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Stationary Electromagnetic Fields Around Black Holes. II. General Solutions and the Fields of Some Special Sources Near a Kerr Black Hole[1]

J. BIČÁK and L. DVOŘÁK

Department of Mathematical Physics, Faculty of Mathematics and Physics, The Charles University, Prague

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Abstract

The electromagnetic field of a general stationary source, occurring in the vicinity of a rotating (Kerr) black hole, is obtained by solving the Maxwell and Teukolsky equations. The field is expressed both outside and inside the radius at which the source is located. As examples the fields of point charges, charged rings, current loops, and magnetic dipoles not necessarily located in axisymmetric positions are calculated. The electromagnetic field occurring when a Kerr black hole is placed in an originally uniform magnetic field is derived without assuming the alignment of the direction of the magnetic field and the axis of symmetry of the black hole.

§(1): *Introduction*

The purpose of this paper is to study stationary electromagnetic fields on Kerr background: although we fully take into account the influence of the Kerr geometry on electromagnetic fields, we neglect the influence of these fields on the geometry.

[1] We have recently learned that A. King from the University of Hamburg has independently found the vacuum solutions given in Section 2 of the present paper. His work will be published in *Mathematics Proceedings of the Cambridge Philosophical Society*. In another paper (to be published in *Lettere al Nuovo Cimento*) he gives the fields of a stationary point charge and a stationary magnetic monopole.

In the first paper of this series [1], general stationary electromagnetic fields around a Schwarzschild black hole were constructed and the fields of some special sources were explicitly given.[2] As indicated in Ref. 1 the assumption of stationarity may be a good approximation in realistic situations provided the sources are not located too close (in proper distance) to the event horizon so that the force pulling a stationary source into the hole does not grow to infinity. The fields of stationary currents possibly arising in accretion disks around black holes or, asymptotically uniform (interstellar, intergalactic) magnetic fields, in the vicinity of a black hole (distorted by curved geometry) may serve as examples of astrophysically plausible strictly stationary fields. The magnetic field of a current loop in the equatorial plane of a Schwarzschild black hole was obtained by Petterson [2]; also fields of other types of stationary current loops in the Schwarzschild geometry were constructed in Ref. 1 (using the Newman-Penrose formalism). Here, we generalize most of the results given in Ref. 1 to the Kerr geometry.

The solution for the electromagnetic field occurring when a rotating black hole is placed in an originally uniform magnetic field aligned along the symmetry axis of the hole was recently derived by Wald [3]. We will give such a solution without assuming the alignment of the field along the axis of symmetry.

The only further solution known for a stationary electromagnetic field on the Kerr background was found by Cohen et al. [4]; it describes the field of a point charge at rest on the hole's axis of symmetry. Solutions corresponding to point charges will also be constructed here without assuming axial symmetry. All the special solutions mentioned above follow from our procedure of finding the electromagnetic field of a general stationary source occurring in Kerr space-time outside the event horizon.

In Section 2 the Maxwell equations and the Teukolsky equations [5] for electromagnetic perturbations are written down and their general stationary vacuum solutions are obtained. Special source terms describing point charges, current loops, and magnetic dipoles in various positions are constructed in Section 3. The procedure of finding the fields of general stationary sources is described in Section 4; here also the explicit forms of the fields of the special sources (considered in Section 3) are presented. In Section 5 the results are reformulated for the case of an extreme Kerr black hole. The solution for the field, generated by placing a rotating black hole in an originally uniform magnetic field, is given in Section 6. A few concluding remarks are added in Section 7.

[2] Both the motivations and calculations are described in greater detail in [1] than in the present paper. The manner of solving the field equations is, however, somewhat different here (even in performing the limit to the Schwarzschild hole). Furthermore, certain points, e.g., the classification of fields into purely "electric" and purely "magnetic," are typical only for the nonrotating case.

§(2): *General Stationary Vacuum Fields*

We start with the general form of the Maxwell equations in the Newman-Penrose (NP) formalism [1, 5]. Following Teukolsky [5], we choose the null tetrad in such a manner that its components in the Boyer-Lindquist coordinate system $\{t, r, \theta, \varphi\}$ read

$$l^\mu = [(r^2 + a^2)/\Delta, 1, 0, a/\Delta], n^\mu = \frac{1}{2\Sigma}[r^2 + a^2, -\Delta, 0, a]$$

$$m^\mu = \frac{1}{\sqrt{2}\,(r + ia\cos\theta)}[ia\sin\theta, 0, 1, i/\sin\theta], \tag{2.1}$$

where $\Delta = r^2 - 2Mr + a^2$, $\Sigma = r^2 + a^2\cos^2\theta$, M is the mass of the hole, and the constant $a\,(\leqslant M)$ (the angular momentum per unit mass of the hole) is (without any loss of generality) assumed to be nonnegative. With this choice of the tetrad, the spin coefficients and the NP differential operators are given in Ref. 5. We substitute them into the general NP form of the Maxwell equations. Instead of the usual NP electromagnetic field components,

$$\Phi_0 = F_{\mu\nu}l^\mu m^\nu, \quad \Phi_1 = \tfrac{1}{2}F_{\mu\nu}(l^\mu n^\nu + \overline{m}^\mu m^\nu), \quad \Phi_2 = F_{\mu\nu}\overline{m}^\mu n^\nu \tag{2.2}$$

($F_{\mu\nu}$ denoting the conventional electromagnetic field tensor, $\overline{m}^\mu$ the complex conjugate of m^μ), we shall use the quantities

$$\tilde{\Phi}_0 = \Phi_0, \quad \tilde{\Phi}_1 = \frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\Phi_1, \quad \tilde{\Phi}_2 = \frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\Phi_2 \tag{2.3}$$

where $r_\pm = M \pm (M^2 - a^2)^{1/2}$ are the coordinate radii of the outer and inner horizon, respectively. Putting all time derivatives equal to zero, we arrive at the Maxwell equations in the form

$$\sqrt{2}\,(r_+ - r_-)^2\left(\frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\varphi}\right)\tilde{\Phi}_1 - (r - ia\cos\theta)\left(\frac{\partial}{\partial\theta} + \cot\theta - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\tilde{\Phi}_0$$
$$+ ia\sin\theta\,\tilde{\Phi}_0 = \sqrt{2}\,(r - ia\cos\theta)^2\,2\pi J_l \tag{2.4}$$

$$\sqrt{2}\,(r_+ - r_-)^2\left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\tilde{\Phi}_1 + (r - ia\cos\theta)\left(\frac{\partial}{\partial r} - \frac{a}{\Delta}\frac{\partial}{\partial\varphi}\right)\Delta\tilde{\Phi}_0$$
$$- \Delta\tilde{\Phi}_0 = \sqrt{2}\,(r - ia\cos\theta)\,\Sigma\,2\pi J_m \tag{2.5}$$

$$\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\tilde{\Phi}_1 - (r - ia\cos\theta)\left(\frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial\varphi}\right)\tilde{\Phi}_2 + \tilde{\Phi}_2$$
$$= -\sqrt{2}\,\frac{(r - ia\cos\theta)^2}{(r_+ - r_-)^2}\,2\pi J_{\overline{m}} \tag{2.6}$$

$$\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial r}-\frac{a}{\Delta}\frac{\partial}{\partial\varphi}\right)\tilde{\Phi}_1+(r-ia\cos\theta)\left(\frac{\partial}{\partial\theta}+\cot\theta+\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)\frac{\tilde{\Phi}_2}{\Delta}$$

$$-\frac{ia\sin\theta}{\Delta}\tilde{\Phi}_2=-\sqrt{2}\,\frac{\Sigma(r-ia\cos\theta)}{\Delta(r_+-r_-)^2}\,2\pi J_n \qquad (2.7)$$

Here, the source terms are given by $J_l=l_\mu(j^\mu+i\mathfrak{M}^\mu)$, $J_m=m_\mu(j^\mu+i\mathfrak{M}^\mu)$, etc., with j^μ being the four current, and $\mathfrak{M}^\mu$ the "magnetic" four current (formed from "magnetic charges" similarly as j^μ is from electric charges); $\mathfrak{M}^\mu$ will be used for constructing magnetic dipoles. In equations (2.4)-(2.7), different Φs are combined in each equation, however, Teukolsky [5] found decoupled equations of the second order for Φ_0, Φ_2; moreover, he showed these equations to be completely separable in terms of "spin weighted spheroidal harmonics." In stationary cases the angular parts are given by the NP spin-weighted spherical harmonics ${}_sY_{lm}$ (see [6] for details on ${}_sY_{lm}$), so that we first expand the Φs in terms of ${}_sY_{lm}$s with the appropriate spin weights,

$$\tilde{\Phi}_0=\sum_{l,m}{}^0R_{lm}(r)\,{}_1Y_{lm}(\theta,\varphi) \qquad (2.8)$$

$$\tilde{\Phi}_2=\sum_{l,m}{}^2R_{lm}(r)\,{}_{-1}Y_{lm}(\theta,\varphi) \qquad (2.9)$$

($\Sigma_{l,m}$ is an abbreviation for $\Sigma_{l=1}^{\infty}\,\Sigma_{m=-l}^{l}$). From the Teukolsky equation for Φ_2 we obtain (using the orthonormality of ${}_{-1}Y_{lm}$s) the following equation for the radial part ${}^2R_{lm}$:

$$(r^2-2Mr+a^2)\frac{d^2({}^2R_{lm})}{dr^2}+\left[\frac{a^2m^2-2iam(r-M)}{r^2-2Mr+a^2}-l(l+1)\right]{}^2R_{lm}$$

$$=-4\pi\,{}^2J_{lm} \qquad (2.10)$$

Here the source term is given by

$${}^2J_{lm}(r)=\int_0^{2\pi}\int_0^{\pi}\frac{(r-ia\cos\theta)^2}{(r_+-r_-)^2}\,\Sigma\,J_2\;{}_{-1}\overline{Y}_{lm}\sin\theta\,d\theta\,d\varphi \qquad (2.11)$$

where $J_2(r,\theta,\varphi)$ is the combination of the NP components of the four current and their derivatives:

$$J_2=\frac{-\Delta}{2\sqrt{2}\,\Sigma(r-ia\cos\theta)^2}\left[\sqrt{2}\left(\frac{\partial}{\partial r}-\frac{a}{\Delta}\frac{\partial}{\partial\varphi}+\frac{1}{r-ia\cos\theta}\right)(r-ia\cos\theta)^2J_{\overline{m}}\right.$$

$$\left.+2\left(\frac{\partial}{\partial\theta}-\frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}+\frac{ia\sin\theta}{r-ia\cos\theta}\right)\frac{\Sigma(r-ia\cos\theta)}{\Delta}J_n\right] \qquad (2.12)$$

