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Power Law Singularities in Orthogonal Spatially Homogeneous Cosmologies

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Abstract

We investigate the structure of the so-called power asymptote singularities in orthogonal spatially homogeneous solutions of the Einstein field equations with perfect fluid source. We first give a systematic survey of the different possible power asymptotes, some of which are well known, and some new, and characterize them in a coordinate-independent manner. The known orthogonal spatially homogeneous exact solutions with perfect fluid source are then classified on the basis of which power asymptote they admit. In many cases this leads to simpler forms of the known solutions, and suggests methods for deriving new solutions.

§(1): *Introduction*

It is well known [1] that all the orthogonal spatially homogeneous (SH) cosmologies with zero cosmological constant, perfect fluid source, and "reasonable" equation of state, which are expanding at some instant, originate at a "big-bang," i.e., a singularity at which the energy density of the fluid diverges. Two distinct types of behavior near the singularity have been identified, using power series expansions (see [2] for a recent survey), and the qualitative theory of ordinary differential equations (see, for example [3]). The simplest type of behavior is that of the power law singularity or power asymptote [4]. In simple terms, the leading time dependence of the metric tensor of the fluid near the singularity $t = 0$ is in powers of t. The different power asymptotes are characterized by various restrictions on the exponents of these powers of t. This type of singularity can occur in all Bianchi types, in the usual classification of orthogonal spatially homogeneous cosmologies [5]. In Bianchi types VIII and IX, however, it has been shown that the solution can also display an indefinitely oscillatory behavior near the singularity (see for example [6-8]).

This paper deals with the power asymptotes. Our interest in this type of singularity stems from the fact that they may be of interest in connection with the early universe [9]. Our first aim is to review the known [4] spatially homogeneous power asymptotes. We also give several new types. As mentioned earlier, power asymptotes have been discovered on the one hand by qualitative analysis of the field equations and on the other hand, by simply substituting a power series expansion for the field variables (e.g., metric components) into the field equations [4]. In our work, we have essentially taken the second approach, but have found it convenient to use the orthonormal tetrad form of the field equations as given by Ellis and MacCallum [5]. This avoids the introduction of local coordinates, which are irrelevant as regards the structure of the singularities. We also do not restrict our considerations to an equation of state of the form $p = (\gamma - 1)\mu$; it is sufficient to assume that p/μ has a (constant) limit as the singularity is approached. To ensure compatibility with the γ-law equation of state, however, we write

$$\lim_{t \to 0^+} (p/\mu) = \gamma - 1 \tag{1}$$

In addition, we characterize the different power asymptotes in a coordinate-independent manner using the expansion tensor of the fluid and the Ricci and Weyl tensors of the space-time. Our results thus augment and clarify reference [4].

Before continuing, we mention the restrictions that are imposed on the constant γ in equation (1). The structure of the field equations leads naturally to the restriction

$$2/3 < \gamma < 2 \tag{2}$$

which corresponds to the inequalities

$$\mu + 3p > 0 \quad \text{and} \quad \mu - p > 0$$

holding as $t \to 0^+$. The case $\gamma = 2$, which is possibly of interest from a physical point of view, can also occur, but requires separate treatment, and is not considered in this paper. Of course the requirement of nonnegative pressure as $t \to 0^+$ leads to the stronger restriction

$$1 \leqslant \gamma < 2 \tag{3}$$

We do not give any details of the calculations, which are elementary and somewhat tedious. The method is simply to assume that the diagonal tetrad components of the expansion tensor are of the form

$$\theta_\alpha = \frac{q_\alpha}{t} \, [1 + O(t^r)] \tag{4}$$

as the singularity $t = 0$ is approached, where t is clock time along the fluid flow lines, the q_α are constants, and the symbol $O(t^r)$ denotes higher-order terms which tend to zero as a power of t. The field equations, as given in [5] Appendix I, can then be integrated to give the asymptotic t dependence of the other variables. The field equations also lead to consistency conditions, which determine the form of the different power asymptotes. We have been unable to complete the analysis of the consistency conditions in one special case, namely, the subclass Bb(ii), in the terminology of [5]. It is thus conceivable that there are additional power asymptotes.

It should be noted that the above power law analysis enables one to deduce properties of the different power asymptotes, but it does not prove existence of solutions with this type of singularity. However, in all cases except one, the existence of the power asymptote is established by the existence of an appropriate exact perfect fluid solution. In addition, in some cases existence is confirmed by the more rigorous qualitative analysis.

In all the known exact orthogonal SH cosmologies with perfect fluid matter content, the singularities are of the power asymptote type. Our second aim in this paper is to present these known solutions (mostly with a γ-law equation of state) in a unified manner, and to use them to illustrate and establish the existence of the various power asymptote singularities.

The plan of this paper is as follows. In Section 2 we briefly introduce the concepts which are needed to describe and characterize the power asymptotes, and give a summary of the properties of the various power asymptotes. In Section 3 we present and classify the known exact orthogonal Bianchi cosmologies with perfect fluid source, according to which power asymptote they admit. In Section 4 we summarize some aspects of our results in a table, and discuss some related issues.

