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Junction Conditions in General Relativity

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Abstract

We examine the three sets of junction conditions commonly used in general relativity: those due to Darmois, to O'Brien and Synge, and to Lichnerowicz. We show that those due to Darmois and Lichnerowicz are equivalent. The O'Brien and Synge set is stronger than the other two and is unsatisfactory in that it may rule out physically plausible junctions. We conclude that the Darmois set is the most convenient and reliable.

§(1): *Introduction*

In this paper we discuss ordinary discontinuities across nonnull hypersurfaces in general relativity. By this we mean that surface layers are excluded: for consideration of the latter see [1].

Three sets of junction conditions are in use: those of Darmois [2] (hereafter denoted by D), O'Brien and Synge [3] (hereafter O), and Lichnerowicz [4] (hearafter L).[1] Almost certainly the set most commonly used are O: a study of the Science Citation Index from 1970 to 1979 shows that [3] was quoted in 26 papers during that period. Inspection of these papers reveals that, in about half, the conditions O were being applied to boundary value problems. It is therefore fair to say that O plays a significant part in general relativity today.

D and L are evidently somewhat similar, and in both cases arose from a mathematical approach based on differentiability classes of the metric coefficients g_{ik}. The method of O was quite different, relying on considerations of a hypothetical boundary layer whose thickness was allowed to tend to zero,

[1] Another approach, not to our knowledge so far used in problems, is that of Synge [5].

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similar to that sometimes used in electromagnetism. The junction conditions which they derived were not obviously equivalent to D or L.

We shall show in this paper that D and L are equivalent but that O are not equivalent to the others. Although a junction satisfying O necessarily satisfies D and L, the converse is not true. This is made clear in Section 3, where we write out D and compare them with O, but we also show it in Section 4 by an example, taken from cosmology, in which there is a boundary satisfying D but not O. We shall argue that in this case the requirements of O are too restrictive, since they would rule out a physically reasonable situation. These conclusions are different from those of Israel [6] and of Robson [7]. We shall discuss their work in Section 5.

In our opinion the neatest formulation of boundary conditions in general relativity is that of Darmois because it is manifestly covariant, and elegantly geometrical. It is also by far the most convenient because it does not demand the continuity of the normal coordinate. Examples of its use are given by Cocke [8], Bonnor and Faulkes [9], Vickers [10], and Bonnor [11]. A discussion of it is to be found in [12].

Throughout the work we shall suppose that the bounding hypersurface is nonnull, though it may be spacelike or timelike. Latin indices run from 1 to 4, Greek indices from 1 to 3.

§(2): *The Three Sets of Junction Conditions*

Let V and $\bar{V}$ be two regions of space-time, separated by a hypersurface S. We shall suppose that the metric coefficients g_{ik} are of differentiability class C^3 except on S. In O and L the same coordinate system is used on both sides of S, but in D this need not be the case.

Darmois Conditions (D) Let $x^i, \bar{x}^i$ be the coordinates in V, $\bar{V}$, and $g_{ik}, \bar{g}_{ik}$ the corresponding metrics. Let S be given by the functions

$$f(x^i) = 0 \quad \text{in } V, \quad \bar{f}(\bar{x}^i) = 0 \quad \text{in } \bar{V} \tag{2.1}$$

of class C^2. Then the unit normals can be calculated:

$$n_i = f_{,i}(g^{ab} f_{,a} f_{,b})^{-1/2} \quad \text{in } V$$
$$\bar{n}_i = \bar{f}_{,i}(\bar{g}^{ab} \bar{f}_{,a} \bar{f}_{,b})^{-1/2} \quad \text{in } \bar{V}$$

where $,i$ means $\partial/\partial x^i$ or $\partial/\partial \bar{x}^i$. These are needed to construct the second fundamental form of S (see Section 3). We need, in addition to (2.1), the two parametric representations of S:

$$x^i = g^i(u^1, u^2, u^3) \quad \text{in } V, \quad \bar{x}^i = \bar{g}^i(u^1, u^2, u^3) \quad \text{in } \bar{V} \tag{2.2}$$

where $g^i, \bar{g}^i$ are of class C^3 (see below), such that S is covered by the same domain of $u^\alpha (\alpha = 1, 2, 3)$ in both representations. Then V and $\bar{V}$ are said to match across S if the first and second fundamental forms of S, calculated as functions of u^α by means of g_{ik} and $\bar{g}_{ik}$, are identical.

Lichnerowicz Conditions (L). V and $\bar{V}$ are said to match across S if for every point P of S there exists a system of coordinates such that their domain contains P, and such that the metric components and their first derivatives are continuous across S. Such coordinates are called admissible.

O'Brien and Synge Conditions (O). Let the coordinates be chosen so that S is given by x^4 = const (x^4 need not be a timelike coordinate). Then V and $\bar{V}$ are said to match across S if

$$g_{ik}, \frac{\partial g_{\mu\nu}}{\partial x^4}, \quad T_k{}^4 \tag{2.3}$$

are continuous across S, $T^i{}_k$ being the energy tensor.

