physics, University of Oslo, Oslo, Norway.

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Florides considered two classes of solutions, one representing charged, perfect-fluid spheres, and one representing charged distributions with no radial stress.

In Section 2 of the present paper Florides' class of solutions [1] will be generalized to the charged case under the assumption of isotropic pressure. A particular solution is found in Section 3, which represents an electromagnetic mass-model of a neutral spherically symmetric system, in the sense that the mass vanishes when the charge density is set equal to zero everywhere. The results are summarized in Section 4.

2. CHARGED GENERALIZATION OF FLORIDES' INTERIOR SCHWARZSCHILD SOLUTION

A class of static and spherically symmetric solutions of the Einstein–Maxwell equations will be investigated. The line element, as expressed in curvature coordinates takes the form

$$ds^2 = e^{\alpha} dr^2 + r(d\theta^2 + \sin^2\theta\, d\phi^2) - e^{\gamma}\, dt^2 \tag{1}$$

where α and γ are functions of r. As shown by Synge [5] the metric of an arbitrary static, spherically symmetric solution of Einstein's field equations is then given by

$$e^{-\alpha} = 1 + \frac{8\pi}{r} \int_0^r T^t{}_t r^2\, dr \tag{2}$$

$$\gamma = \int_R^r \left(\frac{e^{\alpha} - 1}{r} + 8\pi T^r{}_r e^{\alpha} r \right) dr + \gamma(R) \tag{3}$$

Florides' class of solutions is defined by the condition $T^r{}_r = 0$. This gives

$$e^{-\alpha} = 1 - 2m(r)/r \tag{4}$$

$$\gamma = \int_R^r \frac{2m(r)/r}{r - 2m(r)}\, dr + \gamma(R) \qquad \text{where} \tag{5}$$

$$m(r) = -4\pi \int_0^r T^t{}_t r^2\, dr \tag{6}$$

is the mass inside r.

The energy-momentum tensor of the charged medium plus the electrical field has components

$$T^t{}_t = -\left(\rho + \frac{E^2}{8\pi} \right), \qquad T^r{}_r = p_r - \frac{E^2}{8\pi}, \qquad T^\theta{}_\theta = T^\phi{}_\phi = p_\Omega + \frac{E^2}{8\pi} \tag{7}$$

where ρ is the mass density of the matter, E the electrical field strength, p_r the radial pressure and p_Ω the tangential pressure. The condition $T^r{}_r = 0$ gives

$$p_r = E^2/8\pi \tag{8}$$

showing that although the radial fluid pressure vanishes for Florides' class of solutions in the electrically neutral case, this is not so when the system is charged.

Einstein's field equations, with $T^r{}_r = 0$, gives [5]

$$T^\theta{}_\theta = -\tfrac{1}{4}(e^\alpha - 1)\, T^4{}_4 \tag{9}$$

Thus, by means of (7) and (8)

$$p_\Omega + p_r = \tfrac{1}{4}(e^\alpha - 1)(\rho + p_r) \tag{10}$$

In the present case Maxwell's equations reduce to

$$E(r) = \frac{4\pi}{r^2}\int_0^r \sigma(r)\, e^{\alpha/2} r^2\, dr \tag{11}$$

where σ is the charge density.

The above equations can be combined to give

$$\{[r^2(r - re^{-\alpha})' - 8\pi\rho r^4]^{1/2}\}' = 4\pi\sigma e^{\alpha/2} r^2 \tag{12}$$

The charged generalization of Florides' class of solutions is obtained by solving this differential equation for α. This demands specification of charge and mass distributions and of boundary conditions. Alternatively, one may specify a relation between p_Ω and p_r and calculate the mass distribution from (10). The condition for local flatness at $r = 0$, $\alpha(o) = 0$, provides a boundary condition for α.

In the case of a vanishing charge density, $\sigma = 0$, integration of (12) gives

$$(r - re^{-\alpha})' = 8\pi\rho r^2 \tag{13}$$

and Florides' class of solutions, with vanishing radial pressure, is recovered.

The radial pressure is always nonvanishing when $E \neq 0$. The most natural physical condition for this case is obtained by assuming an isotropic pressure, $p_r = p_\Omega = p$. Then (10) gives

$$p = [(9 - e^\alpha)/(e^\alpha - 1)]\, \rho \tag{14}$$

Equation (6) may now be written

$$m(r)=4\int_0^r \frac{E^2r^2}{9-e^\alpha}\,dr \tag{15}$$

This equation shows that the mass vanishes when the charge density is zero everywhere. In this sense the solutions of the charged generalization of Florides' class with isotropic pressure represents general relativistic electromagnetic mass-models [6].