As a summary of results, this paper is relatively self-contained; the reader is occasionally referred to [10–12] for additional background. In order to verify the calculations of Section 2, however, familiarity with the orthonormal tetrad formalism of [5] is required. The exact solutions of Section 3 have all been checked using the symbolic computing language CAMAL [13, 14]. We use geometrized units with $c = 1$, $8\pi G = 1$, and our sign conventions as regards metric signature and curvature tensors are as in [10–12].

§(2): *The Power Asymptotes*

Relative to its eigenframe the rate of expansion tensor θ_{ab} has the form

$$\theta_{ab} = \operatorname{diag}(0, \tilde{\theta}_1, \tilde{\theta}_2, \tilde{\theta}_3)$$

where the $\tilde{\theta}_\alpha$, $\alpha = 1, 2, 3$ are the eigenvalues that are associated with the spacelike eigenvectors. The length scales l_α in the spatial eigendirections are defined, up to constant scale factors, by

$$\dot{l}_\alpha / l_\alpha = \tilde{\theta}_\alpha$$

where the dot denotes differentiation along the fluid flow lines.

In the solutions with a power asymptote singularity, the $\tilde{\theta}_\alpha$ satisfy

$$\tilde{\theta}_\alpha = \frac{p_\alpha}{t}\,[1 + O(t^r)]\,, \qquad \alpha = 1, 2, 3 \tag{5}$$

as the singularity $t = 0$ is approached, where t is clock time along the flow lines, and the p_α are constants which characterize the power asymptote. The symbol $O(t^r)$ denotes higher-order terms which tend to zero as a power of t. Equation (5) implies that the length scales satisfy

$$\lim_{t\to 0^+} \frac{l_\alpha}{t^{p_\alpha}} = b_\alpha$$

where the b_α are nonzero constants which can be scaled to equal 1.

In order to characterize the different power asymptotes we consider the dimensionless ratios μ/θ^2, σ^2/θ^2, and R^*/θ^2, where μ, θ, and σ are the matter density, expansion scalar, and shear scalar of the fluid [10], and R^* is the curvature scalar of the hypersurfaces orthogonal to the fluid flow [11]. Each of these ratios has a finite limit as $t \to 0^+$, and so we define

$$\beta_m = \lim_{t\to 0^+} \frac{3\mu}{\theta^2}, \qquad \beta_s = \lim_{t\to 0^+} \frac{3\sigma^2}{\theta^2}, \qquad \beta_c = \lim_{t\to 0^+} \frac{-3R^*}{2\theta^2}$$

If $\beta_m \neq 0$, the matter is said to be dynamically significant near the singularity. Similarly if $\beta_s \neq 0$ or $\beta_c \neq 0$, the shear or spatial curvature, respectively, are dynamically significant near the singularity.

These constants are not independent, however. On account of the following first integral of the Einstein field equations with irrotational perfect fluid source [11]:

$$\tfrac{1}{3}\theta^2 = \sigma^2 + \mu - \tfrac{1}{2}R^*$$

these constants satisfy

$$\beta_m + \beta_s + \beta_c = 1 \tag{6}$$

The constant β_s is nonnegative by definition, and β_m is nonnegative since μ is assumed to be positive. Finally, the analysis of the field equations in the various cases shows that β_c is also nonnegative. It thus follows from equation (6) that each constant is bounded above by 1. The constants β_m and β_s can actually attain this upper bound, in which case the other two constants are zero. If $\beta_m = 1$, the matter is said to be dynamically dominant, while if $\beta_s = 1$, the shear is said to be dynamically dominant. The singularities with $\beta_c = 0$, i.e., spatial curvature dynamically negligible, are precisely the singularities which are called "velocity dominated" by Eardley, Liang, and Sachs [15, 16]. One can infer from their work that if $\beta_c = 0$ and $\beta_m \neq 0$ then $\beta_s = 0$. Furthermore, it follows from the field equations that $\beta_s = 0$ implies $\beta_c = 0$. Thus, based on the dynamical significance of the matter, shear, and spatial curvature, there are four possible types of singularity. The corresponding restrictions on the exponents p_α are given below:

Case 1: $\beta_m \neq 0, \beta_s = \beta_c = 0$

$$2/\gamma = p_1 + p_2 + p_3 > p_1^2 + p_2^2 + p_3^2 = 4/(3\gamma^2) \tag{7a}$$

Case 2: $\beta_m \neq 0, \beta_s \neq 0, \beta_c \neq 0$

$$2/\gamma = p_1 + p_2 + p_3 > p_1^2 + p_2^2 + p_3^2 > 4/(3\gamma^2) \tag{7b}$$

Case 3: $\beta_m = 0, \beta_s \neq 0, \beta_c \neq 0$

$$2/\gamma > p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 > 1 \tag{7c}$$

Case 4: $\beta_m = 0, \beta_s \neq 0, \beta_c = 0$

$$2/\gamma > p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \tag{7d}$$

A number of subcases arise, corresponding to different explicit expressions for the p_α. These are given below together with the expressions for the parameters β_m, β_s, and β_c. The values of the p_α also determine the singularity type, i.e., point, cigar, barrel, or pancake, in the usual terminology (see for example [11], p. 131).

