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General Relativistic Electromagnetic Mass Models of Neutral Spherically Symmetric Systems

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The recent work of Grøn [1] concerning charged analogues of Florides' class of solutions is discussed and generalized. The properties of this kind of model are investigated. In particular it is shown that the ratio m/r as well as the acceleration of gravity are maximum inside the body rather than at the boundary. Some exact solutions of the Einstein–Maxwell equations illustrating these properties are presented. The solutions are matched continuously to the exterior Schwarzschild solution and they represent electromagnetic mass models of neutral systems. All physical quantities are finite inside the distributions. The energy density is positive and decreases monotonically from its maximum value at the center to zero at the boundary.

1. INTRODUCTION

In a recent publication [1], Grøn considered charged generalizations of Florides' [2] interior Schwarzschild solution. Specifically, Grøn studied static, spherically symmetric distributions of charged matter under the assumptions that: (a) the matter is a perfect fluid; (b) the component T^r_r of the energy momentum tensor vanishes everywhere, and (c) the distribution has a finite extent and is surrounded by empty space. He showed that these assumptions lead to an interesting class of solutions of the Einstein–Maxwell equations representing electromagnetic mass models of neutral spherically symmetric systems, in the sense that all physical quantities vanish when the charge density vanishes everywhere.

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However, in the work of Grøn there is an unfortunate algebraic error in the computations which leads to incorrect expressions for the mass as well as for the charge distribution [1, Eqs. (15) and (16)]. Consequently this error makes his solutions to the field equations incorrect [1, Eqs. (17)–(26)]. In view of this and of the intrinsic interest in such models [3–5] it is our purpose in this note to discuss in more detail Grøn's class of solutions.

In Sec. 2 we write the (corrected) field equations and show some general properties of the models. In Sec. 3 we show that the metric given by Grøn can only be used as a core solution. We also construct a family of exact solutions which represent perfect fluid spheres with vanishing total charge but with nonvanishing charge density within the source. Furthermore we generalize Grøn's models to include matter with anisotropic "pressures." The conclusions and the summary of the results are given in Sec. 4. Some details of the calculations are shown in the appendix.

2. FIELD EQUATIONS

We choose the line element in the form (with $c=1$)

$$ds^2 = e^{\nu(r)}\,dt^2 - e^{\lambda(r)}\,dr^2 - r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{1}$$

Denoting differentiation with respect to r by a dash and letting $(t, r, \theta, \phi) \equiv (0, 1, 2, 3)$, the Einstein–Maxwell equations read

$$8\pi T_0^0 = 8\pi\rho + E^2 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2} \tag{2}$$

$$8\pi T_1^1 = -8\pi p_r + E^2 = -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2} \tag{3}$$

$$8\pi T_2^2 = -8\pi p_\perp - E^2 = -\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) \tag{4}$$

$$(r^2E)' = 4\pi\sigma e^{\lambda/2} r^2 \tag{5}$$

where ρ is the energy density of matter, σ is the charge density, E is the electric field intensity, and p_r and $p_\perp$ are, respectively, the radial and tangential "pressure."

The condition $T_1^1 = 0$ yields

$$p_r = E^2/8\pi \tag{6}$$

$$\nu' = (e^\lambda - 1)/r \tag{7}$$

Substituting (7) into (4) and using (2) we find

$$p_r + p_\perp = [(e^\lambda - 1)/4](\rho + p_r) \tag{8}$$

We show in the appendix that (8) is the Tolman–Oppenheimer–Volkov equation of hydrostatic equilibrium (generalized to the case of charged anisotropic matter) after (7) has been substituted into it. For perfect fluid, (8) gives

$$\rho = [(9 - e^\lambda)/(e^\lambda - 1)]\, p \tag{9}$$

Using (6) and (9), the mass distribution $m(r)$ for perfect fluid becomes

$$m(r) = 4\pi \int_0^r T_0^0 r^2\, dr = 4 \int_0^r \frac{E^2 r^2}{(e^\lambda - 1)}\, dr \tag{10}$$

This is the equation which shows that the perfect fluid solutions with $T_1^1 = 0$ represent general relativistic electromagnetic mass models.

