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Charged Fluid Sphere in General Relativity

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Abstract

In this paper a solution for the interior metric of a uniformly charged static fluid sphere has been obtained. The model sphere obtained has a physically reasonable equation of state. It is found that both the central density and the pressure become infinite when $\epsilon = \frac{2}{5}(1 + \sigma)$. Here $\epsilon = m_0/r_0$, $\sigma = Q_0^2/r_0^2$; m_0, r_0, and Q_0 are, respectively, the mass, radius, and charge of the sphere. In the limit $\sigma \to 0$ the solution becomes identical to the Adler solution.

§(1): *Introduction*

Because of the nonlinearity of Einstein's field equations exact solutions, in closed analytic form, are difficult to obtain. A model approach to the solution to these equations in the case of an uncharged perfect fluid sphere is well known [1-3].

Recently Adler [4] has obtained the solution to Einstein's equations for the interior of a static fluid sphere in closed analytic form. In this paper we generate a closed analytic solution to the Einstein's equations for a uniformly charged fluid sphere by a method similar to that used by Adler. When $\sigma \equiv 0$ (uncharged case) our results are in exact agreement with those of Adler. This work may then be considered as a generalization of Adler's paper.

§(2): *Solutions of the Field Equations*

The Einstein–Maxwell field equations for an ideal matter fluid are [5]

$$G_{ij} = -8\pi T_{ij} \tag{2.1}$$

$$T_{ij} = \rho u_i u_j + p(u_i u_j - g_{ij}) + \tfrac{1}{4}[g^{kl}F_{ik}F_{jl} - \tfrac{1}{4}g_{ij}F_{kl}F^{kl}] \tag{2.2}$$

$$[(-g)^{1/2}F^{ij}]_{,j} = 4\pi J^i(-g)^{1/2} \tag{2.3}$$

and

$$F_{[ij,k]} = 0 \tag{2.4}$$

where G_{ij} is the Einstein tensor, $u^i = dx^i/ds$ is the four velocity of a fluid element, F^{ij} is the electromagnetic field tensor, J^i is the electric current, and g_{ij} is the metric. (In this paper units are chosen so that $c = \kappa = 1$.) Spherical symmetry requires only the radial component of the electric field, $F^{01} = -F^{10}$, to be nonvanishing.

The appropriate line element for a static spherically symmetric system is

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.5}$$

where ν and λ are functions of r only which vanish as $r \to \infty$. The field equations may now be written in the forms [5]

$$8\pi\rho - \frac{Q^2}{r^4} = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) \tag{2.6}$$

$$8\pi p + \frac{Q^2}{r^4} = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) - \frac{1}{r^2} \tag{2.7}$$

$$8\pi p - \frac{Q^2}{r^4} = e^{-\lambda}\left[\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu'\lambda'}{4} + \frac{\nu' - \lambda'}{2r}\right] \tag{2.8}$$

where

$$Q = 4\pi \int_0^r J^0 r^2 e^\alpha dr \tag{2.9}$$

is the charge up to radius r. The corresponding electric field is given by

$$F^{01} = (Q/r^2)e^{-\alpha}, \qquad \alpha = (\lambda + \nu)/2 \tag{2.10}$$

In these equations a prime denotes differentiation with respect to r. Using equation (2.7) we may eliminate p from equation (2.8) and thereby write the field equations in the final forms:

$$8\pi\rho - \frac{Q^2}{r^4} = \frac{1}{r^2} - e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) \tag{2.11}$$

$$8\pi p + \frac{Q^2}{r^4} = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) - \frac{1}{r^2} \tag{2.12}$$

and

$$\left(\frac{1}{r^2} + \frac{2Q^2}{r^2}\right)e^\lambda = \frac{\nu'\lambda'}{4} - \frac{\nu''}{2} - \frac{\nu'^2}{4} + \frac{\lambda' + \nu'}{2r} + \frac{1}{r^2} \tag{2.13}$$

To solve equation (2.13) we introduce the following functions:

$$\gamma(r) = e^{\nu/2}$$
$$\tau(r) = e^{-\lambda} \tag{2.14}$$

Equation (2.13) may now be written as a linear first-order equation for τ:

$$\tau' + f(r)\tau = g_0(r) \tag{2.15}$$

where

$$f(r) = -\frac{2(\gamma + r\gamma' - r^2\gamma'')}{r(r\gamma' + \gamma)} \tag{2.16}$$

$$g_0(r) = -\frac{(2r^2\gamma + 4\gamma Q^2)}{r^3(r\gamma' + \gamma)} \tag{2.17}$$