Remarks (1) A further condition given by O'Brien and Synge, namely, that

$$g_{i\alpha}T^\alpha{}_k - g_{k\alpha}T^\alpha{}_i$$

shall be continuous across S, is automatically fulfilled, because of $T_{ik} = T_{ki}$, if O are satisfied. It has been proved by Kumar and Singh [13] that the continuity of $T^4{}_k$ is implied by the continuity of g_{ik} and $\partial g_{\mu\nu}/\partial x^4$.

(2) It has in the past been assumed, usually tacitly, that in O and L the g_{ik} as functions on S of the surface coordinates are sufficiently smooth for the existence of at least the second tangential derivatives. In the case of D the corresponding assumption is that on S $g_{ik}(u^\alpha)$ are of class C^2 at least, which entails that g^i and $\bar{g}^i$ in (2.2) are C^3 at least. These assumptions will be made in this paper also.

§(3): *Equivalence of Sets of Conditions*

The equivalence of the Darmois and Lichnerowicz conditions is easy to see. If the first and second fundamental forms of S are the same when approached from either side, then it follows, as shown by Darmois [2], that if one uses in V and $\bar{V}$ Gaussian coordinates with S given by $x^4 = 0, \bar{x}^4 = 0$, then $g_{\alpha\beta} = \bar{g}_{\alpha\beta}$ and $\partial g_{\alpha\beta}/\partial x^4 = \bar{g}_{\alpha\beta}/\partial \bar{x}^4$ on S. Since in Gaussian coordinates $g_{44} = \bar{g}_{44} = 1$ and $g_{4\alpha} = \bar{g}_{4\alpha} = 0$, all the g_{ik} and $g_{ik,4}$ are continuous, and Gaussian coordinates constitute admissible coordinates in the sense of L. This shows that if D are satisfied so are L.

Next, assuming L, i.e., that there exists a system of coordinates in which g_{ik} and $g_{ik}/\partial x^m$ are continuous, across S, we at once see that the first and second fundamental forms of S, which depend only on g_{ik} and $\partial g_{ik}/\partial x^m$, will be the same when approached from either side, so D are satisfied.

Turning now to the comparison of D with O let S be given by

$$x^4 = \text{const}, \quad \bar{x}^4 = \text{const} \tag{3.1}$$

in V and $\bar{V}$. Suppose that the other coordinates and the parametrization of S are such that on S

$$x^\alpha = \bar{x}^\alpha = u^\alpha \tag{3.2}$$

The first fundamental forms of S, obtained from V and $\bar{V}$, are

$$ds^2 = g_{\alpha\beta}\, du^\alpha\, du^\beta, \qquad d\bar{s}^2 = \bar{g}_{\alpha\beta}\, du^\alpha\, du^\beta$$

and according to D we have

$$g_{\alpha\beta} = \bar{g}_{\alpha\beta} \tag{3.3}$$

so $g_{\alpha\beta}$ is continuous on S, which is part of O.

The second fundamental form of S is

$$d_{\alpha\beta}\, du^\alpha\, du^\beta = -n_{i;k} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^k}{\partial u^\beta} du^\alpha\, du^\beta \tag{3.4}$$

n_i being the unit normal, which in case (3.1) is

$$n_i = \delta^4{}_i (g^{44})^{-1/2}$$

(assuming $g^{44} > 0$, otherwise the modulus must be taken). Using (3.1) and (3.2) we have for the equality of the second fundamental forms required by D

$$0 = \bar{d}_{\alpha\beta} - d_{\alpha\beta} = n_{\alpha;\beta} - \bar{n}_{\alpha;\beta} = \bar{\Gamma}^4_{\alpha\beta}\bar{n}_4 - \Gamma^4_{\alpha\beta} n_4$$

A short calculation gives

$$\{(\bar{g}^{44})^{1/2}\, \overline{[\alpha\beta, 4]} - (g^{44})^{1/2} [\alpha\beta, 4]\} = [\alpha\beta, \gamma]\, \{\bar{g}^{4\gamma}(\bar{g}^{44})^{-1/2} - g^{4\gamma}(g^{44})^{-1/2}\} = 0 \tag{3.5}$$

where $[\alpha\beta, i]$ is the Christoffel symbol of the first kind. Because of (3.3) we have put $\overline{[\alpha\beta, \gamma]} = [\alpha\beta, \gamma]$. Evidently if O are satisfied so is (3.5), but not the converse, since (3.5) do *not* imply, for example, that $g^{44} = \bar{g}^{44}$, which is a consequence of O. Hence *if O are satisfied D are satisfied, but not the converse.* In other words, O are more restrictive than D.