Next, from (2), (5), (6), and (9), we obtain

$$8\pi(2)^{1/2} r^2 e^{\lambda/2} \sigma = \{[r^2(r - re^{-\lambda})'\,(e^\lambda - 1)]^{1/2}\}' \tag{11}$$

Our equations (8)–(11) correspond to [1, Eqs. (10), (14)–(16), respectively]. Nevertheless, there is a discrepancy between them, due to the incorrect manipulation of [1, Eq. (8)].

Some general features of the solutions with $T_1^1 = 0$ may be seen from the above equations, namely:

a. The regulatity conditions as well as the condition of local flatness demand $p_r(0) = p_\perp(0)$ and $\lambda(0) = 0$ at $r = 0$. Consequently, (6) and (8) show that

$$p_r = p_\perp = E = 0 \qquad \text{at} \quad r = 0 \tag{12}$$

b. At the boundary (say r_o) of the distribution, $p(r_o) = 0$. Therefore, (9) implies that for perfect fluid

$$\rho = 0 \qquad \text{at the boundary} \quad r = r_o \tag{13}$$

c. For perfect fluid the positiveness of ρ in the models under consideration requires that $\varepsilon \equiv M/r_o \equiv$ (total mass/radius) $< 4/9$. In fact, since $p > 0$ and $e^\lambda > 1$ it follows from (9) that $e^\lambda < 9$. Moreover from (2) and (10)

$$e^{-\lambda} = 1 - \frac{2m(r)}{r}.$$

Consequently, $m(r)/r < 4/9$ throughout the distribution. However, from (8) we see that for anisotropic matter ε can be larger than 4/9. It is worthwhile to recall that there are situations where even for arbitrarily large values of anisotropy, M/r_o remains always less than 4/9 [6–7].

d. An interesting feature of the models is that the ratio $[m(r)/r]$ is maximum inside the distribution rather than at the boundary. To see this let us consider the derivative $[m(r)/r]' = \lambda' e^{-\lambda}/2$. The regularity conditions imply that λ must vanish at least like r^2 as $r \to 0$. Then, from (2) and (12), it follows that

$$e^{-\lambda} = 1 - [8\pi\rho(0)/3]r^2 + O(r^3) \qquad \text{as} \quad r \to 0 \tag{14}$$

On the other hand, substracting (3) from (2), we obtain

$$8\pi(\rho + p_r) = (e^{-\lambda}/r)(\nu' + \lambda') \tag{15}$$

According to (13) for perfect fluid (as well as for the anisotropic solutions we show in Sec. 3), $\rho = p_r = 0$ at the boundary. Consequently

$$\lambda'(r_o) = -\nu'(r_o) = -(2M/r_o^2)[1 - (2M/r_o)]^{-1} \tag{16}$$

Now, (14) and (16) indicate that $(m/r)'$ changes sign inside the body. Showing that m/r is maximum (since $m > 0$) somewhere inside the body and not at the boundary.

e. Another interesting feature refers to the acceleration of gravity. In fact, the acceleration of gravity g can be defined as

$$g = -\tfrac{1}{2}e^{(\nu - \lambda)/2}\nu' = -(M_G/r^2) \tag{17}$$

where M_G is the Tolman–Whittaker active gravitational mass of the system (see, e.g., Ref. 8). Now, using (7), (14), (16), and (17), it is easy to verify that $g'(0) = -[8\pi\rho(0)/3]e^{\nu(0)/2}$ and $g'(r_o) = 2M/r_o^3$ implying that g changes sign inside the body. This shows that the acceleration of gravity (or more exactly $|g|$) attains its maximum value (for all M/r_o) inside the source rather than at the boundary.