Physically acceptable solutions must have vanishing pressure at the surface of the matter distribution. From (8) it then follows that there is no electrical field outside the source. Thus, even if there is a nonvanishing charge density within the source, its net charge must be zero. These solutions of the Einstein–Maxwell equations therefore represent electromagnetic mass models of *neutral* spherically symmetric systems.

In the present case (12) takes the form

$$\{[r^2(9-e^\alpha)(r-re^{-\alpha})']^{1/2}\}'=8\pi\, 2^{1/2}\sigma e^{\alpha/2}r^2 \tag{16}$$

3. A PARTICULAR SOLUTION

In order to give a simple example of a solution of the charged generalization of Florides' class, I will assume a charge distribution given by

$$\sigma e^{\alpha/2}=(p_0/\pi)^{1/2}(4r^2)^{-1}[r^2(9-e^\alpha)^{1/2}]' \tag{17}$$

where p_0 is a constant. Integration of (16) then leads to

$$e^\alpha=[1-(8\pi/3)\,p_0r^2]^{-1}, \qquad r\leqslant R_H=(3/8\pi p_0)^{1/2} \tag{18}$$

Equation (3) now gives

$$e^\gamma=\{[1-(8\pi/3)\,p_0R^2]/[1-(8\pi/3)\,p_0r^2]\}^{1/2}\,e^{\gamma(R)} \tag{19}$$

Substituting (17) into (11) gives the electrical field strength

$$E=[\pi p_0(9-e^\alpha)]^{1/2} \tag{20}$$

The fluid pressure is now found from (8)

$$p=(1/8)(9-e^\alpha)\,p_0 \tag{21}$$

Inserting (18) gives

$$p = \{(1 - 3\pi p_0 r^2)/[1 - (8\pi/3)\, p_0 r^2]\}\, p_0 \tag{22}$$

Thus the constant p_0 is the pressure at the center.

The surface radius R of the matter distribution is given by $p(R) = 0$. Thus

$$R = (3\pi p_0)^{-1/2} \tag{23}$$

Since $R < R_H$ the condition $r < R_H$ is not violated inside the source. From (22) it follows that the pressure is a monotonously decreasing function of r.

Since the electrical field vanishes at $r = R$, the total charge of the source is zero. This is possible due to the particular charge distribution (17), which implies that the charge density changes sign at $R_1 = (0.28/\pi p_0)^{1/2} = 0.92 \cdot R$.

The mass density is

$$\rho = (\pi/3)\, p_0^2 r^2/[1 - (8\pi/3)\, p_0 r^2] \tag{24}$$

The density increases from zero at the center to $\rho(R) = p_0$ at the boundary of the matter distribution. From (4) and (18) the mass inside r is

$$m = (4\pi/3)\, p_0 r^3 \tag{25}$$

Outside the matter distribution there is Schwarzschild space-time. Thus $\gamma(R) = \ln(1 - 2M/R)$, where $M = m(R) = (4\pi/3)\, p_0 R^3$. The line element of the internal solution may now be written

$$ds^2 = \frac{dr^2}{1 - 2Mr^2/R^3} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2) - \frac{(1 - 2M/R)^{3/2}}{(1 - 2Mr^2/R^3)^{1/2}}\, dt^2 \tag{26}$$

Thus the internal solution is continuously joined to the external Schwarzschild solution at $r = R$.

4. DISCUSSION

The components of the energy-momentum tensor for the solution considered in Section 3 are

$$T^t{}_t = -p_0, \qquad T^\theta{}_\theta = T^\phi{}_\phi = (2\pi/3)\, p^2{}_0 r^2 (1 - 2Mr^2/R^3)^{-1} \tag{27}$$

These components have the same mathematical form as the ones corresponding to the constant density solution of Florides' uncharged class

[1]. Thus there is no surprise that the solution (26) describes an identical space-time to that of this solution.

However, the matter distributions are radically different for these two gravitationally equivalent solutions. Florides' solution describes space-time in a medium that is electrically neutral everywhere and that is kept in static equilibrium by tangential stresses although there is no radial pressure. The solution above, on the other hand, describes space-time in a medium with an isotropic pressure and a nonvanishing charge density. In fact, there is no medium if the charge-density vanishes everywhere.

Florides [4] has recently given a solution of the Einstein–Maxwell equations [his equation (3.9)] describing a space-time equivalent to the one described by the line element (26). Florides solution is, however, interpreted as describing a system with a net charge, and with the Reissner–Nordström space-time external to the source. This interpretation was made possible by giving up the condition that the pressure vanishes at the surface of the fluid distribution. It has been shown here that if this condition is maintained for a class of static, spherically symmetric solutions of the Einstein–Maxwell equations, defined by the condition $T^r{}_r = 0$, one obtains a class of electromagnetic mass-models of *neutral* systems.

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