Equation (2.15) has the immediate solution

$$\tau = e^{-F(r)}\left[\int^r e^{F(r')}g(r')\,dr' + \int^r e^{F(r')}g_1(r')\,dr' + C_1\right] \tag{2.18}$$

where

$$g(r) = -2\gamma/r(r\gamma' + \gamma)$$
$$g_1(r) = -4\gamma Q^2/r^3(r\gamma' + \gamma)$$
$$F(r) = \int^r f(r')\,dr', \qquad C_1 = \text{const}$$

Equation (2.18) together with equations (2.11) and (2.12) represent all solutions for charged static spherically symmetric fluid bodies. Not all such solutions will be physically reasonable. Only a subclass of these solutions, corresponding to certain functions $\gamma(r)$, will admit a physically reasonable equations of state. Thus the choice of $\gamma(r)$ is critical if one desires a physically meaningful solution.

§(3): *The Model Solution*

We focus on the case of a fluid sphere with uniform charge density. We assume that the sphere has radius r_0 and carries a charge Q_0. It then follows that $Q = Kr^3$ with $K = Q_0^2/r_0^3$. The solution (2.18) for τ will be particularly simple if $f = g$. This leads to requiring that $r\gamma' - r^2\gamma'' = 0$, or equivalently that

$$\gamma(r) = A + Br^2 \tag{3.1}$$

A and B being constants of integration. Using equations (2.18) and (2.19) we obtain $\tau(r)$ in the form

$$\tau(r) = 1 - (4K^2A/5B)r^2 - \tfrac{2}{5}K^2r^4 + C_1r^2(A + 3Br^2)^{-2/3} \tag{3.2}$$

The constants A, B, and C_1 are specified by matching the solution to the exterior Nordström-Reissner solution for a mass m_0, charge Q_0, and radius r_0. We obtain the following solutions:

$$\gamma(r) = (1 - 2\epsilon + \sigma)^{-1/2}(1 - \tfrac{5}{2}\epsilon + \sigma + \tfrac{1}{2}\epsilon y^2) \tag{3.3}$$

$$\tau(r) = 1 - \frac{8\sigma(1 - \frac{5}{2}\epsilon + \sigma)y^2}{5\epsilon} - \frac{2}{5}\sigma y^4 + \frac{[\epsilon(7\sigma - 10\epsilon) + 8\sigma(1 - \frac{5}{2}\epsilon + \sigma)](1 - \epsilon + \sigma)^{2/3}y^2}{5\epsilon(1 - \frac{5}{2}\epsilon + \sigma + \frac{3}{2}\epsilon y^2)^{2/3}} \tag{3.4}$$

$$\rho(r) = \frac{1}{4\pi r_0^2}\left\{\frac{3}{2}\sigma y^2 + \frac{6\sigma(2 - 5\epsilon + 2\sigma)}{5\epsilon} - \frac{[\epsilon(7\sigma - 10\epsilon) + 4\sigma(2 - 5\epsilon + 2\sigma)]}{10\epsilon(1 - \frac{5}{2}\epsilon + \sigma + \frac{3}{2}\epsilon y^2)^{2/3}} \times (1 - \epsilon + \sigma)^{2/3}[3 - 2\epsilon y^2(1 - \tfrac{5}{2}\epsilon + \sigma + \tfrac{3}{2}\epsilon y^2)^{-1}]\right\} \tag{3.5}$$

$$p(r) = \frac{1}{4\pi r_0^2}\left\{\frac{\epsilon e^{-\lambda}}{(1 - \frac{5}{2}\epsilon + \sigma + \frac{1}{2}\epsilon y^2)} + \frac{[\epsilon(7\sigma - 10\epsilon) + 4\sigma(2 - 5\epsilon + 2\sigma)](1 - \epsilon + \sigma)^{2/3}}{10\epsilon(1 - \frac{5}{2}\epsilon + \sigma + \frac{3}{2}\epsilon y^2)^{2/3}} - \frac{7}{10}\sigma y^2 - \frac{2\sigma(2 - 5\epsilon + 2\sigma)}{5\epsilon}\right\} \tag{3.6}$$

where $\epsilon = m_0/r_0$, $\sigma = Q_0^2/r_0^2$, and $y = r/r_0$. The mass distribution, defined as $m(r) = [1 - \exp(-\lambda)]r/2$ is

$$m(r) = r_0\left\{\frac{2\sigma(2 - 5\epsilon + 2\sigma)}{5\epsilon} + \frac{\sigma}{5} - \frac{[\epsilon(7\sigma - 10\epsilon) + 4\sigma(2 - 5\epsilon + 2\sigma)](1 - \epsilon + \sigma)^{2/3}}{10\epsilon(1 - \frac{5}{2}\epsilon + \sigma + \frac{3}{2}\epsilon y^2)^{2/3}}\right\} y^3 \tag{3.7}$$