3. SOLUTIONS TO THE FIELD EQUATIONS

3.1. Grøn's Metric

We now proceed to show that choosing appropriately the charge distribution one can obtain a metric similar to that given by Grøn. However, we will see that such a metric represents a perfect fluid distribution without boundary. We assume the charge distribution as follows

$$\sigma = \frac{e^{-\lambda/2}}{4r^2}\left(\frac{p_o}{\pi}\right)^{1/2} [r^2(e^{\lambda} - 1)^{1/2}]' \tag{18}$$

where p_o is a constant. Using this equation in Eq. (11), we find e^{λ}; then, from (7), we obtain e^{ν}. The result is

$$e^{-\lambda} = 1 - (r^2/R^2), \qquad r < R \equiv (3/8\pi p_o)^{1/2} \tag{19}$$

$$e^{-\nu} = \text{const.} \times [1 - (r^2/R^2)]^{1/2} \tag{20}$$

These metric functions have the same form as those given by [1, Eqs. (18) and (19)]. However, the correct expression for the density and pressure (for $T_1^1 = 0$) are given by

$$8\pi p = E^2 = (8\pi^2 p_o^2/3)\, r^2 e^{\lambda} = 3(e^{\lambda} - 1)/8R^2 \tag{21}$$

$$\rho = p_o[1 - (9r^2/8R^2)]\, e^{\lambda}, \qquad r < (\tfrac{8}{9})^{1/2} R \tag{22}$$

We see that neither the pressure nor the electric field vanish for any value of $r < (8/9)^{1/2} R$. Consequently the distribution given by Eqs. (19)–(22) cannot be matched continuously to the Schwarzschild exterior metric. Therefore, (19)–(12) is a core solution and should be joined to an envelope over which the pressure drops to zero. There is also another possibility consisting in joining the distribution to the Schwarzschild exterior metric across a singular hypersurface of order one [9] having surface concentration of charge.

3.2. Perfect Fluid Sources with Boundary

Equations (10) and (21) suggest that a family of solutions may be generated by assuming the electric field as follows

$$E^2 = k^2(e^{\lambda} - 1)[1 - (r/r_o)^a]^b \tag{23}$$

where k^2, a, and b are positive constants and r_o defines the boundary of the body. This assumption assures the fulfillment of the conditions at the center and at the boundary. Now, given a and b, we can easily integrate (10) to obtain $m(r)$ and e^{λ}. Then from (2) and (6) we get the pressure and density. Finally, the metric function e^{ν} is obtained by straightforward integration of (7). We have studied the solutions for various values of a and b and found that generally e^{ν} is given in terms of elliptic functions. We show here the particular solution which arises from (23) by putting $a = 2$, $b = 1$. In this case e^{ν} becomes expressible entirely in terms of elementary functions.

Substituting (23) with $a = 2$, $b = 1$, in (10) we find

$$m(r) = 4k^2 \left[\frac{r^3}{3} - \frac{r^5}{5r_o^2}\right] \tag{24}$$

From the boundary condition $m(r_o) = M$ we get

$$8k^2 r_o^2 = 15\varepsilon \equiv 15(M/r_o) \tag{25}$$

Using (24) and (25) and integrating (7) we obtain the final form of the metric functions as follows

$$e^{-\lambda} = 1 - 5\varepsilon v^2 + 3\varepsilon v^4 \tag{26}$$

$$e^{\nu} = \frac{(1-2\varepsilon)^{5/4}}{(1 - 5\varepsilon v^2 + 3\varepsilon v^4)^{1/4}} \exp\left\{\frac{5\beta}{2}\left[\tan^{-1}(6\beta v^2 - 5\beta) - \tan^{-1}\beta\right]\right\} \tag{27}$$

where

$$v \equiv \frac{r}{r_0}, \qquad \beta = \left(\frac{\varepsilon}{12 - 25\varepsilon}\right)^{1/2}, \qquad \varepsilon \equiv \frac{M}{r_o} < 0.48$$

