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Isotropic Singularities and Isotropization in a Class of Bianchi Type-VI$_h$ Cosmologies

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Abstract

The evolution of a class of exact spatially homogeneous cosmological models of Bianchi type VI$_h$ is discussed. It is known that solutions of type VI$_h$ cannot approach isotropy asymptotically at large times. Indeed the present class of solutions become asymptotic to an anisotropic vacuum plane wave solution. Nevertheless, for these solutions the initial anisotropy can decay, leading to a stage of finite duration in which the model is close to isotropy. Depending on the choice of parameters in the solution, this quasi-isotropic stage can commence at the initial singularity, in which case the singularity is of the type known as "isotropic" or "Friedmann-like." The existence of this quasi-isotropic stage implies that these models can be compatible in principle with the observed universe.

§(1): *Introduction*

In this paper we analyze the evolution and singularities of a class of spatially homogeneous cosmological models with equation of state $p = (\gamma - 1)\mu$. These exact solutions were discovered by Collins [1] and are of type VI$_h$ in the Bianchi-Behr classification [2].

Since the microwave background is observed to be highly isotropic, it is usually inferred that the Universe must be very close to a Friedmann-Robertson-Walker (FRW) model. Thus in studies of spatially homogeneous cosmological models consideration is usually restricted to those Bianchi group types which admit FRW solutions [3], and this excludes models of type VI$_h$. For example, it is known [4] that models of Bianchi types which do not admit FRW solutions

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cannot isotropize completely at arbitrarily large times, and hence cannot asymptotically approach an FRW model. However, as pointed out by Barrow [5], it is possible that models of type VI_h could become arbitrarily close to isotropy *over some finite time interval.* Indeed this is the sense in which Doroshkevich et al. [6] use the word "isotropize" in their investigation of models of types VII, VIII, and IX. We show that this type of isotropization can occur in the solutions under consideration, and hence that these models can be compatible in principle with the observed universe.

Our second aim is to use these solutions to illustrate certain aspects of the type of singularity that is referred to as "isotropic" or "Friedmann-like" [7]. This type of cosmological singularity, though highly specialized, is of interest in connection with realistic models of the universe, e.g., Barrow's quiescent cosmology concept [8] (the universe is a perturbed FRW model near the singularity), and Penrose's Weyl tensor hypothesis (the Weyl tensor should tend to zero at an initial singularity), which arose from the consideration of gravitational entropy [9].

The present models, when restricted to have an isotropic singularity, remain close to isotropy for a certain time interval after the initial singularity, but then grow increasingly anisotropic. Such models can be compatible with the observed isotropy of the Universe, and illustrate the scenario discussed by Barrow [5], namely, that the Universe is "young," and is observed to be isotropic because it originated in an isotropic state, and instabilities have had insufficient time to grow.

Secondly, as the initial singularity is approached both the Weyl and Ricci tensors diverge, but in such a way that the Ricci tensor dominates the Weyl tensor. This suggests that the Penrose hypothesis should be slightly weakened in the sense that the Weyl tensor should be required to tend to zero, not in an absolute sense, but relative to the Ricci tensor (indeed Penrose mentioned this as a possibility in an earlier paper [10]).

Although the idea of an isotropic singularity first arose in 1963 [11], it has not been studied in detail, and indeed the concept has not been clearly defined. The study of the present class of solutions suggested that the key feature of an isotropic singularity is that *the physical metric is conformal to a metric which is regular at the singularity*, i.e., *the singularity arises solely due to a singular conformal factor.* We illustrate this idea in the present paper, and will discuss the idea in general in a subsequent paper.

In Section 2 we present the Collins solutions in a new form, which clarifies their relation to the FRW solutions. The question of isotropization is dealt with in Section 3, and Section 4 contains the discussion of isotropic singularities. Our sign conventions for the Riemann and Ricci tensors are those of [12] and we use geometrized units so that $8\pi G = 1 = c$.

§(2): *The Solutions*

The following solution satisfies the Einstein field equations with perfect fluid source, and zero cosmological constant. The line element and fluid 4-velocity have the form

$$ds^2 = -dt^2 + T^{4/(3\gamma)}[A^{2q_1}(w^1)^2 + A^{2q_2}(w^2)^2 + A^{2q_3}(w^3)^2]$$

$$u = \partial/\partial t \tag{1}$$

The functions T and A satisfy

$$A = [\alpha_m + \gamma^2\alpha_c T^{2-4/(3\gamma)} + \alpha_s T^{1-2/\gamma}]^{1/(2-\gamma)}$$

$$\frac{dT}{dt} = A^{1-\gamma} \tag{2}$$

where α_m, α_c, and α_s are constants with $\alpha_m \geqslant 0$, $\alpha_c \geqslant 0$, and the exponents q_α are given by

$$q_1 = (1/2)\gamma, \qquad q_2 = (2 - \gamma + s)/4, \qquad q_3 = (2 - \gamma - s)/4$$

with

$$s^2 = (3\gamma + 2)(2 - \gamma) \tag{3}$$

The energy density and pressure of the fluid are given by

$$\mu = \frac{4\alpha_m}{3\gamma^2 T^2 A^\gamma}, \qquad p = (\gamma - 1)\mu, \qquad 1 \leqslant \gamma < 2 \tag{4}$$