It is also of interest to consider the limiting behavior of the Weyl tensor as the singularity is approached. The Weyl tensor itself diverges, but the limit of the ratio of the Weyl tensor to θ^2 is finite. We thus define

$$\tilde{E}_{\alpha\beta} = \lim_{t\to 0^+} E_{\alpha\beta}/\theta^2, \qquad \tilde{H}_{\alpha\beta} = \lim_{t\to 0^+} H_{\alpha\beta}/\theta^2$$

where $E_{\alpha\beta}$ and $H_{\alpha\beta}$, $\alpha, \beta = 1, 2, 3$, are the components of the electric and magnetic parts of the Weyl tensor relative to the fluid 4-velocity [12], in an appropriate orthonormal frame (e.g., the eigenframe of the expansion tensor). Then by calculating the complex matrix

$$Z_{\alpha\beta} = \tilde{E}_{\alpha\beta} + i\tilde{H}_{\alpha\beta}$$

we can determine the Petrov type in the usual way [12]. *The Petrov types given below thus relate to this limit of the Weyl tensor, and not to the Weyl tensor of the space-time.*

Case 1 (Lifshitz-Khalatnikov):

$$\begin{gathered} p_1 = p_2 = p_3 = 2/(3\gamma) \\ \beta_m = 1, \qquad \beta_s = 0, \qquad \beta_c = 0 \end{gathered} \tag{8}$$

Singularity type: isotropic point
Petrov type: 0

This type of singularity is also referred to as "isotropic" or "Friedmannlike" [17-19].

Case 2a (Novikov):

$$\begin{gathered} p_1 = (2-\gamma)/(2\gamma), \quad p_2 = p_3 = (2+\gamma)/(4\gamma) \\ \beta_m = 3(6-\gamma)/64, \quad \beta_s = (3\gamma-2)^2/64, \quad \beta_c = 3(3\gamma-2)(2-\gamma)/64 \end{gathered} \tag{9}$$

Singularity type: axisymmetric point
Petrov type: D

Case 2b (Ellis-MacCallum):

$$\begin{gathered} p_1 = 1, \quad p_2 = p_3 = (2-\gamma)/(2\gamma) \\ \beta_m = 3(2-\gamma)/4, \quad \beta_s = (3\gamma-2)^2/16, \quad \beta_c = 3(3\gamma-2)(2-\gamma)/16 \end{gathered} \tag{10}$$

Singularity type: axisymmetric point
Petrov type: I

Case 2c:

$$\begin{gathered} p_1 = 1, \quad p_2 = (2-\gamma+rs)/(2\gamma), \quad p_3 = (2-\gamma-rs)/(2\gamma) \\ \beta_m = 3(1-r^2)(2-\gamma)/4, \quad \beta_s = (3\gamma-2)(1-\beta_m)/4, \\ \beta_c = 3(2-\gamma)(1-\beta_m)/4 \end{gathered} \tag{11}$$

where

$$s^2 = (3\gamma-2)(2-\gamma)$$

and r is an arbitrary parameter which satisfies

$$0 < r < 1$$

Singularity type:

$$\begin{gathered} \text{point} \Longleftrightarrow r^2 < (2-\gamma)/(3\gamma-2) \\ \text{barrel} \Longleftrightarrow r^2 = (2-\gamma)/(3\gamma-2) \\ \text{cigar} \Longleftrightarrow r^2 > (2-\gamma)/(3\gamma-2) \end{gathered}$$

Petrov type: I

Case 2d:

$$\begin{gathered} p_1 = \tfrac{3}{5}(1+r), \quad p_2 = \tfrac{3}{5}(1-r), \quad p_3 = \tfrac{3}{5} \\ \beta_m = 1-r^2, \quad \beta_s = \tfrac{1}{3}r^2, \quad \beta_c = \tfrac{2}{3}r^2 \end{gathered} \tag{12}$$

where r is an arbitrary parameter which satisfies

$$2/3 < r < 1$$

Singularity type: point
Petrov type: I
The equation of state parameter is

$$\gamma = 10/9$$

on account of (7b) and (12).

Case 3a:

$$\begin{aligned} &p_1 = 1, \quad p_2 = u + [u(1-u)]^{1/2}, \quad p_3 = u - [u(1-u)]^{1/2} \\ &\beta_m = 0, \quad \beta_s = (1-u)/(1+2u), \quad \beta_c = 3u/(1+2u) \end{aligned} \tag{13}$$

where u is an arbitrary parameter which satisfies

$$0 < u < 1$$

Singularity type:

$$\begin{aligned} \text{point} &\Longleftrightarrow \tfrac{1}{2} < u < 1 \\ \text{barrel} &\Longleftrightarrow u = \tfrac{1}{2} \\ \text{cigar} &\Longleftrightarrow 0 < u < \tfrac{1}{2} \end{aligned}$$

Petrov type: N
The equation of state parameter satisfies

$$\gamma < \frac{2}{1+2u}$$

on account of (7c) and (13). A point or barrel singularity thus entails negative pressure, since then $\gamma < 1$.

Case 3b:

$$\begin{aligned} &p_1 = 6/5, \quad p_2 = 0, \quad p_1 = 3/5 \\ &\beta_m = 0, \quad \beta_s = 1/3, \quad \beta_c = 2/3 \end{aligned} \tag{14}$$

Singularity type: barrel
Petrov type: III
The equation of state parameter satisfies

$$\gamma < 10/9$$

on account of (7c) and (14).