[We do not write the analogous equation for $\tilde{\Phi}_0$ (and J_0) because it will not be needed; $\tilde{\Phi}_0$, and also $\tilde{\Phi}_1$ (for $\tilde{\Phi}_1$ no separable equation exists) will be found from the original system of Maxwell equations (2.4)-(2.7).] Equation (2.10) is of the Fuchsian type and can thus be solved by means of hypergeometric functions. Assuming the source term to be equal to zero, we obtain the standard form of the hypergeometric equation for function ${}^2y_{lm}(x)$ which is given by the substitution

$${}^2R_{lm}(x) = \left(1 - \frac{1}{x}\right)^{-iZ_m} {}^2y_{lm}(x)$$

where

$$x = \frac{r - r_-}{r_+ - r_-}, \qquad Z_m = \frac{am}{r_+ - r_-} \tag{2.13}$$

are dimensionless. Two linearly independent solutions of the hypergeometric equation,

$$x(x-1)\,{}^2y''_{lm} - 2iZ_m\,{}^2y'_{lm} - l(l+1)\,{}^2y_{lm} = 0 \tag{2.14}$$

can conveniently be chosen as

$${}^2y^{(\mathrm{I})}_{lm} = \left(1 - \frac{1}{x}\right)^{2iZ_m} x(x-1)F(l+2, 1-l, 2-2iZ_m; x), \tag{2.15}$$

$${}^2y^{(\mathrm{II})}_{lm} = (-x)^{-l}F(l, l+1-2iZ_m, 2l+2; x^{-1}) \tag{2.16}$$

In the following, we need the derivatives of these solutions. Standard formulas for the differentiation of the hypergeometric function (see, e.g., Ref. 7) lead to the following results:

$$\begin{aligned}
\frac{d}{dx}[{}^2y^{(\mathrm{I})}_{lm}] &= -(1-2iZ_m)\left(1 - \frac{1}{x}\right)^{2iZ_m} F(l+1, -l, 1-2iZ_m; x) \\
\frac{d^2}{dx^2}[{}^2y^{(\mathrm{I})}_{lm}] &= 2iZ_m(2iZ_m - 1)\left(1 - \frac{1}{x}\right)^{2iZ_m} [x(x-1)]^{-1} \\
&\quad \cdot F(l, -l-1, -2iZ_m; x) \qquad \text{for } Z_m \neq 0 \\
&= l(l+1)F(l+2, 1-l, 2; x) \qquad \text{for } Z_m = 0 \\
\frac{d}{dx}[{}^2y^{(\mathrm{II})}_{lm}] &= l(-x)^{-l-1}F(l+1, l+1-2iZ_m, 2l+2; x^{-1}) \\
\frac{d^2}{dx^2}[{}^2y^{(\mathrm{II})}_{lm}] &= l(l+1)(-x)^{-l-2}F(l+2, l+1-2iZ_m, 2l+2; x^{-1})
\end{aligned} \tag{2.17}$$

In order to determine ${}^0R_{lm}$ outside a source, we turn to the Maxwell equations (2.4)-(2.7) with all source terms equal to zero. First operate with

$$\left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\varphi}\right)$$

on Eq. (2.4), then with

$$\left(\frac{\partial}{\partial r} + \frac{a}{\Delta}\frac{\partial}{\partial \varphi}\right)$$

on Eq. (2.6). Comparing the results obtained, inserting the expansions (2.8) and (2.9), and employing the properties of ${}_sY_{lm}$, we can express ${}^0R_{lm}$ in terms of ${}^2R_{lm}$.

$${}^0R_{lm} = \frac{2(r_+ - r_-)^2}{l(l+1)}\left(\frac{d}{dr} + \frac{iam}{\Delta}\right)\left(\frac{d}{dr} + \frac{iam}{\Delta}\right){}^2R_{lm} \tag{2.18}$$

Putting [similarly to (2.13)]

$${}^0R_{lm}(x) = \left(1 - \frac{1}{x}\right)^{-iZ_m} {}^0y_{lm}(x) \tag{2.19}$$

relation (2.18) simplifies to

$${}^0y_{lm} = \frac{2}{l(l+1)}\frac{d^2}{dx^2}[{}^2y_{lm}] \tag{2.20}$$

so that ${}^0y_{lm}^{(\mathrm{I}),(\mathrm{II})}$ and ${}^0R_{lm}^{(\mathrm{I}),(\mathrm{II})}$ can easily be obtained from ${}^2y_{lm}^{(\mathrm{I}),(\mathrm{II})}$ by using (2.17).

In this way we found $\tilde{\Phi}_2$ and $\tilde{\Phi}_0$. The calculation of $\tilde{\Phi}_1$ is slightly more complicated. Owing to axial symmetry we may write

$$\tilde{\Phi}_1(x,\theta,\varphi) = \sum_{m=-\infty}^{\infty}\left(1 - \frac{1}{x}\right)^{-iZ_m} e^{im\varphi}\,\tilde{\Phi}_{1m}(x,\theta) \tag{2.21}$$

Insert this expansion and the expansion (2.8) for $\tilde{\Phi}_0$ into (2.4) (with $J_l = 0$). Regarding (2.19) and the orthogonality of $e^{im\varphi}$, we can use (2.4) to express $\tilde{\Phi}_{1m}$ as the integral (over x) of the terms containing ${}^0y_{lm}(x)$ plus an arbitrary function of θ. Taking into account (2.20) and integrating by parts we finally arrive at

$$\begin{aligned}\tilde{\Phi}_1 = \frac{\sqrt{2}}{(r_+ - r_-)}\sum_{l,m}[l(l+1)]^{-1}\left(1 - \frac{1}{x}\right)^{-iZ_m}\Big\{[l(l+1)]^{1/2}\\ \cdot\left[(r - ia\cos\theta)\frac{d}{dx}({}^2y_{lm}) - (r_+ - r_-)({}^2y_{lm})\right]{}_0Y_{lm}(\theta,\varphi)\\ - ia\sin\theta\,\frac{d}{dx}({}^2y_{lm})\,{}_1Y_{lm}(\theta,\varphi)\Big\} + \sum_{m=-\infty}^{\infty} f_m(\theta)\,e^{im\varphi}\left(1 - \frac{1}{x}\right)^{-iZ_m}\end{aligned} \tag{2.22}$$

with $f_m(\theta)$ arbitrary. To determine f_ms we insert (2.22) and (2.8) [regarding (2.19)] into (2.5) (in which $J_m = 0$). Using the properties of ${}_sY_{lm}$, expressing ${}^0y_{lm}$ in terms of ${}^2y_{lm}$ by (2.20), and calculating the terms

$$\frac{d}{dx}\left[x(x-1)\frac{d^2}{dx^2}({}^2y_{lm})\right]$$

by differentiating (2.14), we find out that (2.5) implies

$$\left[\frac{\partial}{\partial\theta} - \frac{m}{\sin\theta}\right] f_m(\theta) = 0.$$

Similarly, substituting (2.22) into (2.6) (in which $J_{\overline{m}} = 0$), we arrive at the condition

$$\left[\frac{\partial}{\partial\theta} + \frac{m}{\sin\theta}\right] f_m(\theta) = 0$$

so that

$$f_m(\theta) = C\delta_{m0} \tag{2.23}$$

with C constant. It can now be shown that the last equation—(2.7) with $J_n = 0$—is satisfied identically.

In order to select physically appropriate solutions we have to investigate their asymptotic behavior at infinity and at the horizon. By using standard asymptotic formulas for hypergeometric functions [7] we find that (2.13) with (2.15), (2.16), and (2.19) with (2.20) and (2.17), imply ${}^2R_{lm}^{(\text{I})} \sim x^{l+1}$, ${}^0R_{lm}^{(\text{I})} \sim x^{l-1}$, ${}^2R_{lm}^{(\text{II})} \sim x^{-l}$, and ${}^0R_{lm}^{(\text{II})} \sim x^{-l-2}$ at $x \to \infty$. Thus only the solutions $R^{(\text{II})}$ are well-behaved at infinity. Near the horizon ($r \to r_+$, or $x \to 1$) we get ${}^2R_{lm}^{(\text{I})} \sim (x-1)\,[1-(1/x)]^{iZ_m}$, ${}^0R_{lm}^{(\text{I})} \sim (x-1)^{-1}\,[1-(1/x)]^{iZ_m}$ for $Z_m \neq 0$, and ${}^0R_{lm}^{(\text{I})} \sim$ const for $Z_m = 0$; further, ${}^2R_{lm}^{(\text{II})} \sim$ const, ${}^0R_{lm}^{(\text{II})} \sim (x-1)^{-1}\,[1 - 1/x)]^{iZ_m}$. The conditions on R physically required at the horizon, are [5] ${}^2R \lesssim (\text{const})\,(x-1)$, ${}^0R \lesssim (\text{const})\,\Delta^{-1}$, so that only $R^{(\text{I})}$ are admissible at the horizon. Finally, substituting ${}^2y_{lm}$ into (2.22), we can write

$$\tilde{\Phi}_1 = \sum_{l,m} \tilde{\Phi}_{1lm}\,(r,\theta,\varphi) + \text{const},$$

and make sure that $\tilde{\Phi}_{1lm}^{(\text{I})}$ (obtained from ${}^2y_{lm}^{(\text{I})}$) are admissible at the horizon, while $\tilde{\Phi}_{1lm}^{(\text{II})}$ (obtained from ${}^2y_{lm}^{(\text{II})}$) are admissible at infinity.