The differential forms w^α, $\alpha = 1, 2, 3$ are given by

$$w^1 = dx, \qquad w^2 = e^{r[s+(3\gamma-2)]x}\,dy, \qquad w^3 = e^{r[s-(3\gamma-2)]x}\,dz \tag{5}$$

where

$$r^2 = \alpha_c(3\gamma + 2)/[36(2 - \gamma)] \tag{6}$$

The solution is defined up to a quadrature in terms of clock time t along the fluid flow lines, but is given explicitly if one uses T as time coordinate. As regards coordinate ranges we assume that x, y, z take on all real values. The range of values of T is determined by the initial singularity. If $\alpha_s \geqslant 0$ the initial singularity will occur when $T = 0$, while if $\alpha_s < 0$, the initial singularity will occur when $A = 0$, at some positive value of T. Thus, we write the range of T as

$$T > T_s \geqslant 0$$

where $T_s = 0$ if $\alpha_s \geqslant 0$ and T_s is defined by $A(T_s) = 0$ if $\alpha_s < 0$.

The solution depends on three parameters α_m, α_c, and α_s. The parameter α_m relates to the energy density of the fluid, while α_s determines the nature of the initial singularity (see Sections 3 and 4). Finally, α_c relates to the spatial curvature of the hypersurfaces orthogonal to the fluid flow lines, and also determines the Bianchi type of the 3-parameter isometry group of the solutions. In particular, if $\alpha_c \neq 0$, the solution is of Bianchi type VI_h, but if $\alpha_c = 0$, it specializes to type I. These three parameters are not all essential and one of them, if nonzero, can be fixed by rescaling the coordinates. We will normally use this freedom to set

$$\alpha_m = 1 \tag{7}$$

As mentioned in Section 1, these solutions were first given by Collins [1] (example 3a(ii), on p. 156; see also [13], p. 151, equation (12.31)). We have redefined the parameters and functions in the metric, in order to facilitate comparison with the FRW solutions, and to clarify their evolution and singularity structure. Three special cases of this solution, which correspond to the three stages in their evolution (to be discussed in Section 3), arise as follows. When $\alpha_s = 0 = \alpha_c$, and we use the normalization (7) the solution specializes to the FRW solution with the flat spatial sections ($k = 0$):

$$\begin{aligned} ds^2 &= -dt^2 + t^{4/(3\gamma)}(dx^2 + dy^2 + dz^2) \\ \mu &= 4/(3\gamma^2 t^2), \qquad p = (\gamma - 1)\mu \end{aligned} \tag{8}$$

When $\alpha_m = 0 = \alpha_c$, the solution specializes to a Kasner *vacuum* solution. (see, for example, [13], p. 135). After redefining the time coordinate, and rescaling the spatial coordinates in an obvious way, one obtains

$$ds^2 = -dt^2 + t^{2p_1}\, dx^2 + t^{2p_2}\, dy^2 + t^{2p_3}\, dz^2 \tag{9}$$

$$p_1 = (4 - 3\gamma)/6, \qquad p_2 = (2 + 3\gamma + 3s)/12, \qquad p_3 = (2 + 3\gamma - 3s)/12$$

and s is defined by equation (3). When $\gamma = 4/3$, the exponents simplify to

$$p_1 = p_3 = 0, \qquad p_2 = 1$$

and this solution is flat.

Finally, when $\alpha_m = 0 = \alpha_s$, the solution specializes to a *vacuum* solution of Bianchi type VI_h. After an obvious coordinate transformation, the line element can be written in the form

$$ds^2 = -dt^2 + t^2 dx^2 + t^{2p_2} e^{2p_2 x}\, dy^2 + t^{2p_3} e^{2p_3}\, dz^2 \tag{10}$$

where

$$\begin{aligned} p_2 &= s[s + (3\gamma - 2)]/[2(8 - s^2)] \\ p_3 &= s[s - (3\gamma - 2)]/[2(8 - s^2)] \end{aligned}$$

and s is defined by equation (3). If $\gamma = 4/3$ we obtain $p_2 = 1$ and $p_3 = 0$, and the line element describes flat space-time. If $\gamma \neq 4/3$ this vacuum solution is a member of the class of plane wave spatially homogeneous solutions described by Siklos [14] (see p. 400). It is obtained by setting $k = 0$ in the type VI$_h$ solution and performing a coordinate transformation $x' = (x + y)/\sqrt{2}, y' = (-x + y)/\sqrt{2}$. It follows from [14] that this vacuum solution is of Petrov type[1] N, and that the repeated principal null direction[1] l, which is given by

$$l = \partial/\partial t - t^{-1} \partial/\partial x$$

is covariantly constant. In addition the metric admits a six-parameter group of isometries acting transitively on space-time. This solution was apparently first given by Lifshitz and Khalatnikov [11] (p. 232; see also the recent review [15], p. 655 and 662), and also appears in Collins [1] (example 3a(iii); see also [13], p. 136).