Case 4a (Taub):

$$p_1 = 1, \quad p_2 = p_3 = 0$$
$$\beta_m = 0, \quad \beta_s = 1, \quad \beta_c = 0 \tag{15}$$

Singularity type: pancake
Petrov type: 0

Case 4b (Kasner):

$$p_1 = (1+u)/(1+u+u^2), \quad p_2 = -u/(1+u+u^2),$$
$$p_3 = u(1+u)/(1+u+u^2) \tag{16}$$
$$\beta_m = 0, \quad \beta_s = 1, \quad \beta_c = 0$$

where u is an arbitrary parameter which satisfies

$$0 < u \leqslant 1$$

Singularity type: cigar
Petrov type: $D(\Longleftrightarrow u = 1)$, or I

We conclude this section with some comments on the origins of the power asymptotes. The Kasner asymptote 4b, and its special case the Taub asymptote 4a, are the best known of the power asymptotes. They were first considered as a single case by Lifshitz and Khalatnikov [20]. Their names are taken from the well-known Kasner vacuum solution and apparently from the Taub vacuum type IX solution. The LK asymptote (case 1) was also discussed in some detail in this paper (hence the name). Case 3a was also briefly mentioned here, as an example of a non-Kasner vacuum power asymptote (see also [2], pp. 622 and 655). Case 2a was discovered by Novikov (see [3]), while case 2b was apparently first discussed in [4]. The name refers to an exact solution of Ellis and MacCallum [5] (see Section 3). The remaining cases are apparently new. The survey [4] covers cases 1, 2a, 2b, 4a, and 4b. We find ourselves in disagreement with [4] on one point. We find that at a Taub asymptote, the spatial curvature is not dynamically significant ($\beta_c = 0$), while the opposite is stated in [4]. On the other hand we distinguish between the Kasner and Taub asymptotes using the Weyl tensor, which was not considered in [4].

§(3): *Exact Solutions with Power Asymptote Singularities*

For each power asymptote of Section 2, we give an exact solution in which the quantities $3\mu/\theta^2$, σ^2/θ^2, and $-3R^*/(2\theta^2)$ are constants [21]. These solutions do not provide realistic models of the evolution of the universe over a large time scale, since the restriction $\sigma^2/\theta^2 = \text{const}$ does not permit them to isotropize.

However, they do serve as prototypes for the different power asymptotes in the sense that they give a simple description of the time evolution near the singularity, and thus may be of interest in studies of the early universe. We will refer to these solutions as *exact power law solutions.* We also give examples of exact solutions with power asymptote singularities which are not exact power law solutions. Indeed the classification of power asymptotes of Section 2 enables one to give a systematic survey of the known orthogonal SH cosmologies with perfect fluid matter content.

In each of the examples in this section, the coordinates are comoving, so that in each case the fluid 4-velocity is proportional to the vector field $\partial/\partial t$.

Case 1 (Lifshitz-Khalatnikov): The exact power law solution in this case is the well-known FRW solution with flat spatial geometry, and equation of state $p = (\gamma - 1)\mu$:

$$\begin{aligned} ds^2 &= -dt^2 + t^{4/(3\gamma)}(dx^2 + dy^2 + dz^2) \\ \mu &= 4/(3\gamma^2 t^2) \end{aligned} \tag{17}$$

This solution is of Bianchi type I and type VII_0. An example which is not an exact FRW solution is provided by a solution of Kantowski-Sachs type, with equation of state $p = \frac{1}{3}\mu$, and the related model of Bianchi type III (i.e., type VI_h, with $h = -1$), first given by Kantowski [22]. These solutions can be written jointly in the form

$$\begin{aligned} ds^2 &= -A\, dt^2 + t[A^{-1}\, dx^2 + A^2 b^{-2}(dy^2 + f^2\, dz^2)] \\ \mu &= 3/(4t^2A^2), \qquad p = \mu/3 \end{aligned}$$

where

$$A = 1 - 4\epsilon b^2 t/9$$

and

$$f(y) = \begin{cases} \sin y & \text{if } \epsilon = +1 \text{ (Kantowski-Sachs)} \\ \sinh y & \text{if } \epsilon = -1 \text{ (Bianchi type III)} \end{cases}$$

In this form, their relationship to the prototype (17) is obvious, since $\gamma = 4/3$, and $A \to 1$ as $t \to 0$, so that in this limit, t approximates clock time along the fluid flow lines.

The only other SH exact solutions with this type of singularity, of which we are aware, are discussed in detail in [18]. These solutions, first given by Collins [23], are of Bianchi type VI_h, and specialize to the above Bianchi type III solution when $\gamma = 4/3$.

Case 2a (Novikov): The exact power law solution is a solution of Bianchi type II, given by Collins and Stewart [24]:

$$\begin{aligned} ds^2 &= -dt^2 + t^{2p_1}[dx + (k/2\gamma)z\,dy]^2 + t^{2p_2}\,dy^2 + t^{2p_3}\,dz^2 \\ \mu &= (6-\gamma)/(4\gamma^2 t^2), \qquad p = (\gamma-1)\mu \end{aligned} \tag{18}$$

where the p_α are given by (9), and

$$k^2 = (2-\gamma)(3\gamma-2)$$

A solution of Collins *et al.* [25] (p. 807) is the only other solution with this type of singularity of which we are aware. The limiting equation of state near the singularity is $p \approx -\frac{1}{7}\mu$, and at late times it is $p \approx -\frac{2}{7}\mu$, and hence the solution is of limited interest.