We summarize the results of this section by giving the original NP components (2.2) of the electromagnetic field of a source located between r_1 and r_2, with $r_+ < r_1 < r_2 < \infty$. In the region $r_+ \leqslant r < r_1$ the field reads as follows

$$\Phi_0 = \sum_{l,m} a_{lm}\,2[l(l+1)]^{-1}\left(1-\frac{1}{x}\right)^{-iZ_m}\frac{d^2}{dx^2}\,[{}^2y_{lm}^{(\text{I})}]\;{}_1Y_{lm}$$

$$\Phi_1 = \frac{\sqrt{2}\,(r_+ - r_-)}{(r - ia\cos\theta)^2} \sum_{l,m} a_{lm}\,[l(l+1)]^{-1} \left(1 - \frac{1}{x}\right)^{-iZ_m}$$

$$\cdot \left\{ [l(l+1)]^{1/2} \left[(r - ia\cos\theta) \frac{d}{dx} ({}^2y_{lm}^{(\mathrm{I})}) - (r_+ - r_-)\,{}^2y_{lm}^{(\mathrm{I})} \right] {}_0Y_{lm} \right. \tag{2.24}$$

$$\left. - ia\sin\theta \frac{d}{dx}({}^2y_{lm}^{(\mathrm{I})})\,{}_1Y_{lm} \right\} + \frac{E_a}{(r - ia\cos\theta)^2}$$

$$\Phi_2 = \frac{(r_+ - r_-)^2}{(r - ia\cos\theta)^2} \sum_{l,m} a_{lm} \left(1 - \frac{1}{x}\right)^{-iZ_m} {}^2y_{lm}^{(\mathrm{I})}\,{}_{-1}Y_{lm}$$

while for $r > r_2$,

$$\Phi_0 = \sum_{l,m} b_{lm}\, 2\,[l(l+1)]^{-1} \left(1 - \frac{1}{x}\right)^{-iZ_m} \frac{d^2}{dx^2} [{}^2y_{lm}^{(\mathrm{II})}]\,{}_1Y_{lm}$$

$$\Phi_1 = \frac{\sqrt{2}\,(r_+ - r_-)}{(r - ia\cos\theta)^2} \sum_{l,m} b_{lm}\,[l(l+1)]^{-1} \left(1 - \frac{1}{x}\right)^{-iZ_m}$$

$$\cdot \left\{ [l(l+1)]^{1/2} \left[(r - ia\cos\theta) \frac{d}{dx} ({}^2y_{lm}^{(\mathrm{II})}) - (r_+ - r_-)\,{}^2y_{lm}^{(\mathrm{II})} \right] {}_0Y_{lm} \right. \tag{2.25}$$

$$\left. - ia\sin\theta \frac{d}{dx}({}^2y_{lm}^{(\mathrm{II})})\,{}_1Y_{lm} \right\} + \frac{E_b}{(r - ia\cos\theta)^2}$$

$$\Phi_2 = \frac{(r_+ - r_-)^2}{(r - ia\cos\theta)^2} \sum_{l,m} b_{lm} \left(1 - \frac{1}{x}\right)^{-iZ_m} {}^2y_{lm}^{(\mathrm{II})}\,{}_{-1}Y_{lm}$$

Here, x and Z_m are given by (2.13), ${}^2y_{lm}^{(\mathrm{I}),(\mathrm{II})}$ and their derivatives by (2.15)-(2.17). The constants a_{lm}, b_{lm}, E_a, E_b are to be determined by the particular character of a source. Several concrete sources will be studied in the following section; the corresponding fields are constructed in Section 4.

§(3): *The Sources: Point Charges, Charged Rings, Current Loops, and Magnetic Dipoles*

In order to describe the sources from the point of view of local observers, we first introduce two sets of local proper reference frames [9].

(i) Local static frames (LSFs) which are at rest with respect to static observers at infinity and have orthonormal axes directed along the coordinate lines r, θ, φ.

(ii) Locally nonrotating frames (LNRFs—see, e.g., Refs. 8 and 9), which are orbiting around the black hole with r, θ fixed, and

$$\frac{d\varphi}{dt} = \omega = \frac{2Mar}{\mathcal{A}}, \qquad \mathcal{A} = (r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \tag{3.1}$$

The time-like axes of the LNRFs are perpendicular to the hypersurfaces $t =$ const, and two of the space-like axes are chosen along the coordinate lines r, θ. At a given point, r_0, θ_0, and φ_0, the transformation between the coordinates $\xi^{\hat{\mu}}$ of the LSF and the Boyer-Lindquist coordinates is given by

$$\begin{aligned} d\xi^{\hat{0}} &= [\Sigma_0 (\Sigma_0 - 2Mr_0)]^{-1/2} \, [(\Sigma_0 - 2Mr_0)\, dt + 2Mar_0 \sin^2 \theta_0 \, d\varphi] \\ d\xi^{\hat{1}} &= (\Sigma_0/\Delta_0)^{1/2} \, dr \\ d\xi^{\hat{2}} &= (\Sigma_0)^{1/2} \, d\theta \\ d\xi^{\hat{3}} &= [\Delta_0 \, \Sigma_0/(\Sigma_0 - 2Mr_0)]^{1/2} \sin \theta_0 \, d\varphi \end{aligned} \tag{3.2}$$

where $\Sigma_0 = r_0^2 + a^2 \cos^2 \theta_0$, $\Delta_0 = r_0^2 - 2Mr_0 + a^2$. The transformation between the coordinates $\eta^{\hat{\mu}}$ of the LNRF and the Boyer-Lindquist coordinates reads

$$\begin{aligned} d\eta^{\hat{0}} &= (\Delta_0 \, \Sigma_0/\mathcal{A}_0)^{1/2} \, dt \\ d\eta^{\hat{1}} &= (\Sigma_0/\Delta_0)^{1/2} \, dr \\ d\eta^{\hat{2}} &= (\Sigma_0)^{1/2} \, d\theta \\ d\eta^{\hat{3}} &= (\mathcal{A}_0/\Sigma_0)^{1/2} \sin \theta_0 \, [d\varphi - (2Mar_0/\mathcal{A}_0)\, dt] \end{aligned} \tag{3.3}$$

with $\mathcal{A}_0 = (r_0^2 + a^2)^2 - \Delta_0 a^2 \sin^2 \theta_0$. Neither LSF, nor LNRF is a local inertial frame. However, at a given point r_0, θ_0, and φ_0 we can momentarily choose an inertial frame, the origin of which is at rest at r_0, θ_0, and φ_0 and its axes coincide with the axes of the LSF (though the ones are rotating with respect to the others), so that the components of the four current of a source located at the common origin of the systems are the same in both systems. Similarly, another locally inertial frame can be chosen in which the four current has components equal to the ones measured in the LNRF at the point in question. (Owing to the superluminal velocity of static frames inside the ergosphere, LSFs have no good physical meaning there but LNRFs are physically meaningful for every $r > r_+$.)

The contravariant components of four currents transform according to

$$j^{\hat{\mu}}_{\mathrm{LSF}} = \frac{\partial \xi^{\hat{\mu}}}{\partial x^{\nu}} j^{\nu}, \qquad \mathfrak{M}^{\hat{\mu}}_{\mathrm{LSF}} = \frac{\partial \xi^{\hat{\mu}}}{\partial x^{\nu}} \mathfrak{M}^{\nu}$$
$$j^{\hat{\mu}}_{\mathrm{LNRF}} = \frac{\partial \eta^{\hat{\mu}}}{\partial x^{\nu}} j^{\nu}, \qquad \mathfrak{M}^{\hat{\mu}}_{\mathrm{LNRF}} = \frac{\partial \eta^{\hat{\mu}}}{\partial x^{\nu}} \mathfrak{M}^{\nu} \tag{3.4}$$

To construct the source terms we need the equation of continuity,

$$j^{\mu}{}_{;\mu} = \frac{1}{(-g)^{1/2}} \frac{\partial}{\partial x^{\mu}} \left((-g)^{1/2} j^{\mu}\right) = 0 \tag{3.5}$$

where $(-g)^{1/2} = \Sigma \sin \theta$. (The same equation holds for $\mathfrak{M}^{\mu}$.) The total charge contained in volume V is given by

$$e = \int_V (-g)^{1/2} j^0 \, dr \, d\theta \, d\varphi \tag{3.6}$$

(Since we assume that there are no magnetic monopoles, $\int_V (-g)^{1/2} \mathfrak{M}^0 \, dr \, d\theta \, d\varphi = 0$.) Note that the vanishing of the charge density in the LNRFs, $j^{\hat{0}}_{\mathrm{LNRF}} = 0$, implies $j^0 = 0$, so that the total charge is zero.

The covariant components of the null tetrad vectors are immediately obtained from (2.1) and then, knowing j^{μ} and $\mathfrak{M}^{\mu}$, the quantities $J_l = l_{\mu}(j^{\mu} + i\mathfrak{M}^{\mu})$, $J_m = m_{\mu}(j^{\mu} + i\mathfrak{M}^{\mu})$, etc., can be calculated. In order to find the field outside the source, we only need to know the source term in equation (2.10), i.e., ${}^2J_{lm}$, which is given in terms of $J_{\overline{m}}$ and J_n by (2.11) and (2.12). The explicit forms of $J_{\overline{m}}$ and J_n read as follows:

$$\begin{aligned} J_{\overline{m}} &= \overline{m}_{\mu}(j^{\mu} + i\mathfrak{M}^{\mu}) = [\sqrt{2}\,(r - ia\cos\theta)]^{-1} [-ia \sin\theta \, (j^0 + i\mathfrak{M}^0) \\ &\quad - \Sigma\,(j^2 + i\mathfrak{M}^2) + i\,(r^2 + a^2)\sin\theta\,(j^3 + i\mathfrak{M}^3)] \\ J_n &= n_{\mu}(j^{\mu} + i\mathfrak{M}^{\mu}) = \tfrac{1}{2}\,[(\Delta/\Sigma)\,(j^0 + i\mathfrak{M}^0) + j^1 + i\mathfrak{M}^1 \\ &\quad - (a\Delta/\Sigma)\sin^2\theta\,(j^3 + i\mathfrak{M}^3)] \end{aligned} \tag{3.7}$$

Now we find the source terms ${}^2J_{lm}$ for some special types of sources.