In Figures 1 and 2 we have plotted $\rho(r)$ as well as the equation of state p vs ρ for this solution for the values $m_0 = 3$ km, $r_0 = 10$ km, and $Q_0 = 0.5$ km. As in the Adler solution the density peaks sharply at $r = 0$, and the equation of state is physically reasonable. This situation obtains up to $Q_0 \leqslant 2.0$ km. For $Q_0 > 2.0$ km the figures change dramatically. Figures 3 and 4 are representatives of this case. Figure 3 shows that $\rho(r)$ is monotonically increasing while Figure 4 shows that $p(\rho)$ is monotonically decreasing. Thus for $Q_0 > 2$ km the behavior is unphysical and the model is unstable. To obtain a physically reasonable solution Q_0 must be restricted to $Q_0 < 2$ km, so that the solution has a maximum allowed charge.

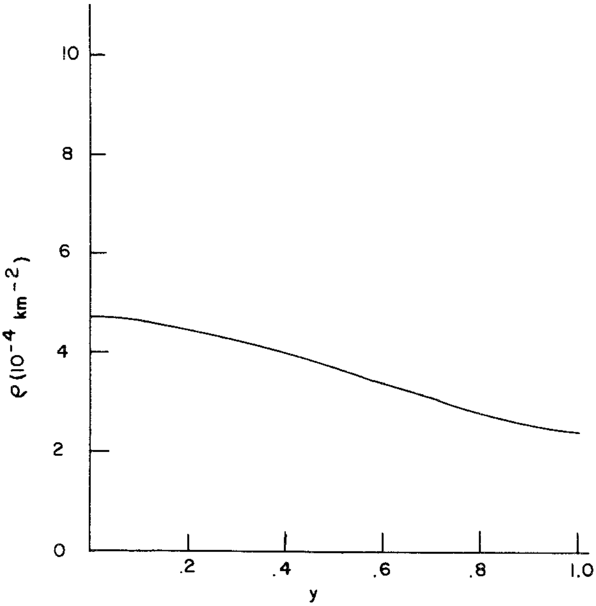

Fig. 1. Density distribution for $m_0 = 3$ km, $r_0 = 10$ km, and $Q_0 = 0.5$ km.

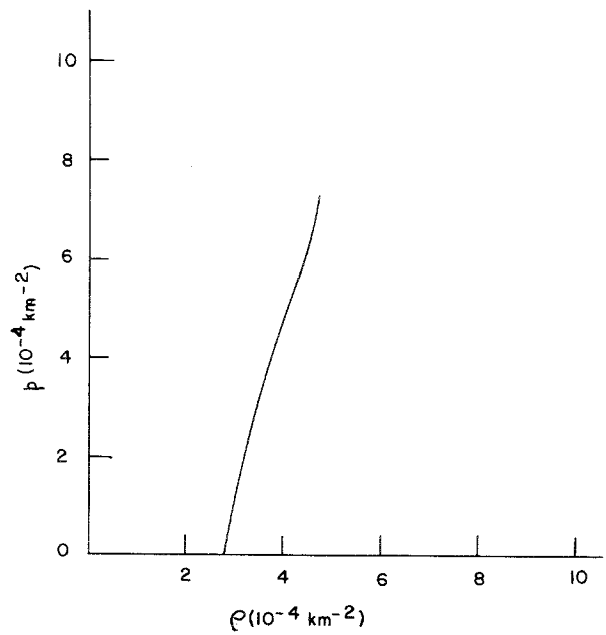

Fig. 2. Equation of state $p(\rho)$ for $m_0 = 3$ km, $r_0 = 10$ km, and $Q_0 = 0.5$ km.