§(3): *Evolution of the Solutions*

In this section we restrict our considerations to the case $\alpha_s > 0$ and $\alpha_c > 0$. Then the time coordinate T assumes the values $0 < T < +\infty$. The most important features of the evolution are determined by the dimensionless scalar σ/θ, with

$$\sigma^2 = (1/2)\, \sigma_{ij} \sigma^{ij}$$

where σ_{ij} is the rate of shear tensor, and θ is the rate of expansion of the fluid (see, for example, [16]). This scalar is called the *distortion* of the fluid by Barrow [5]; it measures the significance of the shear relative to the expansion. For the solution of Section 2, $\theta > 0$ and since σ is defined to be the positive square root of σ^2, the distortion is nonnegative.

For an irrotational fluid, there are space-like hypersurfaces orthogonal to the fluid 4-velocity. In this case the field equations relate the distortion to the Ricci scalar R^* of these hypersurfaces, and to the energy density of the fluid according to

$$3\sigma^2/\theta^2 = 1 - 3\mu/\theta^2 + 3R^*/2\theta^2 \tag{11}$$

(see, for example, [16]). Since $R^* \leqslant 0$ for the solution of Section 2 (see equation (A9)) and $\mu > 0$, equation (11) implies that

$$0 \leqslant \frac{3\sigma^2}{\theta^2} < 1$$

for all values of T. It follows from the expression (A7) for the distortion, and

[1] See [13], Chap. 4, for this terminology.

the limits (A19) and (A20) that

$$\lim_{T \to 0^+} \frac{3\sigma^2}{\theta^2} = 1, \qquad \lim_{T \to +\infty} \frac{3\sigma^2}{\theta^2} = \frac{1}{16}(3\gamma - 2)^2 \tag{12}$$

The first of these limits depends on $\alpha_s > 0$. Thus the distortion is a maximum at the initial singularity $T = 0$, and then decreases initially as the model evolves. Since $\alpha_s > 0$ and $\alpha_c > 0$, it follows from (A7) that at a finite value of T, given by

$$T_*^{1 + 2/(3\gamma)} = 3(2 - \gamma)\,\alpha_s / [2\gamma^2(3\gamma - 2)\,\alpha_c] \tag{13}$$

the distortion reaches zero, and subsequently increases.

Since σ/θ is zero at $T = T_*$, and it is continuous, it follows that there is some interval of T in which the distortion is "small." More precisely, given any bound $0 < \epsilon \ll 1$, the values of T such that

$$\frac{\sigma}{\theta} < \epsilon$$

form an interval (T_1, T_2) with $T_1 < T_* < T_2$. Thus the evolution of these cosmological models falls into three stages:

I. the stage $0 < T < T_1$, in which the distortion is significant, but decreasing,
II. the stage $T_1 < T < T_2$, in which the distortion is small,
III. the stage $T_2 < T < +\infty$, in which the distortion is significant and increasing.

The minimum duration of region II is determined by the bound ϵ. It can be shown (see the Appendix) that for $\epsilon \ll 1$, say, $\epsilon \lesssim 0.1$, the values of T for which $\sigma/\theta < \epsilon$ satisfy

$$|T - T_*| \lesssim \epsilon T_* \tag{14}$$

as an order of magnitude estimate.

The matter distribution is not the only source of anisotropy in the model. One has also to consider the anisotropy in the spatial curvature. This is described by the trace-free Ricci tensor S^*_{ij} of the hypersurfaces orthrogonal to the fluid flow lines (see, for example, [16], p. 34). We define the magnitude $S^* \geqslant 0$ of this tensor by

$$(S^*)^2 = (1/2)\, S^*_{ij} S^{*ij}$$

and as a measure of the anisotropy of the spatial curvature we use the dimensionless scalar

$$S^*/\theta^2$$

It follows from (A8), (A19), and (A20) that

$$\lim_{T\to 0^+} S^*/\theta^2 = 0, \qquad \lim_{T\to\infty} S^*/\theta^2 = \frac{1}{32\sqrt{3}}(3\gamma+2)(3\gamma-2)(2-\gamma) \quad (15)$$

and from (A8), (A17), and (7) that

$$\left.\frac{S^*}{\theta^2}\right|_{T=T_*} = \left[\frac{3\gamma-2}{6\sqrt{3}}\right]\left[\frac{\alpha}{1+\alpha}\right]$$