a. A Point Charge. The four current corresponding to a point charge e at rest at r_0, θ_0, and φ_0 is given by $j^0 = e\,(\Sigma_0 \sin\theta_0)^{-1}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0)$, $j^1 = j^2 = j^3 = 0$. This leads—via $J_{\overline{m}}$, J_n, and J_2—to

$$\begin{aligned} {}^2J_{lm} &= e\Delta[2\sqrt{2}\,(r_+ - r_-)^2(r_0 + ia\cos\theta_0)]^{-1}(ia\sin\theta_0 \;{}_{-1}\overline{Y}_{lm}\,(\theta_0,\varphi_0)\delta'(r - r_0) \\ &\quad + \{(a^2 m/\Delta_0)\sin\theta_0 \;{}_{-1}\overline{Y}_{lm}(\theta_0,\varphi_0) - [l(l+1)]^{1/2}\;{}_0\overline{Y}_{lm}(\theta_0,\varphi_0)\}\delta(r - r_0)) \end{aligned} \tag{3.8}$$

b. A Static Charged Axial Ring. Consider a thin ring composed of charges at rest, distributed symmetrically around the axis $\theta = 0, \pi$ at $r = r_0$. If the total charge of the ring is e, then

$$j^0 = e(2\pi\Sigma_0 \sin\theta_0)^{-1}\delta(r - r_0)\delta(\theta - \theta_0), \qquad j^1 = j^2 = j^3 = 0,$$

and

$$\begin{aligned} {}^2J_{lm} &= e\Delta\delta_{m0}[2\sqrt{2}\,(r_+ - r_-)^2(r_0 + ia\cos\theta_0)]^{-1}\{ia\sin\theta_0 \\ &\quad \cdot {}_{-1}\overline{Y}_{l0}(\theta_0, 0)\,\delta'(r - r^0) - [l(l+1)]^{1/2}\,{}_0\overline{Y}_{l0}(\theta_0, 0)\,\delta(r - r_0)\} \end{aligned} \tag{3.9}$$

c. An Axial Current Loop. Further, consider a thin current loop, symmetric around the axis $\theta = 0, \pi$, with a total zero charge. The four current of such a source is given by $j^0 = j^1 = j^2 = 0, j^3 = C\delta(r - r_0)\delta(\theta - \theta_0)$, where C is a constant.

Let us first focus on an infinitesimal current loop. Both the current and the magnetic dipole moment associated with such a loop can be expressed in the LSF located on the axis of symmetry. (Here the LSF coincides with the LNRF.) Regarding (3.2) with $\theta_0 \to 0$ and with the azimuthal coordinate φ replaced by ζ, such that $d\zeta = \sin\theta\, d\varphi$, we obtain

$$\begin{aligned} j^{\hat 0}_{\rm LFS} &= j^{\hat 1}_{\rm LFS} = j^{\hat 2}_{\rm LFS} = 0, \qquad j^{\hat 3}_{\rm LFS} = \frac{d\xi^{\hat 3}}{d\zeta}\, j^3 \\ &= C\Sigma_0^{1/2}\sin\theta_0\,\delta(\xi^{\hat 1} - \xi^{\hat 1}_0)\delta(\xi^{\hat 2} - \xi^{\hat 2}_0)\left\{\left|\frac{dr}{d\xi^{\hat 1}}\right|_{\xi^{\hat 1}_0}\left|\frac{d\theta}{d\xi^{\hat 2}}\right|_{\xi^{\hat 2}_0}\right\}^{-1} \\ &= C(\Sigma_0/\Delta_0)^{1/2}\Sigma_0\sin\theta_0\,\delta(\xi^{\hat 1} - \xi^{\hat 1}_0)\delta(\xi^{\hat 2} - \xi^{\hat 2}_0) \end{aligned}$$

Therefore, the total current as measured in the LSF is equal to $\mathcal{J} = C(\Sigma_0/\Delta_0)^{1/2}\Sigma_0 \sin\theta_0$. Since the area of the loop is given by $S = \pi\,(r_0^2 + a^2)\sin^2\theta_0$, the magnetic dipole moment of the loop can be expressed as $\mathfrak{M} = \mathcal{J}\cdot S = C\pi\sin^3\theta_0\Delta_0^{-1/2}(r_0^2 + a^2)^{5/2}$, which conversely yields C in terms of $\mathfrak{M}$. (In all these calculations we neglect the terms of higher order in $\sin\theta_0$.) The procedure described above leads to the source term

$$\begin{aligned} {}^2J_{lm} &= \frac{\mathfrak{M}\Delta\Delta_0^{1/2}(r_0 - ia)^2}{2\sqrt{2}\,(r_+ - r_-)^2(r_0^2 + a^2)^{5/2}}\,\delta_{m0}[l(l+1)]^{1/2}\,{}_0\overline{Y}_{l0}(0, 0) \\ &\quad \cdot [i(r_0 + ia)\delta'(r - r_0) + i\delta(r - r_0)] \end{aligned} \tag{3.10}$$

[In taking the limit $\theta_0 \to 0$, we used the relation

$$\lim_{\theta_0 \to 0} {}_1\overline{Y}_{l0}(\theta_0, 0)/\sin\theta_0 = \tfrac{1}{2}[l(l+1)]^{1/2}\,{}_0Y_{l0}(0, 0).]$$

Next consider a current loop lying in the equatorial plane around the hole ($r = r_0 > r_+, \theta = \pi/2$). The loop may be located inside the ergosphere, therefore, we will interpret constant C in terms of the total current $\mathcal{J}$ measured in the LNRFs. Since we will also study the thin ring of charges rotating around the black hole in the equatorial plane, we assume that in general the charge e of the loop is nonzero. Thus, in Boyer-Lindquist coordinates,

$$j^0 = (e/2\pi\Sigma_0)\delta(r - r_0)\delta(\theta - \pi/2), \quad j^1 = j^2 = 0, \quad \text{and} \quad j^3 = C\delta(r - r_0)\delta(\theta - \pi/2).$$

The transformation (3.3) leads to

$$\begin{aligned} j^{\hat{3}}_{\text{LNRF}} &= (\mathcal{A}_0/\Sigma_0)^{1/2}[C - Mar_0 e/\pi\Sigma_0\mathcal{A}_0]\,\delta(r - r_0)\delta(\theta - \pi/2) \\ &= (\mathcal{A}_0/\Delta_0)^{1/2} r_0 [C - Mae/\pi r_0 \mathcal{A}_0]\,\delta(\eta^{\hat{1}} - \eta_0^{\hat{1}})\delta(\eta^{\hat{2}} - \eta_0^{\hat{2}}). \end{aligned}$$

Since

$$j^{\hat{3}}_{\text{LNRF}} = \mathcal{J}\delta(\eta^{\hat{1}} - \eta_0^{\hat{1}})\delta(\eta^{\hat{2}} - \eta_0^{\hat{2}}),$$

we obtain

$$C = r_0^{-1}[(Mae/\pi\mathcal{A}_0) + \mathcal{J}(\Delta_0/\mathcal{A}_0)^{1/2}].$$

Taking into account the component j^3 only, we arrive at the source term in the form

$$\begin{aligned} {}^2J_{lm} = &-\frac{\Delta\delta_{m0}}{\sqrt{2}\,(r_+ - r_-)^2}\,[(Mae/\mathcal{A}_0) + \pi\,\mathcal{J}(\Delta_0/\mathcal{A}_0)^{1/2}] \\ &\cdot\left[i(r_0^2 + a^2)\,{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\delta'(r - r_0)\right. \\ &\left.+\left\{ir_0\,{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) - a[l(l+1)]^{1/2}\,{}_0\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right\}\delta(r - r_0)\right] \end{aligned} \tag{3.11}$$

If the loop does not carry a net charge (purely current loop), then (3.11) with $e = 0$ represents the final form of the source term. For $e \neq 0$ the total source term is given by the sum of (3.11) and (3.9) (with $\theta_0 = \pi/2$) [(3.9) corresponding to j^0].

Finally, consider a ring of charges rotating in the equatorial plane around the black hole with an angular velocity $d\varphi/dt = \Omega$. Owing to axial symmetry this system will not radiate, in spite of the motion of the individual charges. Each charge moves with respect to the LNRFs (located at the same $r = r_0$) with the velocity $v = d\eta^{\hat{3}}/d\eta^{\hat{0}} = r_0^{-2}\Delta_0^{-1/2}(\mathcal{A}_0\Omega - 2Mar_0)$. By focusing on a fixed (but arbitrary) event and, together with the LNRF at this event, also considering the proper reference frame in which a linear element of charges is at rest, we can easily calculate the four current in the LNRF. In this way we find that the total current $\mathcal{J}$, the linear density of charge ρ, and the velocity v (all quantities being measured in the LNRF) are related by

$$\mathcal{J} = \rho v = ver_0/2\pi\mathcal{A}_0^{1/2} \tag{3.12}$$

Thus, the source term corresponding to the rotating charged ring is given by the sum of (3.9) and (3.11) in which $\mathcal{J}$ is expressed by means of (3.12).

d. An Elementary Magnetic Dipole. It is of interest to compare the results for a small current loop around the axis of symmetry with those obtained for a corresponding elementary magnetic dipole. But we shall also study more general types of elementary magnetic dipoles.

