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Anisotropic Spheres with Uniform Energy Density in General Relativity

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An ansatz is developed to obtain interior solutions of the Einstein field equations for anisotropic spheres. This procedure necessitates a choice for the energy-density and the radial pressure. A class of solutions for a uniform energy-density source is presented. These anisotropic spheres match smoothly to the Schwarzschild exterior and are well-behaved in the interior of the sphere.

1. INTRODUCTION

The study of static anisotropic spheres is important for relativistic astrophysics [1], and several solutions have been found using various ansätze [1–5]. We use a new ansatz to find a class of static anisotropic spheres in the idealized case of incompressibility (i.e., we assume that the energy-density is constant). This could be a good approximation for small stars in which the pressures are not too large. Note that even though the energy-density is uniform our solution does not contain the Schwarzschild interior solution as a special case. The analytic solutions presented are physically reasonable, well-behaved in the interior of the star, and match smoothly to the Schwarzschild exterior solution at the boundary of the star. The pressure is always positive and finite at the center of the sphere. The radial pressure is monotonically decreasing outward and vanishes at the boundary of the sphere. The surface redshift and mass-radius ratio may

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exceed the isotropic (perfect fluid) limit. For notational convenience we put $8\pi G = 1$ and $c = 1$ where G is Newton's gravitational constant and c is the speed of light in vacuum.

2. FIELD EQUATIONS

The appropriate field equations and related conservation equations are given by Bowers and Liang [1], among others. Here we briefly derive the field equations geometrically by supposing that the energy-momentum tensor is invariant (i.e., the matter content is restricted by the space-time symmetries).

The metric for static spherically symmetric space-times is

$$ds^2 = -e^{\nu(r)}\,dt^2 + e^{\lambda(r)}\,dr^2 + r^2(d\theta^2 + r^2 \sin^2\theta\, d\phi^2) \tag{1}$$

For the special case of the external Schwarzschild solution the metric (1) becomes

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + r^2 \sin^2\theta\, d\phi^2)$$

The energy-momentum tensor is of the form

$$T^{ij} = \mu u^i u^j + p h^{ij} + \pi^{ij} \tag{2}$$

where $u^i = e^{-\nu/2}\delta^i{}_t$, μ is the energy-density, p is the isotropic (kinetic) pressure, $h^{ij} = g^{ij} + u^i u^j$ is the projection tensor and π^{ij} is the anisotropic pressure (stress) tensor. The invariance of u^i and T^{ij} with respect to the Killing vectors of (1) imply that the dynamical quantities constructed from them are also invariant. This implies that the dynamical quantities assume the form [6]

$$\mu = \mu(r) \tag{3}$$

$$p = p(r) \tag{4}$$

$$\pi^{ij} = \sqrt{3}\, S(r)(c^i c^j - \tfrac{1}{3}h^{ij}) \tag{5}$$

where $c^i = e^{-\lambda/2}\delta^i{}_r$ is a unit radial vector and $|S(r)|$ is the magnitude of the stress tensor.

With the aid of equations (1)–(5) the Einstein field equations become

$$\frac{1}{r^2}-\frac{1}{e^\lambda}\left(\frac{1}{r^2}-\frac{\lambda'}{r}\right)=\mu$$

$$-\frac{1}{r^2}+\frac{1}{e^\lambda}\left(\frac{1}{r^2}+\frac{\nu'}{r}\right)=p+\frac{2S}{\sqrt{3}}$$

$$\frac{1}{4e^\lambda}\left[2\nu''+(\nu'-\lambda')\left(\nu'+\frac{2}{r}\right)\right]=p-\frac{S}{\sqrt{3}}$$

where $p+2S/\sqrt{3}\equiv p_r$ is the radial pressure and $p-S/\sqrt{3}\equiv p_\perp$ is the tangential pressure. The momentum conservation equation can be written in the form

$$(\mu+p_r)\,\nu'+2p_r'+4\sqrt{3}\,\frac{S}{r}=0 \tag{6}$$

We define the mass function to be

$$m(r)\equiv\tfrac{1}{2}\int_0^r x^2\mu(x)\,dx \tag{7}$$

A form of the mass function similar to (7) has been used by Stephani [7] to study the critical mass of an isotropic star. Utilizing the conservation equation (6) and the mass function (7) we can replace the field equations by the equivalent system

$$\frac{1}{e^\lambda}=1-\frac{2m}{r} \tag{8}$$

$$r(r-2m)\,\nu'=p_r r^3+2m \tag{9}$$

$$2r(r-2m)\left(p_r'+2\sqrt{3}\,\frac{S}{r}\right)=-(p_r r^3+2m)(\mu+p_r) \tag{10}$$

The procedure to obtain an anisotropic solution is now simplified. Specify a form for μ and obtain the mass function (7): equation (8) then gives λ. On assuming a form for p_r we obtain ν from (9). The magnitude of the stress S, and hence $p_\perp$, then follows from Eq. (10).