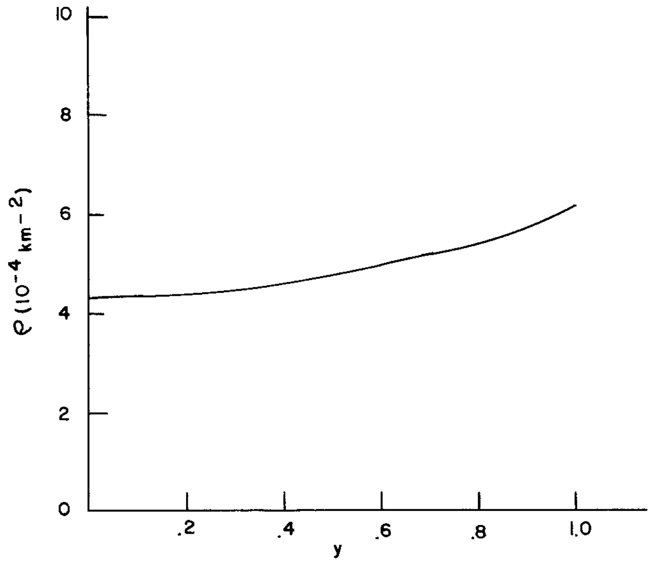

Fig. 3. Density distribution for $m_0 = 3$ km, $r_0 = 10$ km, and $Q_0 =$ 4 km.

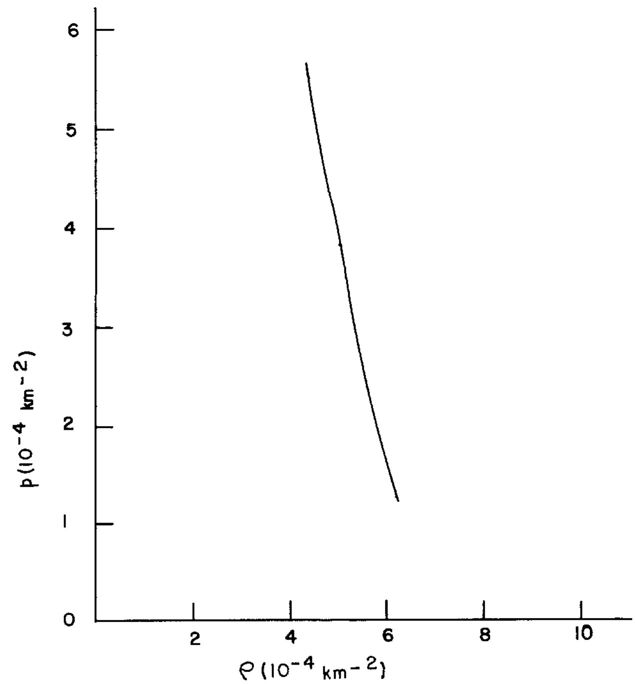

Fig. 4. Equation of state $p(\rho)$ for $m_0 = 3$ km, $r_0 = 10$ km, and $Q_0 =$ 4 km.

§(4): *Properties of the Solution*

It is evident from equations (3.5) and (3.6) that $\rho(0)$ and $p(0)$ both become infinite at the same value of $\epsilon = \frac{2}{5}(1 + \sigma)$. The Adler interior solution has a similar property in that the central presure becomes infinite when $\epsilon = \frac{2}{5}$. It is then clear that this singularity occurs at a slightly higher mass than in the uncharged case.

In the low mass limit our solution yields

$$e^{\nu} = 1 - 3\epsilon + \sigma + \epsilon y^2 \tag{4.1}$$

$$\tau(r) = e^{-\lambda} = 1 - (10\epsilon - 7\sigma)y^2/5 - \tfrac{2}{5}\sigma y^4 \tag{4.2}$$

$$p = \frac{(15\epsilon^2 + 7\sigma)(1 - y^2)}{40\pi r_0^2} \tag{4.3}$$

and

$$\rho = \frac{3}{4\pi r_0^2}\left[\epsilon\left(1 + \epsilon - \frac{5}{3}\epsilon y^2\right) - \frac{3}{2}\sigma\left(1 - \frac{11}{9}y^2\right)\right] \tag{4.4}$$

We may now eliminate y^2 between equations (4.3) and (4.4) to obtain an approximate equation of state for small ϵ:

$$p = \tfrac{3}{10}[\rho - (3\epsilon + \sigma)/4\pi r_0^2] \tag{4.5}$$

If in equations (4.1)-(4.5) we put $\sigma \equiv 0$, we regain the Adler results.

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