First consider a "radial" magnetic dipole occurring at rest at r_0, θ_0, and φ_0. Such a source can be described by the "magnetic four current" with the components $\mathfrak{M}^{\hat{1}}_{\text{LSF}} = \mathfrak{M}^{\hat{2}}_{\text{LSF}} = \mathfrak{M}^{\hat{3}}_{\text{LSF}} = 0$ and $\mathfrak{M}^{\hat{0}}_{\text{LSF}} = -\mathfrak{M}\,\delta'(\xi^{\hat{1}} - \xi^{\hat{1}}_0)\delta(\xi^{\hat{2}} - \xi^{\hat{2}}_0)\,\delta(\xi^{\hat{3}} - \xi^{\hat{3}}_0)$, where $\mathfrak{M}$ is the magnitude of the dipole moment. (For $\mathfrak{M}$ positive, the dipole is pointing radially outwards.) In Boyer-Lindquist coordinates we find $\mathfrak{M}^1 = \mathfrak{M}^2 = \mathfrak{M}^3 = 0$, $\mathfrak{M}^0 = -\mathfrak{M}(\Delta_0/\Sigma_0)^{1/2}\,[(r^2 + a^2\cos^2\theta)\sin\theta_0]^{-1}\delta'(r - r_0)\delta(\theta - \theta_0)\delta(\varphi - \varphi_0)$. (Note that, indeed, $\int_V \mathfrak{M}^0\,(-g)^{1/2}\,dr\,d\theta\,d\varphi = 0$—the total magnetic "charge" vanishes.) After some calculations (using the well-known properties of ${}_sY_{lm}$) we arrive at the source term in the form

$$\begin{aligned}
{}^2J_{lm} = {} & \frac{\mathfrak{M}\Delta(\Delta_0/\Sigma_0)^{1/2}}{2\sqrt{2}\,(r_+ - r_-)^2(r_0 + ia\cos\theta_0)}\left(a\left[\delta''(r - r_0) + \frac{\delta'(r - r_0)}{r_0 + ia\cos\theta_0}\right]\right. \\
& \cdot \sin\theta_0\ {}_{-1}\overline{Y}_{lm}(\theta_0, \varphi_0) + i\left[\delta'(r - r_0) + \frac{\delta(r - r_0)}{r_0 + ia\cos\theta_0}\right] \\
& \cdot [l(l+1)]^{1/2}\ {}_0\overline{Y}_{lm}(\theta_0, \varphi_0) - m(ia^2/\Delta_0)\,\{\delta'(r - r_0) \\
& + [2(r_0 - M)\,\Delta_0^{-1} + (r_0 + ia\cos\theta_0)^{-1}]\ \delta(r - r_0)\} \\
& \left.\cdot \sin\theta_0\ {}_{-1}\overline{Y}_{lm}(\theta_0, \varphi_0)\right)
\end{aligned} \tag{3.13}$$

If $\theta_0 = 0$, this expression goes over to (3.10) so that the equivalence of a small current loop and an elementary magnetic dipole is established. (This equivalence could not have been proved so easily had we started from an equation other than that for Φ_2—see Ref. 1 for the Schwarzschild case.)

Second, consider an elementary magnetic dipole located in the equatorial plane and oriented perpendicularly to this plane. This source is described by

$\mathfrak{M}^{\hat{0}}_{\mathrm{LSF}} = \mathfrak{M}\delta(\xi^{\hat{1}} - \xi^{\hat{1}}_0)\delta'(\xi^{\hat{2}} - \xi^{\hat{2}}_0)\delta(\xi^{\hat{3}} - \xi^{\hat{3}}_0)$, $\mathfrak{M}^{\hat{1}}_{\mathrm{LSF}} = \mathfrak{M}^{\hat{2}}_{\mathrm{LSF}} = \mathfrak{M}^{\hat{3}}_{\mathrm{LSF}} = 0$. It follows that $\mathfrak{M}^0 = \mathfrak{M}\Sigma_0^{-1/2}(\Sigma \sin\theta)^{-1}\delta(r - r_0)\delta'(\theta - \pi/2)\delta(\varphi - \varphi_0)$ and $\mathfrak{M}^1 = \mathfrak{M}^2 = \mathfrak{M}^3 = 0$. The calculations then yield

$$\begin{aligned} {}^2J_{lm} = \frac{\mathfrak{M}\Delta}{2\sqrt{2}\,(r_+ - r_-)^2 r_0^2} \Bigg(a \Bigg[(m + iar_0^{-1})\, {}_{-1}\overline{Y}_{lm}\left(\frac{\pi}{2}, \varphi_0\right) - [l(l+1)]^{1/2} \\ \cdot {}_0\overline{Y}_{lm}\left(\frac{\pi}{2}, \varphi_0\right) \Bigg] \delta'(r - r_0) + i \Bigg\{ [l(l+1) - 1 - ma^2\Delta_0^{-1} \\ \cdot (m + iar_0^{-1})]\, {}_{-1}\overline{Y}_{lm}\left(\frac{\pi}{2}, \varphi_0\right) + (ma^2\Delta_0^{-1} + iar_0^{-1} - m) \\ \cdot [l(l+1)]^{1/2}\, {}_0\overline{Y}_{lm}\left(\frac{\pi}{2}, \varphi_0\right) \Bigg\} \delta(r - r_0) \Bigg) \end{aligned} \tag{3.14}$$

In Ref. 1 we also considered more complicated sources and found their fields. Since, however, the procedure is clear in principle and the calculations become rather involved in the Kerr case we will not construct any other source explicitly here.

§(4): *The Fields of Given Sources*

For all sources studied in the preceding section, in fact for all physically realistic stationary sources outside a Kerr black hole, r_1 and r_2 ($r_+ < r_1 < r_2 < \infty$) between which the sources are located, exist. The field outside this region is given by (2.24) and (2.25), in which the constant coefficients a_{lm} and b_{lm}, as yet undetermined, follow from solving the inhomogeneous equation (2.10). The present section is devoted to the procedure of calculating these coefficients and constants E_a, E_b in the case of a general source and, in particular, it gives the resulting fields of the sources considered in the preceding section.

Since the functions ${}^2R^{(\mathrm{I})}_{lm}$ and ${}^2R^{(\mathrm{II})}_{lm}$, given by (2.13) with ${}^2y_{lm}$ substituted from (2.15) and (2.16), represent the fundamental system of the homogeneous equation corresponding to (2.10), the solution of the full inhomogeneous equation (2.10) reads [10]

$$\begin{aligned} {}^2R_{lm}(x) = {}^2R^{(\mathrm{I})}_{lm}(x) \int \frac{4\pi\, {}^2J_{lm}(\xi)\, {}^2R^{(\mathrm{II})}_{lm}(\xi)}{\xi(\xi - 1) W({}^2R^{(\mathrm{I})}_{lm}, {}^2R^{(\mathrm{II})}_{lm}, \xi)}\, d\xi \\ - {}^2R^{(\mathrm{II})}_{lm}(x) \int \frac{4\pi\, {}^2J_{lm}(\xi)\, {}^2R^{(\mathrm{I})}_{lm}(\xi)}{\xi(\xi - 1) W({}^2R^{(\mathrm{I})}_{lm}, {}^2R^{(\mathrm{II})}_{lm}, \xi)}\, d\xi \end{aligned} \tag{4.1}$$

where $W({}^2R^{(\mathrm{I})}_{lm}, {}^2R^{(\mathrm{II})}_{lm}, \xi)$ is the Wronskian of ${}^2R^{(\mathrm{I})}_{lm}, {}^2R^{(\mathrm{II})}_{lm}$ at the point ξ. Since ${}^2J_{lm} = 0$ outside the region $\langle x_1 = (r_1 - r_-)/(r_+ - r_-)$ and $x_2 =$

$(r_2 - r_-)/(r_+ - r_-)\rangle$, we may integrate from $x_2 + \epsilon$ to x in the first integral in (4.1), and from $x_1 - \epsilon$ to x in the second integral, with ϵ being an arbitrary positive constant which only indicates that x_1 and x_2 are to be included into the integration. It is then easily seen that the boundary conditions required at infinity and at the horizon are satisfied by (4.1), and that the addition of any nontrivial linear combination of fundamental solutions would spoil the boundary conditions. Comparing (4.1) with (2.24) and (2.25) [and regarding (2.3), (2.9), (2.13)], we obtain

$$a_{lm} = -4\pi \int_{x_1-\epsilon}^{x_2+\epsilon} \frac{{}^2J_{lm}(\xi)\, {}^2R_{lm}^{(\mathrm{II})}(\xi)}{\xi(\xi-1)\, W({}^2R_{lm}^{(\mathrm{I})}, {}^2R_{lm}^{(\mathrm{II})}, \xi)}\, d\xi$$

$$b_{lm} = -4\pi \int_{x_1-\epsilon}^{x_2+\epsilon} \frac{{}^2J_{lm}(\xi)\, {}^2R_{lm}^{(\mathrm{I})}(\xi)}{\xi(\xi-1)\, W({}^2R_{lm}^{(\mathrm{I})}, {}^2R_{lm}^{(\mathrm{II})}, \xi)}\, d\xi$$

Now it can be shown that the Wronskian is a constant, the value of which can be determined at $x \to \infty$. The asymptotic behavior of ${}^2R_{lm}^{(\mathrm{I})}$ and ${}^2R_{lm}^{(\mathrm{II})}$ implies that

$$W({}^2R_{lm}^{(\mathrm{I})}, {}^2R_{lm}^{(\mathrm{II})}, \xi) = \frac{(2l+1)!\,\Gamma(2-2iZ_m)}{(l+1)!\,\Gamma(l+1-2iZ_m)}.$$

Thus, in their final form the coefficients a_{lm} and b_{lm} read as

$$a_{lm} = -4\pi \frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)} \int_{x_1-\epsilon}^{x_2+\epsilon} \frac{{}^2J_{lm}(\xi)\, {}^2R_{lm}^{(\mathrm{II})}(\xi)}{\xi(\xi-1)}\, d\xi \tag{4.2a}$$

$$b_{lm} = -4\pi \frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)} \int_{x_1-\epsilon}^{x_2+\epsilon} \frac{{}^2J_{lm}(\xi)\, {}^2R_{lm}^{(\mathrm{I})}(\xi)}{\xi(\xi-1)}\, d\xi \tag{4.2b}$$

In Appendix A, the Gauss theorem is employed in order to show that

$$E_a = 0, \qquad E_b = \tfrac{1}{2}e$$

where e is the total charge of a source bounded by r_1 and r_2. [If also the black hole carries a small net charge Q, then $E_a = \frac{1}{2}Q, E_b = \frac{1}{2}(e+Q)$.]