3. A CLASS OF SOLUTIONS

We suppose that the energy-density is

$$\mu=\frac{6M}{R^3} \tag{11}$$

where R is the stellar radius and $M = m(R)$ gives the stellar mass. By (11) and (7), Eq. (8) gives

$$\frac{1}{e^{\lambda}} = 1 - \frac{2r^2 M}{R^3}$$

which matches the Schwarzschild exterior at $r = R$. On using (7) we find that Eq. (9) reduces to

$$v = \int r p_r \left(1 - \frac{2r^2 M}{R^3}\right)^{-1} dr + \ln A \left(1 - \frac{2r^2 M}{R^3}\right)^{-1/2} \tag{12}$$

where A is a constant of integration.

To continue we need to specify a form for p_r. (Alternatively, we could specify the "degree of anisotropy," $\sqrt{3}\, S = p_r - p_\perp$. Appropriate choices for $p_r - p_\perp$ have been made by Bowers and Liang [1], Cosenza et al. [2], and Herrera and Ponce de Leon [3], leading to classes of static anisotropic spheres with interesting properties.) The form of Eq. (12) suggests the possible choice

$$p_r = C\left(1 - \frac{2r^2 M}{R^3}\right)\left(1 - \frac{r^2}{R^2}\right)^n, \qquad n \geqslant 1 \tag{13}$$

for the radial pressure, where C is the central pressure. This choice is mathematically convenient because the integration in (12) can then be completed. The form (13) is physically reasonable in the sense that $p_r \geqslant 0$ for $0 \leqslant r \leqslant R$ and is a strictly decreasing function from a (finite) maximum value of C at the centre of the sphere. Also we must have $2M/R < 1$ (the same as the Schwarzschild limit) so that p_r vanishes only at the boundary of the sphere and not at any point within the sphere. For isotropic spheres the critical value of $2M/R$ is 8/9 (see, e.g., [7]). The integration in (12) can now be performed using (13). To match v smoothly to the Schwarzschild exterior we must have $e^{v(R)} = 1 - 2M/R$. This gives

$$e^{v} = \left(1 - \frac{2M}{R}\right)^{3/2} \left(1 - \frac{2r^2 M}{R^3}\right)^{-1/2} \exp\left[-\frac{CR^2}{2(n+1)}\left(1 - \frac{r^2}{R^2}\right)^{n+1}\right]$$

Note that the metric function $e^{v(r)}$ and $e^{\lambda(r)}$ are well-behaved in the range $0 \leqslant r \leqslant R$.

The dynamical quantity $S = (p_r - p_\perp)/\sqrt{3}$ now follows from (10). On using (13) we obtain the tangential pressure

$$p_{\perp}=C\left(1-\frac{2r^2M}{R^3}\right)\left(1-\frac{r^2}{R^2}\right)^n$$
$$+\frac{r^2}{2}\left(1-\frac{2r^2M}{R^3}\right)^{-1}\left[\frac{C^2}{2}\left(1-\frac{2r^2M}{R^3}\right)^2\left(1-\frac{r^2}{R^2}\right)^{2n}\right.$$
$$\left.-\frac{2nC}{R^2}\left(1-\frac{2r^2M}{R^3}\right)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}+\frac{6M^2}{R^6}\right] \tag{14}$$

The tangential pressure (14) has the value $C(=p_r)$ at the center of the sphere and the finite (since $2M/R<1$) value $3M^2R^{-4}(1-2M/R)^{-1}$ on the surface of the sphere. Thus the critical value of $2M/R$ corresponds to infinite surface tangential pressure. Note that in the Bowers and Liang solution [1] a critical value of $2M/R$ occurs for infinite central pressure. (Our model does not allow infinite central pressure.) From (13) and (14) we observe that $p_r=p_{\perp}$ at $r=0$. In the Schwarzschild interior solution $p_r=p_{\perp}$ for all values of r in the interior of the sphere. Hence our uniform energy-density solution does not contain the Schwarzschild solution as a special case, unlike the solution of Bowers and Liang [1].