Case 2b (Ellis–MacCallum): The exact power law solution is a solution of Bianchi type VI_0 (with $n_\alpha{}^\alpha = 0$, in the terminology of [5]), with equation of state $p = (\gamma-1)\mu$. The dust case ($\gamma = 1$) was first given by Ellis and MacCallum [5] (p. 125), and the general case ($1 \leqslant \gamma < 2$) by Collins [23] [example 2(a)]. The solution is given as the special case $r = 0$ of the exact power law solution in case 2c, to follow. We are not aware of any other solutions with this type of singularity.

Case 2c: The exact power law solution is a solution of Bianchi type VI_h (with $n_\alpha{}^\alpha = 0$) given by Collins [23] [example 3a(i)]:

$$\begin{aligned} ds^2 &= -dt^2 + t^{2p_1}(w^1)^2 + t^{2p_2}(w^2)^2 + t^{2p_3}(w^3)^2 \\ \mu &= (2-\gamma)(1-r^2)/(\gamma^2 t^2), \qquad p = (\gamma-1)\mu \end{aligned} \tag{19}$$

where

$$w^1 = dx, \qquad w^2 = e^{[r(2-\gamma)+s]\,x/(2\gamma)}\,dy, \qquad w^3 = e^{[r(2-\gamma)-s]\,x/(2\gamma)}\,dz$$

the p_α, $\alpha = 1, 2, 3$ are given by (11),

$$s^2 = (3\gamma-2)(2-\gamma)$$

and r is an arbitrary parameter which satisfies

$$0 < r < 1$$

We are not aware of any other solutions with this type of singularity. Note that the limiting case $r = 0$ in this solution is the exact power law solution in case 2b.

Case 2d: The exact power law solution is

$$\begin{aligned} ds^2 &= -dt^2 + t^2(w^1)^2 + t^{2/5}(w^2 + qt^{4/5}w^1)^2 + t^{6/5}(w^3)^2 \\ \mu &= \frac{27}{25}(1-r^2)/t^2, \qquad p = \frac{1}{9}\mu \end{aligned} \tag{20}$$

where

$$w^1 = dx, \quad w^2 = e^{\sqrt{6}rx/5}\,dy, \quad w^3 = e^{-2\sqrt{6}rx/5}\,dz$$

$$q^2 = 9r^2/4 - 1$$

and r is an arbitrary parameter which satisfies

$$2/3 < r < 1$$

This solution is of Bianchi type $\mathrm{VI}_{h=-1/9}$ (and is in the class Bbii, in the terminology of [5]), and to the best of the writer's knowledge, is new. It is not obvious by inspection that the singularity in this solution is in case 2d, since the natural orthonormal frame is not an expansion eigenframe. However, it is easily verified by direct calculation of the eigenvalues of the expansion tensor, that the p_α for this solution do have the case 2d values as given by (12).

Case 3a: There are three exact power law solutions in this case, namely, the *vacuum* plane wave solutions, as given by Siklos [26], which are of Bianchi types IV, VI_h, and VII_h. We take as our prototype the simplest of these solutions, namely, the special case of the type VI_h solution which satisfies $n_\alpha{}^\alpha = 0$, in the terminology of [5]. This solution can be written in the form

$$ds^2 = -dt^2 + t^{2p_1}\,dx^2 + t^{2p_2}e^{2p_2x}\,dy^2 + t^{2p_3}e^{2p_3x}\,dz^2 \tag{21}$$

where the p_α are given by (13). It has been given in different forms by various authors, and is also obtained as the limiting case $r = 1$, in (19).

A perfect fluid solution with this type of singularity, which generalizes (21), has recently been found by the author [27].

Case 3b: The exact power law solution is the *vacuum* solution given by (20), with $r = 1$. This solution is of Petrov type III, and was first given by Robinson and Trautman [28]. It has also been studied by Collinson and French [29] and by Siklos [26]. Our form of the solution differs from that given by these authors, since we use the coordinates of Ellis-MacCallum [5] (class B, with $n_\alpha{}^\alpha = 0$), which are adapted to the group orbits, while the other authors use null coordinates. We are not aware of any exact perfect fluid solutions with this type of singularity.

Case 4a (Taub): The exact power law solution is the *flat* metric

$$ds^2 = -dt^2 + t^2dx^2 + dy^2 + dz^2 \tag{22}$$

corresponding to the p_α as given by (15). One expects this exact power law solution to be flat, since for this power asymptote, both the Ricci and the Weyl tensor are dynamically negligible near the singularity.

A number of known exact perfect fluid solutions have a singularity of this type. Some occur as special cases of the general solution given in case 4b. A particularly simple example, which is distinct from these, was given by Evans [30]:

$$ds^2 = -dt^2 + [\log(1+t)]^2\, dx^2 + (1+t)^2(dy^2 + e^{2y}\, dz^2)$$

$$\mu = 2/[(1+t)^2 \log(1+t)], \qquad p = 0$$

On noting that $\log(1+t) \approx t$ as $t \to 0$, the relationship with the prototype (22), as $t \to 0$, becomes clear.