Finally, we give the resultant coefficients a_{lm} and b_{lm} for the sources discussed in the preceding section. With these coefficients known, the fields are determined uniquely by (2.24) and (2.25). It is convenient first to find the coefficients corresponding to three idealized source terms ${}^2J_{lm}$: $\Delta\delta(r-r_0)$, $\Delta\delta'(r-r_0)$, and $\Delta\delta''(r-r_0)$. Such coefficients are calculated in Appendix B. With the help of the results of this appendix, the following results can be derived without difficulty by substituting source terms ${}^2J_{lm}$ given by (3.8), (3.9), (3.10), (3.11), (3.13), and (3.14) into (4.2a,b).

a. The Field of a Point Charge e.

$$a_{lm} = \frac{2\pi e}{\sqrt{2}\,(r_+ - r_-)(r_0 + ia\cos\theta_0)} \frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)} \left(1 - \frac{1}{x_0}\right)^{-iZ_m}$$

$$\cdot (-x_0)^{-l} \left\{ -\frac{ia}{r_+ - r_-} \sin\theta_0 \; {}_{-1}\overline{Y}_{lm}(\theta_0, \varphi_0) \frac{l}{x_0} F(l+1, l+1 - 2iZ_m, 2l+2; x_0^{-1}) + [l(l+1)]^{1/2} \; {}_0\overline{Y}_{lm}(\theta_0, \varphi_0) \cdot F(l, l+1-2iZ_m, 2l+2; x_0^{-1}) \right\}$$

$$b_{lm} = \frac{2\pi e}{\sqrt{2}\,(r_+ - r_-)(r_0 + ia\cos\theta_0)} \frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)} \left(1 - \frac{1}{x_0}\right)^{iZ_m}$$

$$\cdot \left[-\frac{ia}{r_+ - r_-}(1 - 2iZ_m)\sin\theta_0 \; {}_{-1}\overline{Y}_{lm}(\theta_0, \varphi_0) \cdot F(l+1, -l, 1-2iZ_m; x_0) + [l(l+1)]^{1/2} \; {}_0\overline{Y}_{lm}(\theta_0, \varphi_0)\, x_0 (x_0 - 1) \cdot F(l+2, 1-l, 2-2iZ_m; x_0) \right] \quad (4.3)$$

$$E_a = 0, \qquad E_b = \tfrac{1}{2} e$$

b. The Field of a Static Charged Axial Ring.

$$a_{lm} = \frac{\delta_{m0}\, 2\pi e}{\sqrt{2}\,(r_+ - r_-)(r_0 + ia\cos\theta_0)} \frac{(l+1)!\,l!}{(2l+1)!} (-x_0)^{-l}$$

$$\cdot \left\{ -\frac{ia}{r_+ - r_-} \sin\theta_0 \; {}_{-1}\overline{Y}_{l0}(\theta_0, 0) \frac{l}{x_0} F(l+1, l+1, 2l+2; x_0^{-1}) + [l(l+1)]^{1/2} \; {}_0\overline{Y}_{l0}(\theta_0, 0) F(l, l+1, 2l+2; x_0^{-1}) \right\} \quad (4.4)$$

$$b_{lm} = \frac{\delta_{m0}\, 2\pi e}{\sqrt{2}\,(r_+ - r_-)(r_0 + ia\cos\theta_0)} \frac{(l+1)!\,l!}{(2l+1)!}$$

$$\cdot \left\{ -\frac{ia}{r_+ - r_-} \sin\theta_0 \; {}_{-1}\overline{Y}_{l0}(\theta_0, 0) F(l+1, -l, 1; x_0) + [l(l+1)]^{1/2} \cdot {}_0\overline{Y}_{l0}(\theta_0, 0)\, x_0 (x_0 - 1) F(l+2, 1-l, 2; x_0) \right\}$$

$$E_a = 0, \qquad E_b = \tfrac{1}{2} e$$

c. The Field of an Axial Current Loop.

1. For a small loop with the magnetic dipole moment $\mathfrak{M}$,

$$a_{lm} = -\frac{\delta_{m0} 2\pi i \mathfrak{M}\sqrt{\Delta_0}\,(r_0 - ia)^2}{\sqrt{2}\,(r_+ - r_-)(r_0^2 + a^2)^{5/2}} \frac{(l+1)!\,l!}{(2l+1)!} (-x_0)^{-l} \,[l(l+1)]^{1/2}$$
$$\cdot {}_0\overline{Y}_{l0}\,(0,0)\left[\frac{r_0 + ia}{r_+ - r_-}\frac{l}{x_0} F(l+1, l+1, 2l+2; x_0^{-1}) + F(l, l+1, 2l+2; x_0^{-1})\right] \tag{4.5}$$
$$b_{lm} = -\frac{\delta_{m0} 2\pi i \mathfrak{M}\sqrt{\Delta_0}\,(r_0 - ia)^2}{\sqrt{2}\,(r_+ - r_-)(r_0^2 + a^2)^{5/2}} \frac{(l+1)!\,l!}{(2l+1)!} [l(l+1)]^{1/2}\,{}_0\overline{Y}_{l0}\,(0,0)$$
$$\cdot\left[\frac{r_0 + ia}{r_+ - r_-} F(l+1, -l, 1; x_0) + x_0\,(x_0 - 1) F(l+2, 1-l, 2; x_0)\right]$$
$$E_a = E_b = 0$$

2. For a current loop in the equatorial plane with current $\mathfrak{J}$ (as measured in LNRF) and zero net charge,

$$a_{lm} = \frac{\delta_{m0} 4\pi^2 \mathfrak{J}}{\sqrt{2}\,(r_+ - r_-)} \frac{(l+1)!\,l!}{(2l+1)!} (\Delta_0/\Sigma_0)^{1/2} (-x_0)^{-l}\left[i\frac{r_0^2 + a^2}{r_+ - r_-}\right.$$
$$\cdot {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\frac{l}{x_0} F(l+1, l+1, 2l+2; x_0^{-1}) + \left\{ir_0\,{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right.$$
$$\left.\left. - a\,[l(l+1)]^{1/2}\,{}_0\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right\} F(l, l+1, 2l+2; x_0^{-1})\right]$$
$$b_{lm} = \frac{\delta_{m0} 4\pi^2 \mathfrak{J}}{\sqrt{2}\,(r_+ - r_-)} \frac{(l+1)!\,l!}{(2l+1)!} (\Delta_0/\Sigma_0)^{1/2}\left[i\frac{r_0^2 + a^2}{r_+ - r_-}\,{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right. \tag{4.6}$$
$$\cdot F(l+1, -l, 1; x_0) + \left\{ir_0\,{}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) - a\,[l(l+1)]^{1/2}\right.$$
$$\left.\left.\cdot\,{}_0\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right\} x_0(x_0 - 1) F(l+2, 1-l, 2; x_0)\right]$$
$$E_a = E_b = 0$$

3. For a charged ring rotating in the equatorial plane with constant angular velocity, the coefficients are given by the sum of the a, b given by (4.6) in the preceding case of an uncharged loop, and of the following expressions:

$$a_{lm} = \frac{\delta_{m0} 2\pi e}{\sqrt{2}\,(r_+ - r_-)} \frac{(l+1)!\,l!}{(2l+1)!} (r_0\, \mathfrak{a}_0)^{-1} (-x_0)^{-l} \left[-\frac{ia}{r_+ - r_-} \Delta_0 r_0^2 \frac{l}{x_0} \right.$$

$$\cdot {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) F(l+1, l+1, 2l+2; x_0^{-1}) + \left\{ 2i\,Mar_0^2\ {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right.$$

$$\left.\left. + (\mathfrak{a}_0 - 2Ma^2 r_0)\,[l(l+1)]^{1/2}\ {}_0\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right\} F(l, l+1, 2l+2; x_0^{-1}) \right]$$

$$b_{lm} = \frac{\delta_{m0} 2\pi e}{\sqrt{2}\,(r_+ - r_-)^3} \frac{(l+1)!\,l!}{(2l+1)!} (\Delta_0 / r_0\, \mathfrak{a}_0) \left[- ia(r_+ - r_-) r_0^2\ {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) \right. \tag{4.7}$$

$$\cdot F(l+1, -l, 1; x_0) + \left\{ 2i\,Mar_0^2\ {}_{-1}\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right) + (\mathfrak{a}_0 - 2Ma^2 r_0)\right.$$

$$\left.\left. \cdot [l(l+1)]^{1/2}\ {}_0\overline{Y}_{l0}\left(\frac{\pi}{2}, 0\right)\right\} F(l+2, 1-l, 2; x_0) \right]$$

$$E_a = 0, \qquad E_b = \tfrac{1}{2} e$$

these coefficients describe the field of a charged current loop at rest with respect to LNRFs (i.e., $\mathcal{J} = 0$). (This is obtained by calculating the a and b from the source term (3.11) with $\mathcal{J} = 0$ and adding the a and b given by (4.4) with $\theta_0 = \pi/2$.)

d. The Field of an Elementary Magnetic Dipole.

1. For a radial magnetic dipole located at r_0, θ_0, and φ_0, the source term (3.13) can be replaced by a simpler one which only contains the δ function and its first derivative, the second derivative being expressed in the manner indicated in Appendix B. If the radial magnetic dipole is located on the axis of symmetry, the coefficients a_{lm} and b_{lm} are, of course, given by (4.5). In a general case, the resulting expressions are rather involved and will not be presented here.