The pressure and density are given by

$$8\pi p_r = E^2 = \frac{15\varepsilon^2 v^2}{8r_o^2} \frac{(1-v^2)(5-3v^2)}{(1 - 5\varepsilon v^2 + 3\varepsilon v^4)} \tag{28}$$

$$8\pi\rho = \frac{15\varepsilon}{8r_o^2} \frac{(1-v^2)(8 - 45\varepsilon v^2 + 27\varepsilon v^4)}{(1 - 5\varepsilon v^2 + 3\varepsilon v^4)} \tag{29}$$

In this solution the metric functions make sense for $\varepsilon < 0.48$. However, as we see from (29), the positiveness of ρ puts a more stringent limit on ε, viz. $\varepsilon < (32/75) \approx 0.426$. This limit is, of course, due to the equation of state, $p_r = p_\perp$, we have assumed.

3.3. Generalization to Anisotropic Sources

We now extend the above solution to obtain more compact distributions, with this aim we assume the following relation between the stresses

$$p_\perp = n p_r, \qquad n = \text{const.} \tag{30}$$

which, for $n = 1$, includes the perfect fluid case.

Substituting (30) in (8) we find

$$\rho = \frac{(4n + 5 - e^{\lambda})}{(e^{\lambda} - 1)} p_r \tag{31}$$

and the mass function is

$$m(r) = 2(n+1)\int_0^r \frac{r^2E^2}{(e^\lambda - 1)}\,dr = 2(n+1)k^2\left[\frac{r^3}{3} - \frac{r^5}{5r_o^2}\right] \tag{32}$$

Consequently, the metric functions are given by (26) and (27). However, in the present case

$$\varepsilon = 4(n+1)\,k^2r_o^2/15 \tag{33}$$

$$8\pi p_r = E^2 = \frac{15\varepsilon^2 v^2}{4(n+1)r_o^2}\frac{(1-v^2)(5-3v^2)}{(1-5\varepsilon v^2 + 3\varepsilon v^4)} \tag{34}$$

$$8\pi\rho = \frac{15\varepsilon}{4(n+1)r_o^2}\frac{(1-v^2)[4n+4-\varepsilon(4n+5)(5-3v^2)v^2]}{(1-5\varepsilon v^2+3\varepsilon v^4)} \tag{35}$$

Now the positiveness of E^2 and ρ demand $n > -1$ and

$$\varepsilon < 0.48\,\frac{(4n+4)}{(4n+5)} \tag{36}$$

Note that for $n = 0$ the tangential pressure $p_\perp$ vanishes everywhere. Moreover, for $n > 1$ $(n < 1)$ one can obtained more (less) compact distributions than that of perfect fluid.

4. SUMMARY AND CONCLUDING REMARKS

The aim of this work was to study the class of spherically symmetric and static solutions of the Einstein–Maxwell equations defined by the condition $T_1^1 = 0$. We have shown that, within this class, it is possible to construct models of gaseous spheres with isotropic and anisotropic "pressures" whose Schwarzschild mass seen at infinity is completely of electromagnetic origin. The explicit examples we have given satisfy the usual regularity requirements and are continuously joined to the external Schwarzschild solution at $r = r_o$ $(v = 1)$. The density ρ is positive and decreases monotonically from its maximum value at the center $\rho = (15\varepsilon/8\pi r_o^2)$ to zero at the boundary. The maximum mass in the sphere depends on the degree of anisotropy and for $n \to \infty$, $M_{\max} \to 0.48\,r_o$. Consequently, the maximum value for the redshift at the boundary is $Z(r_o) = 4$. The central redshift to infinity $Z(0) = e^{-\nu(0)/2} - 1$ is very high even for small values of M/r_o. For example, from (27) we find: for $M = 0.01\,r_o$, $Z(0) = 0.45$; for $M = 0.1\,r_o$, $Z(0) = 78.32$; for $M = 0.3\,r_o$, $Z(0) \cong 4 \times 10^9$. As far we are aware, these values for the central redshift are (for the above-mentioned values of M/r_o)

much greater than those obtained from other models of general relativistic spheres with finite pressure and density (see, e.g., Refs. 10–11).