where

$$\alpha = (1/2)\,[(3\gamma+2)/(3\gamma-2)]\;\alpha_s T_*^{1-2/\gamma} \quad (16)$$

is a dimensionless parameter [see (A15)]. Thus the anisotropy in the spatial curvature will not be less than the prescribed bound $\epsilon \ll 1$ in region II, unless α is restricted. However, if we choose $\alpha \ll 1$, then the interval on which σ/θ and S^*/θ^2 are less than ϵ is no longer required to be of relative length ϵ [see equation (14)]. Indeed one can draw the following stronger conclusion from (A7), (A8), and (A16): for any interval $T_1 \leqslant T \leqslant T_2$ with $0 < T_1 < T_* < T_2$, we can choose the parameter α sufficiently small that

$$\frac{\sigma}{\theta} < \epsilon, \qquad \frac{S^*}{\theta^2} < \epsilon$$

for all $T_1 \leqslant T \leqslant T_2$, where $\epsilon \ll 1$ is a given positive bound. In other words, by choosing α appropriately the "quasi-isotropic" stage II can be made of arbitrary duration.

The relation between T and clock time t should be mentioned at this point. If we scale $\alpha_m = 1$, and choose $\alpha \ll 1$, it follows from equations (13) and (15) that the function A as defined by (2) satisfies $|A - 1| \ll 1$, and hence that T approximates clock time t during stage II.

The energy density μ also helps to distinguish the three stages. It follows from (11) and $R^* \leqslant 0$ that

$$0 \leqslant \frac{3\mu}{\theta^2} \leqslant 1$$

This dimensionless scalar gives a measure of the dynamical significance of the energy density. It follows from (A5), (A2), and (2) that

$$\lim_{T\to 0^+} \frac{3\mu}{\theta^2} = 0, \qquad \lim_{T\to +\infty} \frac{3\mu}{\theta^2} = 0 \quad (17)$$

and from (A5) and (A17) that

$$\left.\frac{3\mu}{\theta^2}\right|_{T=T_*} = \frac{1}{1+\alpha}$$

Thus the matter density is dynamically insignificant near the singularity even though $\lim \mu_{T\to 0^+} = +\infty$, but if $\alpha \ll 1$, it is dominant at $T = T_*$, and $3\mu/\theta^2$ is close to its maximum value of 1 throughout stage II.

On account of equations (17), the asymptotic states of the solution as $T \to 0^+$ and $T \to +\infty$ are both "quasivacuum," and the distortion σ/θ is significant in both. These asymptotic states are quite different, however, on account of the behavior of the spatial curvature as given by equation (15), i.e., spatial curvature is negligible as $T \to 0^+$ [note that equations (11), (12), and (17) imply that $\lim_{T\to 0^+} R^*/\theta^2 = 0$, as well], but is significant as $T \to +\infty$. In fact the *asymptotic state $T \to 0^+$ is described by the Kasner vacuum solution (9), while the asymptotic state $T \to +\infty$ is described by the plane wave vacuum solution (10).*

We have shown that when $\alpha \ll 1$, stage II of the model is close to isotropy both as regards the matter flow and the spatial curvature, and hence is appropriately called a "quasi-isotropic" stage. We thus expect that during stage II, the model in some sense approximates the (isotropic) FRW model with $k = 0$. The well-known characterizations of the FRW models enable us to make this idea more precise, as follows.

The FRW solutions of the Einstein field equations with perfect fluid source and equation of state $p = p(\mu)$, can be characterized in terms of the kinematic quantities of the fluid congruence by

$$\sigma = w = \dot{u} = 0, \qquad \theta \neq 0$$

or in terms of the curvature of space-time by

$$C_{abcd} = 0, \qquad R_{ab} \neq 0$$

(see, for example, [16]). Here C_{abcd} is the Weyl conformal curvature tensor, and w and $\dot{u}$ are the magnitudes of the vorticity tensor and acceleration vector, respectively. It thus seems reasonable to regard as an approximation of an FRW solution any perfect fluid solution such that σ, w, and $\dot{u}$ are small[2] compared to θ, while C_{abcd} is small compared to R_{ab}, in some appropriate sense. We thus consider the dimensionless ratios

$$\sigma/\theta, \qquad w/\theta, \qquad \dot{u}/\theta \tag{18}$$

for the kinematic quantities, and the dimensionless ratios

$$E/\mu, \qquad H/\mu \tag{19}$$

[2] Except near events at which $\theta = 0$, which would occur in a model which passes from a state of expansion to a state of contraction.

for the Weyl and Ricci tensors, where

$$E^2 = (1/2)\,E_{ab}E^{ab}, \qquad H^2 = (1/2)\,H_{ab}H^{ab}$$

and E_{ab}, H_{ab} are the electric and magnetic parts of the Weyl tensor relative to the fluid 4-velocity u. We use the energy density μ of the fluid to represent the Ricci tensor. One could equivalently use the scalar $(R_{ab}R^{ab})^{1/2}$. Thus *our criterion for a perfect fluid solution to approximate an FRW model is that the ratios (18) and (19) should be less than some specified bound* $\epsilon \ll 1$.