Case 4b: The exact power law solution is the well-known Kasner *vacuum* solution

$$ds^2 = -dt^2 + t^{2p_1}\, dx^2 + t^{2p_2}\, dy^2 + t^{2p_3}\, dz^2 \tag{23}$$

where the p_α are given by (16), with $0 < u \leqslant 1$, and hence satisfy (7d). The singularity in most of the known spatially homogeneous exact solutions with equation of state $p = (\gamma - 1)\mu$ is of this type. We now give a unified formulation of these solutions which clearly displays their relation to the prototype (23).

The line element has the general form

$$ds^2 = -A^{2(\gamma-1)}\, dt^2 + t^{2p_1}A^{2q_1}(w^1)^2 + t^{2p_2}A^{2q_2}(w^2)^2 + t^{2p_3}A^{2q_3}(w^3)^2 \tag{24}$$

$$\mu = \mu_0\alpha_m/(t^\gamma A^\gamma), \qquad p = (\gamma - 1)\mu, \qquad 1 \leqslant \gamma < 2$$

where

$$A^{2-\gamma} = \alpha_s + \alpha_c t^{(3\gamma+2)/3} + \alpha_m t^{2-\gamma} \tag{25}$$

Here $\alpha_s, \alpha_c, \alpha_m$ and μ_0 are constants, such that $A > 0$ and $\mu_0 \geqslant 0$ and the p_α and q_α, $\alpha = 1, 2, 3$, are sets of constants, both of which satisfy the Kasner constraints (7d). By restricting the p_α, q_α and μ_0 and α_c, and by specifying the 1-forms w^α, $\alpha = 1, 2, 3$, in a suitable manner, we obtain the three known families of spatially homogeneous solutions with $p = (\gamma - 1)\mu$, which are of Bianchi types I, II, and VI_h (with $n_\alpha{}^\alpha = 0$). These are given below.

Bianchi I:

$$w^1 = dx, \qquad w^2 = dy, \qquad w^3 = dz$$

$$q_\alpha = 2/3 - p_\alpha, \qquad \alpha = 1, 2, 3 \tag{26}$$

$$\mu_0 = 4/3, \qquad \alpha_c = 0$$

The p_α satisfy the Kasner constraint (7d), and can for example be represented by (16), with $0 \leqslant u \leqslant 1$. For given γ, this solution depends on three parameters α_s, α_m, and u. Only two are essential however, since one of α_s and α_m can be specified by rescaling the coordinates.

If $\alpha_m = 0$, we can rescale $\alpha_s = 1$, and obtain the vacuum Kasner solution (23). If $\alpha_s = 0$, we can rescale $\alpha_m = 1$, and obtain the FRW solution (17), with a different time coordinate. If $\alpha_m\alpha_s \neq 0$, we obtain a new form for the general Bianchi

I solution, with equation of state $p = (\gamma - 1)\mu$, first given by Jacobs [31]. If $\alpha_s > 0$, then $A > 0$ for all $t \geqslant 0$ and the singularity occurs when $t = 0$. If $\alpha_s < 0$, the singularity will occur for some positive value t_s such that $A(t_s) = 0$. The cases $\alpha_s > 0$, and $\alpha_s < 0$ are in fact equivalent, however, since A and t appear symmetrically in the solution when we scale $\alpha_m = 1$: note that $p_\alpha + q_\alpha = 2/3$, and that (25) implies

$$A^{\gamma-1}\, dt = t^{\gamma-1}\, dA$$

since $\alpha_c = 0$. Thus without loss of generality, we assume $\alpha_s > 0$ and hence can set $\alpha_s = 1$. Then $A \to 1$ as $t \to 0^+$, so that in this limit, t approximates clock time along the fluid flow lines, which displays the relation of (24) to the prototype (23) if $0 < u \leqslant 1$, or (22) if $u = 0$. Thus *if* $0 < u \leqslant 1$ *the singularity is a Kasner asymptote, and if* $u = 0$, *the singularity is a Taub asymptote.*

Bianchi II:

$$w^1 = dx + \tfrac{1}{2}k(\alpha_m)^{1/2} z\, dy, \qquad w^2 = dy, \qquad w^3 = dz$$

$$\mu_0 = (6 - \gamma)/4, \qquad \alpha_c = 0$$

with

$$k^2 = (3\gamma - 2)(2 - \gamma)$$

The exponents p_α and q_α are given by

$$\begin{aligned} &p_1 = (2 - \gamma)/4, \quad p_2 = (2 + \gamma - q)/8, \quad p_3 = (2 + \gamma + q)/8 \\ &q_1 = p_1, \quad q_2 = p_3, \quad q_3 = p_2 \end{aligned} \tag{27}$$

with

$$q^2 = (2 + \gamma)(10 - 3\gamma)$$

This solution is a simplified form of the Bianchi II solution first given by Collins [23] [(example 1a); see also Kramer et al. [12], p. 149, (12.23)-(12.24)].