From (26) we find

$$\frac{m(r)}{r}=\frac{M}{r_o}\left(\frac{5v^2}{2}-\frac{3v^4}{2}\right) \tag{37}$$

which shows that $[m(r)/r]=(75M/72r_o)$ at $\bar{r}=(5/6)^{1/2}r_o$. Notice that for perfect fluid $[m(\bar{r})/\bar{r}]\to 4/9$ as $(M/r_o)\to 32/75$ and that for anisotropic fluid $[m(\bar{r})/\bar{r}]\to 1/2$ as $(M/r_o)\to 0.48$. According to Bondi [12] the maximum value of the ratio (m/r) for perfect fluid with nonincreasing outward density is 4/9. For anisotropic matter, however, (m/r) can (for larger anisotropies) be as near as one wants to 1/2 as has been shown [13]. As we see in our models, these maximum values are attained inside the body and not at the boundary.

From (17), (26)–(27) it can be verified that for every (M/r_0) there is a value of r (say $\hat{r}$) at which the acceleration of gravity g is greater (in modulus) that at the boundary. For example, for $(M/r_o)=0.01$ we found $\hat{r}=0.85\,r_o$ and $g(\hat{r})=1.09\,g(r_o)$. This is an interesting feature since in other nonsingular solutions (e.g., the Schwarzschild interior solution and Adler's solution [14]) $|g|$ is always monotonically increasing outward.

To complete our discussion we observe that the total charge of the body is zero due to the fact that the charge density σ changes sign inside the source. We omit the explicit form because it is rather complicated. In the explicit examples we constructed here, σ diverges at $r\to r_o$ like $\sigma\sim 1/(r_o-r)^{1/2}$. We recall, however, that this singularity is not important from a physical point of view since the charge in every three-dimensional volume element remains finite. Moreover, examples with finite everywhere charge density may be obtained from (23) by choosing $b\geqslant 2$. The properties of such models are essentially the same as discussed above; however, e^{ν} is given in terms of elliptic functions.

Finally, we point out some interesting aspects of the static equilibrium in our models.

Let r_σ be the position where the charge density changes sign, i.e., $\sigma(r_\sigma)=0$. Then at $r<r_\sigma$ the electrical force on the matter is directed outward and at $r>r_\sigma$ it is directed inward. On the other hand, the fact that the radial "pressure" p_r vanishes at the center and at the boundary implies that there is a value of r (say r_p), inside the body, at which p_r is maximum (since $p_r>0$). Moreover, the slope dp_r/dr is positive for $r<r_p$ and negative for $r>r_p$. This means that the force associated with the pressure gradient is directed inward in the region $r<r_p$ and outward in $r>r_p$.

It is easy to show that if the body is in equilibrium then necessarily $r_\sigma > r_p$. In fact, because $(dp_r/dr) = 0$ at $r = r_p$, from (6) and (A-5) we obtain

$$q/\sigma = 2\pi r^3 e^{\lambda/2} \qquad \text{at} \quad r = r_p \tag{38}$$

This equation shows that q and σ must have the same sign at $r = r_p$. This is so only in the region $r < r_\sigma$; consequently, $r_p < r_\sigma$. In the explicit examples we discussed here, r_p and r_σ depend on the values of ε and n, e.g., for $\varepsilon = 0.1$ and $n = 1$ we found $r_\sigma \cong 0.83 r_o$, $r_p \cong 0.7 r_o$; for $\varepsilon = 0.3$ and $n = 1$, $r_\sigma \cong 0.85 r_o$, $r_p \cong 0.75 r_o$; for $\varepsilon = 0.47$ and $n = 12$, $r_\sigma \cong 0.9 r_o$, $r_p \cong 0.89 r_o$. In all cases $r_\sigma > r_p$ as expected.