Since $w = 0 = \dot{u}$ for the solutions of Section 2, and we have already considered σ/θ, there remain only E/μ and H/μ. It follows from (A12), (A13), and (A5) that if $\alpha \ll 1, E/\mu$ and H/μ will be less than an arbitrarily specified bound ϵ. Thus by choosing α sufficiently small, the model can be made to approximate an FRW model arbitrarily closely, in stage II. Since the ratio $3\mu/\theta^2$ can be made arbitrarily close to 1 throughout stage II by choosing α sufficiently small, the FRW model in question is the $k = 0$ model, since in such a model the density has the critical value given by $\mu = \theta^2/3$. One aspect of this approximation that should be noted is that in stage II *the metric components of the solution are not close to the FRW metric components (8) for all values of the spatial coordinates, owing to the appearance of the exponentials in the differential forms (5).*

The fact that the model can be made close to the exact $k = 0$ FRW model over any interval $[T_1, T_2]$ suggests that these solutions should be closely related to linear perturbations of the FRW solutions. In fact when $\alpha \ll 1$ and $T \simeq T_*$, it follows from (A7), (A12), (A13), (A5), and (A16) that $\sigma/\theta, E/\mu$, and H/μ are approximated by

$$\sigma/\theta \simeq \alpha(\eta_+ - \eta_-)$$

$$E/\mu \simeq \alpha\left(\frac{3\gamma}{2}\,\eta_+ + \eta_-\right)$$

$$H/\mu \simeq \alpha^{3/2}\,(\eta_+)^{1/2}\,(\eta_+ - \eta_-)$$

where we have suppressed overall multiplicative constants, and have written

$$\eta = T/T_*$$

and

$$\eta_+ = \eta^{2-4/(3\gamma)}, \qquad \eta_- = \eta^{1-2/\gamma}$$

These time dependencies are precisely those that arise in the linear perturbations of the $k = 0$ FRW solutions ([17], [18]). A puzzling feature, however, is the fact that H/μ depends not on α but on $\alpha^{3/2}$, whereas in linear perturbation theory, it depends on the first power of the perturbation parameter, as do σ/θ and E/μ [17, 18]. This appears to be due to the fact that when $\alpha \ll 1$ the metric (1) cannot be written in the form

$$g_{ij} \approx g_{ij}^{\text{FRW}} + \alpha h_{ij}$$

i.e., g_{ij} cannot be linearized in terms of α. This is due to the fact that $\sqrt{\alpha}$ appears in the 1-forms w^{α}, on account of (5) and (6). Thus although the solution is close to an FRW solution in a coordinate-independent sense during stage II of the evolution, there is no solution of the linearized Einstein field equations to which it corresponds.

In summary, the model starts at the initial singularity as a Kasner vacuum model, with the exponents p_{α} determined by the equation of state parameter γ. As the model evolves, the energy density of the fluid becomes significant dynamically, and the distortion σ/θ decays to zero. If $\alpha \ll 1$ this gives rise to a quasi-isotropic stage in which the model is close to the $k = 0$ FRW model. At late times, the anisotropic spatial curvature becomes significant, and the model evolves asymptotically as $T \to +\infty$ to a plane wave vacuum state which is dominated by the distortion and the anisotropy in the spatial curvature. There are two arbitrary parameters in the model, T_* and α. The value of T_* determines where stage II actually occurs in the overall evolution of the model, while the value of α determines the duration of stage II, once the bound $\epsilon \ll 1$ has been specified.

§(4): *The Isotropic Singularity*

In this section we consider the case $\alpha_s = 0$. This affects the initial singularity, but not the late stages of the model as described in Section 3. The distortion σ/θ is now a monotone increasing function of T, with

$$\lim_{T \to 0^+} \sigma/\theta = 0 \tag{20}$$

as follows from (A7) and (A8). The limit as $T \to +\infty$ is unchanged [see equation (12)]. In addition it follows from (A12), and (A13), (A5), and (A20) that

$$\lim_{T \to 0^+} E/\mu = 0, \qquad \lim_{T \to 0^+} H/\mu = 0 \tag{21}$$

Thus on the basis of the discussion of Section 3 we can state that the model becomes increasingly close to the FRW model (8) as the initial singularity is approached (into the past). The asymptotic form of the line element as $T \to 0^+$ has the same time dependence, though not the same spatial dependence, as the FRW model (8):

$$ds^2 \simeq -dt^2 + t^{4/(3\gamma)}[dx^2 + e^{2r[s+(3\gamma-2)]x}\,dy^2 + e^{2r[s-(3\gamma-2)]x}\,dz^2] \tag{22}$$

as follows from (1) and (2) with $\alpha_m = 1$ and $\alpha_s = 0$. Thus we recognize the singularity as being isotropic or Friedmann-like [7].