2. For a dipole located in the equatorial plane and oriented perpendicularly to this plane we find

$$a_{lm} = \frac{2\pi \mathfrak{M}}{\sqrt{2}\,(r_+ - r_-)^2 r_0^2} \frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)} \left(1 - \frac{1}{x_0}\right)^{-iZ_m} (-x_0)^{-l}$$

$$\cdot \left[\left\{ [l(l+1)]^{1/2}\ {}_0\overline{Y}_{lm}\left(\frac{\pi}{2}, \varphi_0\right) - \left(m + \frac{ia}{r_0}\right) {}_{-1}\overline{Y}_{lm}\left(\frac{\pi}{2}, \varphi_0\right) \right\} \right.$$

$$\cdot \frac{al}{x_0} F(l+1, l+1-2iZ_m, 2l+2; x_0^{-1}) + i\,(r_+ - r_-) \left\{ [1 - l(l+1)] \right.$$

$$\cdot\ {}_{-1}\overline{Y}_{lm}\left(\frac{\pi}{2},\varphi_0\right)+\left(m-\frac{ia}{r_0}\right)[l(l+1)]^{1/2}\ {}_0\overline{Y}_{lm}\left(\frac{\pi}{2},\varphi_0\right)\Bigg\}$$

$$\cdot\ F(l,l+1-2iZ_m,2l+2;x_0^{-1})\Bigg]$$

$$b_{lm}=\frac{2\pi\mathfrak{M}}{\sqrt{2}\,(r_+-r_-)^2r_0^2}\ \frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left(1-\frac{1}{x_0}\right)^{iZ_m} \tag{4.8}$$

$$\cdot\left[\left\{[l(l+1)]^{1/2}\ {}_0\overline{Y}_{lm}\left(\frac{\pi}{2},\varphi_0\right)-\left(m+\frac{ia}{r_0}\right){}_{-1}\overline{Y}_{lm}\left(\frac{\pi}{2},\varphi_0\right)\right\}\right.$$

$$\cdot\ (1-2iZ_m)F(l+1,-l,1-2iZ_m;x_0)+i\,(r_+-r_-)$$

$$\cdot\left\{[1-l(l+1)]\ {}_{-1}\overline{Y}_{lm}\left(\frac{\pi}{2},\varphi_0\right)+\left(m-\frac{ia}{r_0}\right)[l(l+1)]^{1/2}\right.$$

$$\left.\cdot\ {}_0\overline{Y}_{lm}\left(\frac{\pi}{2},\varphi_0\right)\right\}x_0(x_0-1)F(l+2,1-l,2-2iZ_m;x_0)\Bigg]$$

$$E_a=E_b=0$$

Some remarks concerning the properties of the above solutions are contained in the last section.

§(5): *Stationary Fields on the Background of an Extreme Kerr Black Hole*

We will now outline how the preceding results are modified in case of an extremely rotating black hole ($a=M$). Introducing [in analogy with (2.3)] the quantities

$$\widetilde{\Phi}_0=\Phi_0,\ \widetilde{\Phi}_1=(r-iM\cos\theta)^2M^{-2}\Phi_1,\qquad \widetilde{\Phi}_2=(r-iM\cos\theta)^2M^{-2}\Phi_2 \tag{5.1}$$

we can write the Maxwell equations in a form corresponding to (2.4)-(2.7). Using the expansions (2.8) and (2.9), one arrives at the Teukolsky equation for ${}^2R_{lm}$:

$$(r-M)^2\frac{d^2}{dr^2}({}^2R_{lm})+\left[\frac{M^2m^2-2iMm\,(r-M)}{(r-M)^2}-l(l+1)\right]{}^2R_{lm}=-4\pi\,{}^2J_{lm} \tag{5.2}$$

where

$${}^2J_{lm}=\int_0^{2\pi}\int_0^{\pi}(r-iM\cos\theta)^2M^{-2}\ \Sigma J_2\sin\theta\ {}_{-1}\overline{Y}_{lm}\,d\theta\,d\varphi$$

Notice that all source terms given for $a < M$, can easily be rewritten for $a = M$ by replacing $(r_+ - r_-)$ by M (and by putting $a = M$). Introducing $x = (r/M) - 1$, we first find that, for $m = 0$, two independent solutions of the homogeneous equation following from (5.2) are ${}^2R_{l0}^{(\mathrm{I})} = x^{l+1}$ and ${}^2R_{00}^{(\mathrm{II})} = x^{-l}$. If $m \neq 0$, write

$${}^2R_{lm}(x) = \exp{(im/x)}\, {}^2y_{lm}\,(x)$$

and, further,

$${}^2y_{lm}(x) = x^{-l}\eta_{lm}\,(\xi)$$

where $\xi = -2im/x$. These substitutions lead to the confluent hypergeometric equation

$$\xi\eta_{lm}'' + (2l + 2 - \xi)\,\eta_{lm}' - l\eta_{lm} = 0$$

Two linearly independent solutions can be chosen as

$$\eta_{lm}^{(\mathrm{I})} = (-2im)^{2l+1} \exp{(\xi)}\, \xi^{-(2l+1)}\, \Phi\,(1 - l, -2l; -\xi)$$

$$\eta_{lm}^{(\mathrm{II})} = \Phi\,(l, 2l + 2, \xi)$$

with Φ denoting the confluent hypergeometric function, so that

$$\begin{aligned} {}^2y_{lm}^{(\mathrm{I})} &= x^{l+1} \exp{(-2im/x)}\, \Phi\,(1 - l, -2l; 2im/x) \\ {}^2y_{lm}^{(\mathrm{II})} &= x^{-l}\, \Phi\,(l, 2l + 2, -2im/x) \end{aligned} \tag{5.3}$$

[Since $\Phi\,(1 - l, -2l; 0) = \Phi\,(l, 2l + 2; 0) = 1$, these solutions are also meaningful if $m = 0$.] Employing a procedure analogous to that in Section 2, we find ${}^0R_{lm}$ to be given in terms of ${}^2R_{lm}$ as in (2.18), only $(r_+ - r_-)$ is replaced by M. Then, putting ${}^0R_{lm}\,(x) = \exp{(im/x)}\, {}^0y_{lm}\,(x)$, we obtain the relation (2.20). Further, $\tilde{\Phi}_1$ can be expressed in terms of ${}^2y_{lm}$ and its derivatives as in (2.22), the only difference being that $[(1 - (1/x)]^{-iZ_m}$ is replaced by $\exp{(im/x)}$, and $(r_+ - r_-)$ by M. The final forms of the vacuum solutions describing sources bounded by r_1 and r_2 follow from (2.24) and (2.25), provided that the replacements $a \to M$, $(r_+ - r_-) \to M$, and $[1 - (1/x)]^{-iZ_m} \to \exp{(im/x)}$ are made. Of course, ${}^2y_{lm}^{(\mathrm{I})}$ and ${}^2y_{lm}^{(\mathrm{II})}$ are now given by (5.3). In case of a general source, the coefficients a_{lm} and b_{lm} turn out to be given by simple formulas [compare with (4.2a, b)]:

$$a_{lm} = \frac{4\pi}{2l + 1} \int_{x_1 - \epsilon}^{x_2 + \epsilon} {}^2J_{lm}\,(\xi)\, \xi^{-2}\, {}^2R_{lm}^{(\mathrm{II})}\,(\xi)\, d\xi \tag{5.4a}$$

$$b_{lm} = \frac{4\pi}{2l + 1} \int_{x_1 - \epsilon}^{x_2 + \epsilon} {}^2J_{lm}\,(\xi)\, \xi^{-2}\, {}^2R_{lm}^{(\mathrm{I})}\,(\xi)\, d\xi \tag{5.4b}$$

The values of these coefficients for sources of a δ function type are given in Appendix B. The explicit forms of the coefficients describing the fields of the sources studied in Section 3 are given in Ref. 11.

§(6): *Rotating Black Hole in a Uniform Magnetic Field*

Finally, let us turn to an astrophysically interesting problem of finding the electromagnetic field which is generated by placing a Kerr black hole in an originally uniform magnetic field. In general, the direction of the magnetic field at infinity will not coincide with the direction of the hole's rotation axis. A configuration with both directions aligned will eventually be established, however, this will occur after a time, which is expected to be very long (cf. Ref. 12 for an analogous problem with a scalar field). Here we are going to find out how the hole's geometry modifies the originally uniform magnetic field, without studying how the magnetic field influences the rotation of the hole.

Denoting the components of a uniform magnetic field in asymptotically Minkowskian coordinates $x = r\sin\theta\cos\varphi$, $y = r\sin\theta\sin\varphi$, and $z = r\cos\theta$ by B_0^x, B_0^y, and B_0^z, the electromagnetic field tensor in Boyer–Lindquist coordinates at $r \to \infty$ has the form

$$\begin{aligned}
F^{01} &= F^{02} = F^{03} = 0 \\
F^{12} &= r^{-1}\,(\cos\varphi B_0^y - \sin\varphi B_0^x) \\
F^{13} &= (r\sin\theta)^{-1}\,[\sin\theta\, B_0^z - \cos\theta\,(\sin\varphi B_0^y + \cos\varphi B_0^x)] \\
F^{23} &= (r^2\sin\theta)^{-1}\,[\cos\theta\, B_0^z + \sin\theta\,(\sin\varphi B_0^y + \cos\varphi B_0^x)]
\end{aligned} \tag{6.1}$$

With a view to (2.2) and (2.1) (and the properties of spin s spherical harmonics), we can write the asymptotic form of the corresponding NP components as follows:

$$\begin{aligned}
\Phi_0 &= i\,(4\pi/3)^{1/2}\,[2^{-1/2}\,(B_0^x + iB_0^y)\,{}_1Y_{1\,-1}\,(\theta,\varphi) + B_0^z\,{}_0Y_{10}\,(\theta,\varphi) \\
&\quad - 2^{-1/2}\,(B_0^x - iB_0^y)\,{}_1Y_{11}\,(\theta,\varphi)] \\
\Phi_1 &= (i/2)\,(4\pi/3)^{1/2}\,[2^{-1/2}\,(B_0^x + iB_0^y)\,{}_0Y_{1\,-1}\,(\theta,\varphi) + B_0^z\,{}_0Y_{10}\,(\theta,\varphi) \\
&\quad - 2^{-1/2}\,(B_0^x - iB_0^y)\,{}_0Y_{11}\,(\theta,\varphi)] \\
\Phi_2 &= (i/2)\,(4\pi/3)^{1/2}\,[2^{-1/2}\,(B_0^x + iB_0^y)\,{}_{-1}Y_{1\,-1}\,(\theta,\varphi) + B_0^z\,{}_{-1}Y_{10}\,(\theta,\varphi) \\
&\quad - 2^{-1/2}\,(B_0^x - iB_0^y)\,{}_0Y_{11}\,(\theta,\varphi)]
\end{aligned} \tag{6.2}$$

It is now easy to find the field we are looking for. It must satisfy the same boundary conditions at the horizon as all the fields studied hitherto, however, at infinity it must go over into (6.2). Regarding (2.24), it is seen that such a field is given by (2.24) with $E_a = 0$, and

$$\begin{aligned}
a_{1\,-1} &= i\,(\pi/6)^{1/2}\,(B_0^x + iB_0^y) \\
a_{10} &= i\,(\pi/3)^{1/2}\,B_0^z \\
a_{11} &= -i\,(\pi/6)^{1/2}\,(B_0^x - iB_0^y)
\end{aligned} \tag{6.3}$$

(all the other a_{lm} being equal to zero) everywhere outside the hole. With small modifications (see Section 5), (2.24) represents the field on the background of an extremely rotating black hole.