Thus in order to maintain the static equilibrium of our models, the gravitational attractive force, the electrical force, and the force due to the pressure gradient arrange themselves in a different form in each of the regions $r < r_p$, $r_p < r < r_\sigma$, and $r > r_\sigma$. In the central region $r < r_p$ the pressure acts in conjunction with the gravitation to counteract the electrical repulsion and maintain the equilibrium (we refer here to the perfect fluid case). In the region $r_p < r < r_\sigma$ the gravitational attraction is balanced by the negative pressure gradient and by the Coulomb repulsion. In the outer region, where the electrical force is directed inward $(r > r_\sigma)$, it is the (negative) pressure gradient that keeps the equilibrium with attractive gravitational and electrical forces. At r_p and r_σ the gravitational attraction is balanced only by the Coulomb repulsion and by the pressure gradient, respectively.

The above discussion is easily generalized to the case of anisotropic matter if we take into account that is this case there is an additional force that acts on the matter pushing outward or inward depending on whether $p_\perp > p_r$ or $p_\perp < p_r$, respectively (see A.6).

APPENDIX: THE EQUATION OF HYDROSTATIC EQUILIBRIUM

In this appendix we derive the equation of hydrostatic equilibrium for an anisotropic spherically symmetric charged fluid. Moreover, we show that in the case where $T_1^1 = 0$ this equation reduces to (8).

Using the field equations (2)–(4) or, from the conservation equations $T^\mu_{\nu;\mu} = 0$, we find

$$\frac{dT_1^1}{dr} = \frac{\nu'}{2}(T_0 - T_1^1) + \frac{2}{r}(T_2^2 - T_1^1) \tag{A1}$$

where ν' is given by (3) as

$$\frac{\nu'}{2}=\frac{m(r)-4\pi r^3 T_1^1}{r(r-2m)} \tag{A2}$$

Specializing the choice of the energy-momentum tensor to the case of charged anisotropic matter we obtain

$$\frac{d}{dr}\left(p_r-\frac{E^2}{8\pi}\right)=-\frac{\nu'}{2}(\rho+p_r)+\frac{2}{r}(p_\perp-p_r)+\frac{E^2}{2\pi r} \tag{A3}$$

and

$$\nu'=\frac{2m(r)+8\pi r^3 p_r-r^3E^2}{r(r-2m)} \tag{A4}$$

From (5) the charge $q(r)$ inside a sphere of "radius" r is defined as follows

$$E(r)=\frac{4\pi}{r^2}\int_0^r \sigma e^{\lambda/2} r^2\,dr \equiv \frac{q(r)}{r^2} \tag{A5}$$

In terms of $q(r)$ and σ, (A-3) becomes

$$(\rho+p_r)\frac{\nu'}{2}=-\frac{dp_r}{dr}+\frac{q\sigma}{r^2}e^{\lambda/2}+\frac{2}{r}(p_\perp-p_r) \tag{A6}$$

In the Newtonian limit this equation gives

$$\frac{m\rho}{r^2}=-\frac{dp_r}{dr}+\frac{q\sigma}{r^2}+\frac{2}{r}(p_\perp-p_r) \tag{A7}$$

Equation (A-3) (or, equivalently, A-6) is the generalization of the Tolman–Oppenheimer–Volkov equation of hydrostatic equilibrium to the case of charged matter with anisotropic pressures.

It is noted that in (A-6) and (A-7) there is an additional "force," viz., $2(p_\perp-p_r)/r$. This force is directed outward when $p_\perp>p_r$ and inward when $p_\perp<p_r$. It is precisely the existence of this repulsive force (in the case $p_\perp>p_r$) that allows the construction of more compact distributions when using anisotropic fluid than when using isotropic fluid.

In the case under study, putting $E^2=8\pi p_r$ ($T_1^1=0$) into (A-4), we obtain

$$\nu'=\frac{2m(r)}{r(r-2m)}=\frac{e^\lambda-1}{r} \tag{A8}$$

Substituting this expression into (A-3) we find

$$p_{\perp} + p_r = [(e^{\lambda} - 1)/4](\rho + p_r) \tag{A9}$$

These last two equations are just (7) and (8). Consequently we have shown that (7) and (8) are equivalent to the (generalized) equation of hydrostatic equilibrium of Tolman–Oppenheimer–Volkov.

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