The structure of the singularity becomes particularly clear when one introduces a *conformal time coordinate* τ, related to T by

$$dT = \tau^{2/(3\gamma-2)}\, d\tau$$

After rescaling the spatial coordinates and the parameters τ and α_c in an obvious way, one finds that

$$ds^2 = \tau^{4/(3\gamma-2)}[-A^{2(\gamma-1)}\, d\tau^2 + A^{2q_1}(w^1)^2 + A^{2q_2}(w^2)^2 + A^{2q_3}(w^3)^2] \tag{23}$$

where

$$A^{2-\gamma} = \alpha_m + \alpha_c \tau^2 \tag{24}$$

and the density and pressure are

$$\mu = \frac{12\alpha_m}{(3\gamma-2)^2\, \tau^{6\gamma/(3\gamma-2)}\, A^\gamma}, \qquad p = (\gamma - 1)\,\mu \tag{25}$$

The w^α and the q_α are given by equations (3) and (5) as before, and

$$r^2 = (3\gamma+2)\,\alpha_c/[4(2-\gamma)(3\gamma-2)^2] \tag{26}$$

The essential point is that *the physical metric g is conformal to a metric $\tilde{g}$ which is regular at the singularity* $\tau = 0$:

$$g = \Omega^2 \tilde{g}$$

with

$$\Omega = \tau^{2/(3\gamma-2)} \tag{27}$$

and $\tilde{g}$ defined by the square bracket in equation (23). The singularity is entirely due to the fact that the conformal factor Ω is zero when $\tau = 0$. The conformally related metric $\tilde{g}$ is analytic in τ at $\tau = 0$, and is moreover an even function of τ [cf. equation (24)]. Thus the metric components $\tilde{g}_{ij}$ have power series expansions in even powers of τ. When $\alpha_c = 0$ and α_m is normalized to equal 1, the solution (23)-(25) reduces to the FRW $k = 0$ solution relative to a conformal time coordinate, namely,

$$ds^2 = \tau^{4/(3\gamma-2)}(-d\tau^2 + dx^2 + dy^2 + dx^2)$$

and

$$\mu = \frac{12}{(3\gamma-2)^2\, \tau^{6\gamma/(3\gamma-2)}}, \qquad p = (\gamma-1)\,\mu$$

Thus the conformal factor Ω as given by (27), which determines the isotropic singularity, is simply the expansion factor for this FRW solution, relative to the conformal time coordinate.

Within this framework the singularity is represented by a hypersurface $\tau = 0$ which is a regular spacelike hypersurface relative to the conformally related metric $\tilde{g}$. The fact that the metric $\tilde{g}$ is an even function of τ implies that the extrinsic curvature [12], or second fundamental form [19] of this hypersurface is zero. On the other hand, the intrinsic geometry of the hypersurface $\tau = 0$ is not fully constrained. Indeed the metric induced on $\tau = 0$ by $\tilde{g}$ is given by

$$ds_{(3)}{}^2 = dx^2 + e^{2r[s+(3\gamma-2)]x}\, dy^2 + e^{2r[s-(3\gamma-2)]x}\, dz^2$$

as follows from (23) and (24), with α_m scaled to equal 1. This 3-metric depends on the arbitrary parameter α_c through equation (26), and is of constant curvature if and only if $\alpha_c = 0$, i.e., if and only if the solution is an exact FRW solution. The intrinsic geometry of the hypersurface $\tau = 0$, and the singular conformal factor Ω completely determine the evolution of the solution. Since Ω is determined by the equation of state parameter γ, *we regard the 3-metric induced on $\tau = 0$ by $\tilde{g}$ as being initial data for the solution.*

To summarize, this solution suggests the following description of an isotropic singularity. The singularity is represented by a regular spacelike hypersurface relative to a regular conformally related metric $\tilde{g}$. The extrinsic curvature of this hypersurface is zero, while the intrinsic curvature determines the future evolution of the model, and can be specified in an arbitrary manner. The limits (20) and (21) concerning the shear and the Weyl tensor imply that near the isotropic singularity, the solution is close to an exact FRW solution. Nevertheless, the Weyl tensor itself diverges at the singularity:

$$\lim_{T\to 0^+} E = +\infty, \qquad \lim_{T\to 0^+} H = +\infty$$

unless $\alpha_c = 0$, i.e., unless the solution is exact FRW. [The case $\gamma = 4/3$ is special in that only E diverges, since $H \equiv 0$.] This motivates the weakening of Penrose's Weyl tensor hypothesis, which was mentioned in Section 1. These matters will be discussed in more detail in a subsequent paper.