For given γ this solution depends on one essential parameter, since one of α_s, α_m can be specified by rescaling the coordinates. If $\alpha_s \neq 0$, one can without loss of generality assume $\alpha_s > 0$ and hence scale $\alpha_s = 1$. Then, as in the Bianchi I case, the line element (24) approximates the prototype (23) as $t \to 0^+$, and *the singularity is a particular Kasner asymptote* [note that the p_α as given by (27) satisfy the Kasner constraint (7d)].

For completeness, we note that when $\alpha_s = 0$, the solution is the exact power law solution (18), with Novikov asymptote, but with a different time coordinate.

Bianchi VI_h*:*

$$w^1 = dx, \qquad w^2 = e^{r[k+(3\gamma-2)]x}\, dy, \qquad w^3 = e^{r[k-(3\gamma-2)]x}\, dz$$

$$\mu_0 = 4/3, \qquad \alpha_c > 0 \quad \text{but arbitrary}$$

with

$$k^2 = (3\gamma + 2)(2 - \gamma)$$

$$r^2 = \alpha_c(3\gamma + 2)/[36(2 - \gamma)]$$

The exponents p_α and q_α are given by

$$p_1 = (4 - 3\gamma)/6, \quad p_2 = (2 + 3\gamma - 3k)/12, \quad p_3 = (2 + 3\gamma + 3k)/12 \tag{28}$$

$$q_1 = \gamma/2, \quad q_2 = (2 - \gamma + k)/4, \quad q_3 = (2 - \gamma - k)/4 \tag{29}$$

and the relation (26) is again satisfied.

The solution contains three parameters α_s, α_c, and α_m, one of which can be fixed by rescaling. If $\alpha_m > 0$ we obtain the type VI_h $p = (\gamma - 1)\,\mu$ solution of Collins [23] [example 3(a) (ii)] in a much simpler form {see also Kramer et al. [12], p. 151, equation (12.31)}. There are three subclasses of solutions corresponding to $\alpha_s > 0$, $\alpha_s = 0$ and $\alpha_s < 0$. If $\alpha_s > 0$, we can rescale $\alpha_s = 1$. The singularity occurs at $t = 0$, and as in the Bianchi I case, the line element (24) approximates the prototype (23) as $t \to 0^+$, if $\gamma \neq 4/3$, and the singularity is a particular Kasner asymptote. If $\gamma = 4/3$, it follows from (28) that $p_1 = p_2 = 0$, $p_3 = 1$, and the singularity is a Taub asymptote. If $\alpha_s < 0$, the singularity occurs at a positive value t_s, such that $A(t_s) = 0$. By introducing the function A as the time coordinate and regarding t as a function of A, one finds that the singularity is again a Kasner asymptote, since the q_α, as given by (29), again satisfy the Kasner constraint (7d). If $\gamma = 4/3$ in this case, the Kasner asymptote is degenerate with $(p_\alpha) = (2/3, 2/3, -1/3)$, corresponding to $u = 1$ in (16). Finally if $\alpha_s = 0$, the singularity is a Lifshitz-Khalatnikov asymptote. This is manifestly obvious from the form of the solution given in [18]. In particular, if $\gamma = 4/3$, the solution is equivalent to the Kantowski solution of Bianchi type III, given as an example in case 1 in this section.

For completeness, we note the following vacuum limits for this solution. If $\alpha_m = 0$, $\alpha_s \neq 0$, we obtain the type-VI_h vacuum solution of Ellis and MacCallum [5] {see p. 134; Kramer et al. [12], p. 136, equation (11.54)}, and if in addition $3\gamma = 2$, we obtain the Joseph type V vacuum solution {Kramer et al. [12], p. 136, equation (11.57)}, in a different form. If $\alpha_m = 0 = \alpha_s$, we obtain the vacuum solution (21) in a different form.

§(4): *Discussion*

Table I indicates which of the spatially homogeneous group types are compatible with the various power asymptotes. The group classification is that of Bianchi and Behr (see for example [5] or [11]), and we have included the Kantowski-Sachs (KS) class [32], in which there is no group of isometries which acts simply transitively on the three-dimensional group orbits. A Y indicates that

Table I. Power Asymptotes for the Different Spatially Homogeneous Group Types

Power asymptote	Class A						Class B				
	I	II	VI_0	VII_0	VIII	IX	V	IV	VI_h	VII_h	KS
1 (LK)	$Y(p)$	$Y(q)$	Y	Y	Y	$Y(q)$	Y	Y	$Y(e,q)$	Y	$Y(e,q)$
2a (N)		$Y(p)$	Y	Y	$Y(e)$	$Y(q)$		Y	Y	Y	
2b (EM)			$Y(p)$		Y						
2c									$Y(p)$		
2d									$Y(p)$		
3a								$Y(p)$	$Y(p,e,q)$	$Y(p)$	
3b									$Y(p)$		
4a (T)	$Y(p,e,q)$	$Y(q)$		Y	Y	$Y(q)$			$Y(e,q)$		$Y(e,q)$
4b (K)	$Y(p,e,q)$	$Y(q)$	$Y(q)$	Y			$Y(q)$	Y	$Y(e,q)$	Y	$Y(e,q)$

the power asymptote in question is possible, while a blank indicates not. A (p) indicates that an exact power law solution exists, and is known explicitly (see Section 3). An (e) indicates that an exact spatially homogeneous anisotropic perfect fluid solution,[1] with $\lim_{t\to 0} p/\mu \neq 1$, is known and is given in Section 3. A (q) indicates that existence of the power asymptote in question can be inferred from published work on the qualitative analysis of the field equations.