We find values of n for which the tangential pressure $p_{\perp}$ is positive throughout the sphere. In the open interval $(0, R)$ the quantities $(1-r^2/R^2)$ and $(1-2r^2M/R^3)$ are positive. Hence the only negative term in (14) is $-2nCR^{-2}(1-2r^2M/R^3)^2(1-r^2/R^2)^{n-1}$. We observe that $p_{\perp}$ will be positive if

$$\frac{6M^2}{R^6}-\frac{2nC}{R^2}\left(1-\frac{2r^2M}{R^3}\right)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}>0 \tag{15}$$

Since $0<1-2r^2M/R^3<1$ and $0<1-r^2/R^2<1$ in the interval $(0, R)$ we have

$$\frac{6M^2}{R^6}-\frac{2nC}{R^2}\left(1-\frac{2r^2M}{R^3}\right)^2\left(1-\frac{r^2}{R^2}\right)^{n-1}>\frac{6M^2}{R^6}-\frac{2nC}{R^2}$$

Thus the condition (15) will be satisfied if we make the restriction

$$1\leqslant n<\frac{3M^2}{CR^4}$$

The tangential pressure $p_{\perp}$ is positive for this range of values of n. A plot of $p_{\perp}$ is presented in Fig. 1 for $n=1$.

The surface redshift is given by

$$z=\left(1-\frac{2M}{R}\right)^{-1/2}-1$$

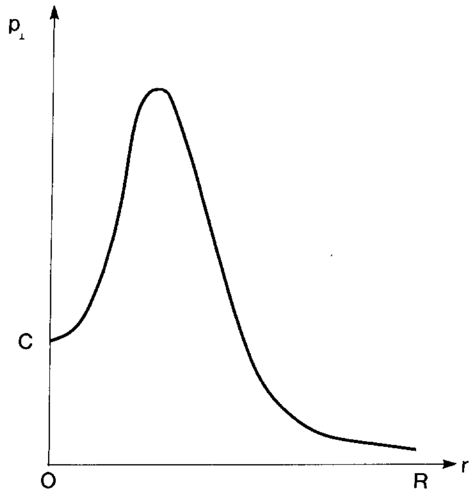

Fig. 1. The tangential pressure for $n = 1$ and M/R small (i.e., small surface redshift).

The critical redshift $z_c = 2$, which is the limiting value for perfect fluid spheres, is attained when $2M/R = 8/9$ (see [7]). For the range of values $8/9 < 2M/R < 1$ the redshift can be greater than z_c. For values of $2M/R$ very close to 1 (i.e., for very large surface $p_\perp$) the surface redshift becomes infinitely large.

4. DISCUSSION

We have found a class of solutions to the Einstein field equations that correspond to anisotropic spheres. Note that it is possible to obtain further classes of analytic solutions for different reasonable choices of the radial pressure (13). For example, the simple linear form

$$p_r = C\left(1 - \frac{r}{R}\right)$$

gives the solution

$$ds^2 = A\left(1 - \frac{r}{D}\right)^{CD^2(D-R)/(2R)}\left(1 + \frac{r}{D}\right)^{-CD^2(D+R)/(2R)}\left(1 - \frac{r^2}{D^2}\right)^{-1/2}$$
$$\times \exp(rCD^2/R)\, dt^2 + \left(1 - \frac{r^2}{D^2}\right)^{-1} dr^2 + r^2(d\theta^2 + r^2 \sin^2\theta\, d\phi^2)$$

where for convenience we have let

$$D^2 = \frac{R^3}{2M}$$

and the constant of integration A becomes

$$A = \left(1 - \frac{R^2}{D^2}\right)^{3/2} \left(1 - \frac{R}{D}\right)^{CD^2(R-D)/(2R)} \left(1 + \frac{R}{D}\right)^{CD^2(R+D)/(2R)} \exp(-CD^2)$$

The circular form for the radial pressure

$$p_r = C \sin\left[\frac{\pi}{2}\left(1 - \frac{r^2}{R^2}\right)\right]$$

also leads to closed form integration involving the elliptic functions **si** and **ci** [8].

Finally we point out that our method may be extended to the case of nonuniform energy-density [9]. This case is more complicated than the uniform energy-density spheres studied in this paper. For a variable matter distribution in the interior of the sphere we need, in addition, to study the behavior of the quantities $dp_r/d\mu$ and $dp_\perp/d\mu$. In fact, the conditions $dp_r/d\mu \leqslant 1$ and $dp_\perp/d\mu \leqslant 1$ have to be satisfied to produce a realistic stellar model.

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