The length of time during which this model is close to the exact FRW solution depends on the parameter α_c. We can define a characteristic time for the model, analogous to T_*, by considering the distortion σ/θ. This quantity is strictly increasing from 0 at $T = 0$, and so for a given bound $\epsilon > 0$, we introduce a time T_ϵ such that

$$0 < T < T_\epsilon \Rightarrow \sigma/\theta < \epsilon \qquad (28)$$

Consideration of (A7), with $\alpha_s = 0$, suggests the definition[3]

$$T_\epsilon^{2-4/(3\gamma)} = \frac{3\sqrt{3}\,(2-\gamma)\,\epsilon\alpha_m}{(3\gamma-2)\,\gamma^2\alpha_c}$$

[3] σ/θ will in fact be less than ϵ on a somewhat larger interval than $(0, T_\epsilon)$.

which implies

$$\frac{\sigma}{\theta} = \frac{\epsilon\eta_+}{1 + [4\sqrt{3}\,\epsilon/(3\gamma - 2)]\,\eta_+}$$

where

$$\eta = T/T_\epsilon$$

and

$$\eta_+ = \eta^{2 - 4/(3\gamma)}$$

It follows that condition (28) is satisfied. Similarly E/μ and H/μ will be small for $0 < T < T_\epsilon$, i.e., less than some numerical multiple of ϵ. Thus, *by choosing the parameter α_c appropriately, we can insure that the model remains close to the FRW model up to some specified time T_ϵ.*

Appendix

In this appendix we give the expressions for the kinematic quantities, the spatial curvature, and the Weyl tensor of the solutions of Section 2. These quantities were calculated, and the solutions themselves were verified, using a library of programs [20] written in the symbolic computation language CAMAL [21]. We use the natural orthonormal frame associated with the line element (1), with the basis 1-forms given by

$$\bar{w}^0 = dt, \qquad \bar{w}^\alpha = T^{2/(3\gamma)} A^{q_\alpha} w^\alpha, \qquad \alpha = 1, 2, 3 \tag{A1}$$

where the q_α and w^α are given by equations (3) and (5).

In addition to the function A, defined by equation (2), we need the following expressions:

$$B = \alpha_m + \frac{4\gamma^2}{3(2-\gamma)}\alpha_c T_+ + \frac{1}{2}\alpha_s T_-$$

$$C = -\frac{2\gamma^2(3\gamma - 2)}{3(2-\gamma)}\alpha_c T_+ + \alpha_s T_-$$

$$D = \frac{1}{9}\left(\frac{3\gamma + 2}{2 - \gamma}\right)\gamma^2 \alpha_c T_+ A^{2-\gamma} \tag{A2}$$

where for convenience we write

$$T_+ = T^{2 - 4/(3\gamma)}, \qquad T_- = T^{1 - 2/\gamma} \tag{A3}$$

The various quantities are given below as dimensionless ratios formed with the expansion scalar θ, since this yields simpler expressions. The expansion scalar

θ is given by

$$\theta = 2B/(\gamma TA) \tag{A4}$$

For example, for the energy density μ, it follows from (4) and (A4) that

$$\frac{3\mu}{\theta^2} = \alpha_m A^{2-\gamma}/B^2 \tag{A5}$$

The nonzero components of the shear tensor relative to the frame (A1) are given by

$$\begin{aligned}
\sigma_{11}/\theta &= (1/12)(2 - 3\gamma)\, C/B \\
\sigma_{22}/\theta &= -(1/24)(2 - 3\gamma + 3s)\, C/B \\
\sigma_{33}/\theta &= -(1/24)(2 - 3\gamma - 3s)\, C/B
\end{aligned} \tag{A6}$$

It follows that the shear scalar is simply

$$\sigma^2/\theta^2 = C^2/(12B^2) \tag{A7}$$

The trace-free spatial Ricci tensor S^*_{ab} is a multiple of σ_{ab}, and hence it suffices to give the magnitude S^* of S^*_{ab}:

$$\frac{S^*}{\theta^2} = \frac{3\gamma - 2}{2\sqrt{3}} \frac{D}{B^2} \tag{A8}$$

The spatial Ricci scalar is given by

$$R^*/\theta^2 = -2D/B^2 \tag{A9}$$

The electric part of the Weyl tensor is expressed in terms of three additional functions:

$$\begin{aligned}
L &= \frac{1}{36}\left[\frac{\gamma^3(3\gamma - 2)}{2 - \gamma}\, \alpha_m \alpha_c T_+ + \alpha_s N T_-\right] \\
M &= \frac{s\gamma(4 - 3\gamma)}{16(3\gamma - 2)}\, C^2
\end{aligned} \tag{A10}$$

where

$$N = \alpha_m + \frac{\gamma^2(3\gamma + 2)}{2 - \gamma}\, \alpha_c T_+ - \frac{(4 - 3\gamma)(3\gamma + 2)}{4(3\gamma - 2)}\, \alpha_s T_-$$