We make the following comments concerning the table:

(i) The LK asymptote can occur in all SH group types, but a perfect fluid solution of Bianchi type I or V with an LK asymptote must be an exact FRW model.

(ii) The 2d and 3b asymptotes occur only in Bianchi type-VI_h solutions with $h = -1/9$, which are also contained in the subclass Bb(ii) in the Ellis-MacCallum [5] classification. For a 3a asymptote in a solution of Bianchi type VI_h or VII_h, the group parameter h is related to the parameter u in (2.9) according to

$$h = -u/(1-u)$$

For a 2c asymptote, the Bianchi type is necessarily VI_h, and h is related to the parameter r in (2.7) according to

$$h = -[(2-\gamma)/(3\gamma-2)]\, r^2$$

(iii) This table extends the results of Borzeszkowski and Müller [4], who considered Bianchi types I, VII_0, VIII, IX, V, and VII_h, and found the asymptotes 1, 2a, 2b, 4a, and 4b. Our results agree with [4], with the exception that we find that the plane wave asymptote 3a is possible in Bianchi type VII_h solu-

[1] Which is not an exact power law solution.

tions, while this is not mentioned in [4]. However, we have not established the existence of nonvacuum solutions of this type.

(iv) The relationship with the work on the qualitative analysis of the field equations, with $p = (\gamma - 1)\mu$, is as follows. Collins [23] analyzed a subclass of the SH models containing solutions of Bianchi type I, II, VI_0, V, and VI_h, and found that the asymptotes 1, 3a, 4a, and 4b, do occur, although the Kasner asymptote 4b is typical. Secondly the analysis of the Kantowski-Sachs solutions by Collins [32] showed that the power asymptotes 1, 4a, 4b occur (and only these) in this class of solutions. Finally, the analysis by Bogoyavlenskii and Novikov [3] established the existence of the asymptotes 1, 2a, and 4a (and only these) in the class of Bianchi type IX solutions. These remarks are probably incomplete, due to the fact that we have not had access to certain works in Russian, e.g., [33] (see [4]).

We conclude with some comments concerning the degree of anisotropy of the power asymptotes. Anisotropy of the singularity manifests itself via the shear tensor of the fluid flow, the Weyl tensor, and the spatial curvature of the hypersurfaces orthogonal to the fluid flow. The anisotropy due to the shear is described by the parameter β_s of Section 2, and the anisotropy due to the Weyl tensor is described by the Petrov type of the limit of the Weyl tensor, as described in Section 2. The anisotropy due to the spatial curvature is described by the trace-free part of the Ricci tensor of the hypersurfaces:

$$S^*_{\alpha\beta} = R^*_{\alpha\beta} - \tfrac{1}{3} R^* \delta_{\alpha\beta}$$

where the components refer to a suitable orthonormal frame. It follows from the detailed calculations, however, that

$$\lim_{T \to 0^+} S^*_{\alpha\beta}/\theta^2 = 0 \Longleftrightarrow \lim_{T \to 0^+} R^*/\theta^2 = 0$$

Thus the anisotropy due to the spatial curvature can be described by the parameter β_c, defined in Section 2.

It follows from the results of Section 2, that the only power asymptote that is isotropic as regards shear ($\beta_s = 0$) is the LK asymptote. Note, however, that β_s can be made arbitrarily close to zero in case 2d. This is also possible in cases 2a, 2b, 2c, and 3a if one permits negative pressures ($3/2 < \gamma < 1$). For the K and T asymptotes, however, the anisotropy in the shear is maximal ($\beta_s = 1$). The LK, T, and K asymptotes (cases 1 and 4) are isotropic as regards spatial curvature ($\beta_c = 0$), and the LK and T asymptotes (cases 1 and 4a) are isotropic as regards the Weyl tensor. Thus *the LK asymptote is the only power asymptote which is isotropic in all respects.* This justifies the name "isotropic singularity," mentioned in Section 2.

We have noted that both the LK and T asymptotes are isotropic as regards the Weyl tensor. These asymptotes differ, however, in that in the former the Ricci tensor is dynamically dominant ($\beta_m = 1$) while in the latter it is dynamically negligible ($\beta_m = 0$). Thus at an LK asymptote the Ricci tensor dominates the Weyl tensor, while the detailed calculations show that at a T asymptote, these tensors have the same order of magnitude. This is exemplified by the behavior of the ratio

$$C_{abcd}C^{abcd}/R_{ab}R^{ab}$$

which tends to zero at an LK asymptote, but has a finite nonzero limit at a T asymptote. Indeed it follows from the detailed analysis that *the LK asymptote is the only power asymptote at which the Weyl tensor is dominated by the Ricci tensor* (although the above ratio cannot always be used to illustrate this fact). Thus the LK asymptote is the only power asymptote which can be regarded as satisfying the Penrose hypothesis [34] of "zero Weyl tensor" at the initial singularity.

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