In calculating the electromagnetic field tensor $F^{\mu\nu}$, we observe that the presence of the hole's angular momentum also gives rise to an electric field though the field is purely magnetic at infinity. If the rotation axis of the hole and the asymptotic direction of the magnetic field are aligned, our results coincide with those of Wald [3]. (Note that Wald's method of finding the field cannot be extended to obtain the field in the general case treated here.)

§(7): *Concluding Remarks*

The expressions (2.24) and (2.25) with a_{lm}, b_{lm}, ${}^2y_{lm}^{(\mathrm{I})}$, and ${}^2y_{lm}^{(\mathrm{II})}$ given by (4.2a,b), (2.15), and (2.16), respectively, and with $E_a = 0$ and $E_b = \frac{1}{2}e$, represent the vacuum electromagnetic field outside and inside the radius at which a given source (with total charge e) is located in Kerr space-time. The special sources studied in this paper had a δ function character. However, the procedure used to obtain the fields of such sources can easily be generalized to render it applicable to finding fields inside spatially extended sources (e.g., in plasma clouds in the vicinity of a black hole).

On inspecting the behavior of the fields of the sources given in Section 4, we find that $b_{lm} \to 0$ as the sources approach the horizon quasistatically in a physically permissible way. For example, if a charge is lowered toward the hole along the axis of symmetry or if current loops contract quasistatically towards the horizon, the fields outside the radii, at which the sources occur, decrease to zero and a Kerr-Newman black hole is created. However, if a charge is lowered to the horizon in the equatorial plane, it moves with a superluminal velocity when entering the ergosphere, and the coefficients b_{lm} do not approach zero (a similar result is true in case of a magnetic dipole in the equatorial plane). Of course, as indicated in Section 1, the fields of sources very near the horizon (in proper distance) are necessarily dynamical so that the above considerations cannot serve as more than an indication of the validity of the "no-hair hypothesis."

More astrophysical problems connected with some of the solutions discussed here will be dealt with in a future paper.

Appendix A: The Gauss Theorem

In the Boyer-Lindquist coordinates the Gauss theorem can be written in the form

$$\int_0^{2\pi} \int_0^{\pi} (-g)^{1/2}\, F^{01}\, d\theta\, d\varphi = 4\pi e(r_0) \tag{A.1}$$

where the integral is taken over the "sphere" $r = r_0$ which contains the total charge $e(r_0)$. Expressing $F_{\mu\nu}$ by means of the Φ and regarding (2.1), we find

$$F^{01} = (r_0^2 + a^2)\,\Sigma^{-1}\,\Phi_1 - ia \sin\theta\,[\sqrt{2}\,(r_0 + ia\cos\theta)]^{-1}\,\Phi_2$$
$$+ ia\Delta_0 \sin\theta\,[2\sqrt{2}\,\Sigma(r_0 - ia\cos\theta)]^{-1}\,\Phi_0 + \text{c.c.}$$

where "c.c." denotes the terms complex conjugated of the preceding terms. Owing to the integration over φ, axially nonsymmetric terms do not contribute to the integral in (A.1). The axially symmetric parts satisfy the relation $\Delta \cdot \Phi_0^{(\text{a.s.})} = -2(r - ia\cos\theta)^2\,\Phi_2^{(\text{a.s.})}$, as can be learned from (2.24), (2.25), and (2.14) with $m = 0$. Now, some calculations (which employ the properties of ${}_sY_{lm}$ and their derivatives) show that, of all terms in (2.24) and (2.25), only those proportional to E_a and E_b give a nonzero contribution to the lhs of (A.1). As the result we find that (A.1) reduces to

$$E + \overline{E} = e(r_0) \tag{A.2}$$

where

$$E = E_a \quad \text{if } r_0 < r_1, \qquad E = E_b \quad \text{if } r_0 > r_2$$

Starting out with the Gauss theorem for "magnetic charges,"

$$\int_0^{2\pi}\int_0^{\pi} (-g)^{1/2}\,F^{*01}\,d\theta\,d\varphi = 0$$

and making calculations similar to the preceding ones, we arrive at the relation $E - \overline{E} = 0$ which, when combined with (A.2), yields

$$E = \tfrac{1}{2}e(r_0)$$

Assuming the charge of the black hole to vanish, and the total charge of a source located in the region $\langle r_1, r_2\rangle$ to be equal to e, we conclude that the constants E_a and E_b in (2.24) and (2.25) are given by

$$E_a = 0, \qquad E_b = \tfrac{1}{2}e$$

We note that the same result can be derived in the case of an extreme Kerr black hole ($a = M$).

Appendix B: The Coefficients a_{lm} and b_{lm} for the Sources of a δ Function Type

I. The case $a < M$.

1. Let ${}^2J_{lm} = C\Delta\delta(r - r_0) = C(r_+ - r_-)x(x - 1)\delta(x - x_0)$, with C constant and r_0 such that $r_1 \leqslant r_0 \leqslant r_2$. Substituting this source term into (4.2), where

the ${}^2R_{lm}$ are expressed from (2.13), (2.15), and (2.16), we obtain

$$a_{lm} = -4\pi C\,(r_+ - r_-)\,\frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left(1-\frac{1}{x_0}\right)^{-iZ_m}(-x_0)^{-l}$$
$$\cdot F(l, l+1-2iZ_m, 2l+2; x_0^{-1})$$
$$b_{lm} = -4\pi C\,(r_+ - r_-)\,\frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left(1-\frac{1}{x_0}\right)^{iZ_m} x_0\,(x_0-1)$$
$$\cdot F(l+2, 1-l, 2-2iZ_m; x_0)$$

2. Let ${}^2J_{lm} = C\Delta\delta'(r-r_0) = Cx\,(x-1)\delta'(x-x_0)$. We then find [using (2.15)-(2.17)] that

$$a_{lm} = -4\pi C\,\frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left(1-\frac{1}{x_0}\right)^{-iZ_m}(-x_0)^{-l}$$
$$\cdot\left[\frac{iZ_m}{x_0\,(x_0-1)}\,F(l, l+1-2iZ_m, 2l+2; x_0^{-1}) + \frac{l}{x_0}\right.$$
$$\left.\cdot F(l+1, l+1-2iZ_m, 2l+2; x_0^{-1})\right]$$
$$b_{lm} = -4\pi C\,\frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left(1-\frac{1}{x_0}\right)^{iZ_m}$$
$$\cdot\left[iZ_m\,F(l+2, 1-l, 2-2iZ_m; x_0) + (1-2iZ_m)F(l+1, -l, 1-2iZ_m; x_0)\right]$$

3. Let ${}^2J_{lm} = C\Delta\delta''(r-r_0) = C\,(r_+ - r_-)^{-1}x\,(x-1)\delta''(x-x_0)$. Instead of substituting for the R's directly from (2.13), (2.15), and (2.16) we can rewrite (4.2) as follows:

$$a_{lm} = -\frac{4\pi C}{(r_+ - r_-)}\,\frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left\{\frac{d^2}{dx^2}\,[{}^2R_{lm}^{(\mathrm{II})}]\right\}_{x_0}$$
$$= \frac{4\pi C}{(r_+ - r_-)}\,\frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left[\frac{Z_m^2 - iZ_m\,(2x_0-1)}{x_0\,(x_0-1)} - l(l+1)\right]{}^2R_{lm}^{(\mathrm{II})}(x_0)$$
$$= -\frac{4\pi C}{(r_+ - r_-)}\,\frac{(l+1)!\,\Gamma(l+1-2iZ_m)}{(2l+1)!\,\Gamma(2-2iZ_m)}\left[-\frac{Z_m^2 - iZ_m\,(2x_0-1)}{x_0\,(x_0-1)} + l(l+1)\right]$$
$$\cdot\int_{x_1-\epsilon}^{x_2+\epsilon} {}^2R_{lm}^{(\mathrm{II})}(x)\delta(x-x_0)\,dx$$

where we have used the homogeneous equation associated with (2.10). An analogous result can be derived for b_{lm}. Therefore we see that the source term

${}^2J_{lm} = C\Delta\delta''\,(r - r_0)$ can be replaced by

$${}^2J_{lm} = \frac{C}{(r_+ - r_-)^2}\left[l(l+1) - \frac{Z_m^2 - iZ_m\,(2x_0 - 1)}{x_0\,(x_0 - 1)}\right]\Delta\delta\,(r - r_0)$$

II. The case $a = M$.

1. For ${}^2J_{lm} = C\Delta\delta\,(r - r_0) = CMx^2\delta\,(x - x_0)$, we obtain

$$a_{lm} = C4\pi M\,(2l+1)^{-1}\exp\,(im/x_0)\,x_0^{-l}\,\Phi\,(l, 2l+2, -2im/x_0)$$

$$b_{lm} = C4\pi M\,(2l+1)^{-1}\exp\,(-im/x_0)\,x_0^{l+1}\,\Phi\,(1 - l, -2l, 2im/x_0).$$

2. If ${}^2J_{lm} = C\Delta\delta'\,(r - r_0) = Cx^2\delta'\,(x - x_0)$, then

$$a_{lm} = C4\pi\,(2l+1)^{-1}\exp\,(im/x_0)\,x_0^{-l}\,[(im/x_0^2)\,\Phi\,(l, 2l+2, -2im/x_0) + (l/x_0)\,\Phi\,(l+1, 2l+2, -2im/x_0)]$$

$$b_{lm} = C4\pi\,(2l+1)^{-1}\exp\,(-im/x_0)\,x_0^{l+1}\,[(im/x_0^2)\,\Phi\,(1 - l, -2l, 2im/x_0) - (l+1)\,x_0^{-1}\,\Phi\,(-l, -2l, 2im/x_0)]$$

3. Finally, we find that ${}^2J_{lm} = C\Delta\delta''\,(r - r_0) = CM^{-1}x^2\delta''\,(x - x_0)$ can be replaced by

$${}^2J_{lm} = CM^{-2}\,[l(l+1) - (m^2 - 2imx_0)\,x_0^{-2}]\,\Delta\delta\,(r - r_0)$$

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