and C is given by (A2). The nonzero components of E_{ab} are given by

$$\begin{aligned}
E_{11}/\theta^2 &= (3\gamma - 2)\, L/B^2 \\
E_{22}/\theta^2 &= [-(1/2)(3\gamma - 2 - 3s)\, L + M]/B^2 \\
E_{33}/\theta^2 &= [-(1/2)(3\gamma - 2 + 3s)\, L - M]/B^2
\end{aligned} \tag{A11}$$

and the magnitude of E_{ab} is

$$E^2/\theta^4 = (12L^2 + 3sLM + M^2)/B^4 \tag{A12}$$

The magnetic part of the Weyl tensor has only one nonzero component, H_{23}, and we obtain

$$H^2/\theta^4 = (1/64)\,\gamma^2\,(4 - 3\gamma)^2\,DC^2/B^4 \tag{A13}$$

When $\alpha_s > 0$ and $\alpha_m > 0$, as in Section 3, it is convenient to rescale $\alpha_m = 1$, and introduce a dimensionless time variable η by

$$\eta = T/T_* \tag{A14}$$

where T_* is the value of T at which $\sigma = 0$ [see equation (13)]. We define a dimensionless parameter α by

$$\alpha = \frac{1}{2}\left(\frac{3\gamma + 2}{3\gamma - 2}\right) T_*^{1 - 2/\gamma}\,\alpha_s = \frac{\gamma^2(3\gamma + 2)}{3(2 - \gamma)}\quad T_*^{2 - 4/(3\gamma)}\alpha_c \tag{A15}$$

the equality of these expressions following from the definition (13) of T_*. These changes simplify the expressions for A, B, C, D, L, and N as follows, where we again use the notation (A3) for powers of η:

$$\begin{aligned}
A^{2-\gamma} &= 1 + \alpha\,[3(2 - \gamma)\,\eta_+ + 2(3\gamma - 2)\,\eta_-]/(3\gamma + 2)\\
B &= 1 + \alpha\,[4\eta_+ + (3\gamma - 2)\,\eta_-]/(3\gamma + 2)\\
C &= -2(3\gamma - 2)\,\alpha(\eta_+ - \eta_-)/(3\gamma + 2)\\
D &= \alpha\eta_+ A^{2-\gamma}/3\\
L &= (1/18)(3\gamma - 2)\,\alpha(3\gamma\eta_+/2 + N\eta_-)/(3\gamma + 2)\\
N &= 1 + (1/2)\,\alpha\,[6\eta_+ - (4 - 3\gamma)\,\eta_-]
\end{aligned} \tag{A16}$$

Note that when $T = T_*$, i.e., $\eta = 1$, we have

$$A^{2-\gamma} = 1 + \alpha = B, \qquad C = 0, \qquad D = \alpha(1 + \alpha)/3 \tag{A17}$$

$$L = (3\gamma - 2)\,\alpha(1 + \alpha)/36, \qquad M = 0, \qquad N = 1 + (3\gamma + 2)\,\alpha/2 \tag{A18}$$

The following limits are used in Section 3, and are valid if $\alpha_s > 0$, $\alpha_m > 0$:

$$\lim_{T\to+\infty} C/B = -(3\gamma - 2)/2, \qquad \lim_{T\to+\infty} D/B^2 = s^2/16 \tag{A19}$$

$$\lim_{T\to+\infty} L/B^2 = 0, \qquad \lim_{T\to+\infty} M/B^2 = -s\gamma(4 - 3\gamma)(3\gamma - 2)/64$$

$$\lim_{T\to0^+} C/B = 2, \qquad \lim_{T\to0^+} L/B^2 = -(4 - 3\gamma)(3\gamma + 2)/[36(3\gamma - 2)]$$

$$\lim_{T\to0^+} D/B^2 = 0, \qquad \lim_{T\to0^+} M/B^2 = s\gamma(4 - 3\gamma)/[4(3\gamma - 2)] \tag{A20}$$

Finally in order to justify (14), we use (A16) to write σ/θ in the form

$$\frac{\sqrt{3}\,\sigma}{\theta} = \left|1 - \eta^{1+2/(3\gamma)}\right| \left[1 + \frac{4}{3\gamma - 2}\,\eta^{1+2/(3\gamma)} + \frac{(3\gamma+2)}{\alpha(3\gamma-2)}\,\eta^{-1+2/\gamma}\right]^{-1}$$

Thus if $\alpha > 0$ is unrestricted, σ/θ will be small if and only if $\eta \simeq 1$. We write $\eta = 1 + \Delta\eta$ with $|\Delta\eta| \ll 1$. Then σ/θ can be approximated by

$$\frac{\sigma}{\theta} \approx \frac{1}{\sqrt{3}}\left(\frac{3\gamma-2}{3\gamma}\right)\left(\frac{\alpha}{1+\alpha}\right)|\Delta\eta|$$

from which (14) follows.

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