§(4): *An Example*

The example, which was first discovered by Vickers [14], refers to a spherically symmetric dust distribution. Using comoving coordinates $x^i = (r, \theta, \phi, t)$, we may take the metric for this as [15]

$$ds^2 = -(R')^2 \Gamma^{-2}\, dr^2 - R^2 (d\theta^2 + \sin^2 \phi\, d\phi^2) + dt^2 \tag{4.1}$$

where, on account of the field equations (without cosmological term), $R(r, t)$ must satisfy

$$\dot{R}^2 = \Gamma^2 - 1 + 2FR^{-1} \tag{4.2}$$

The prime and the dot mean partial differentiation with respect to r and t, respectively, and Γ and F are arbitrary functions of r. Another arbitrary function h of r arises on integration of (4.2). We shall assume

$$\dot{R} \neq 0, \qquad R' \neq 0 \tag{4.3}$$

and number the coordinates

$$x^1 = r, \qquad x^2 = \theta, \qquad x^3 = \phi, \qquad x^4 = t$$

The only nonzero component of $T^i{}_k$ is

$$T^4{}_4 = 8\pi\rho = \frac{2F'}{R^2 R'} \tag{4.4}$$

We consider two regions of space-time, both with metrics of form (4.1) but with different functions Γ, F, and h, separated by a hypersurface S defined by

$$x^1 = r = a \qquad (a \text{ const}) \tag{4.5}$$

so appropriate adaptation is required of the work in Sections 2 and 3, where the boundary was taken as x^4 = const. In this example we may take all four coordinates as continuous across S. It will be convenient to denote by C the set of functions which are continuous across S.

A straightforward application of junction conditions O to S gives

$$R, R', \Gamma \in C \tag{4.6}$$

However, if we differentiate (4.2) with respect to t and use (4.3), we obtain

$$\ddot{R} = -FR^{-2} \tag{4.7}$$

and since from (4.6) R and $\ddot{R}$ are continuous, so must F be:

$$F \in C \tag{4.8}$$

Similarly differentiating (4.7) with respect to r and then changing the order of differentiations we have

$$(R')^{\cdot\cdot} = 2FR'R^{-3} - F'R^{-2}$$

The left-hand side and the first term on the right are continuous at $r = a$ because of (4.6) and (4.8), so

$$F' \in C \tag{4.9}$$

Finally, differentiating (4.2) with respect to r and using (4.6), (4.8), and (4.9) we have

$$\Gamma' \in C \tag{4.10}$$

Collecting these results together we find that O require

$$R, R', \Gamma, \Gamma', F, F' \in C \tag{4.11}$$

We now apply D to this example. The unit normal n_i to S is

$$n_i = \delta_i^{\ 1}(-g_{11})^{1/2}$$

The parameters u^α are (θ, ϕ, t), $(\alpha = 2, 3, 4)$ and the first fundamental form on S is

$$-[R(a,t)]^2\ (d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2$$

Since this has to have the same form whichever side of S we approach, it evidently implies

$$R(r,t) \in C \tag{4.12}$$

The coefficients of the second fundamental form are found from (3.4) to be

$$d_{\alpha\beta} = \Gamma^1{}_{\alpha\beta}(-g_{11})^{1/2}$$

and the continuity of this at $r = a$ requires

$$\Gamma R \in C$$

Once again, from (4.2) we obtain (4.7) with result

$$F \in C$$

No further conditions are required by D.

We sum up as follows:

(a) O *require* $R, R', \Gamma, \Gamma', F, F' \in C$,

(b) D *require* $R, F, \Gamma \in C$.

We note in particular that D do not require that g_{11} be continuous. D are evidently less restrictive.

From (4.4) we see that O require continuity of the density at $r = a$, a result previously noted by Just [16]. This is too restrictive. Our model applies, for instance, to a condensing region in the expanding universe, and it is undesirable to demand that the density be continuous at the boundary. Indeed, if we put the Schwarzschild exterior in Friedmann form [15] and try to match it, using O, with another Friedmann model, the continuity of the density requires that it vanish on the boundary, and therefore, because of the homogeneity, throughout the Friedmann model. Thus it is not possible to satisfy O for this simple problem using comoving coordinates.

We conclude that there are boundary value problems for which O are unsuitable.

§(5): *Conclusion*

We have shown that D and L are equivalent, but that O are different from D and L, and are stronger in that all junctions satisfying O also satisfy D and L, but not conversely. This is at variance with some previous writings [6, 7] and we shall now explain why.

In this paper we have taken O to express boundary conditions across any nonnull hypersurface S specified by $x^4 = a$. We have assumed metrics $g_{ik}, \bar{g}_{ik}$ given on the two sides of S, and *we have not allowed any transformations of coordinates on these metrics.* This is in the original spirit of O'Brien and Synge, but contrary to the treatments of Israel [6] and Robson [7], who use coordinate transformations in attempts to show the equivalence of L and O.

If coordinate transformations are allowed it is obvious that L and O are equivalent in the sense that there exist coordinate systems in which they reduce to the same conditions. Gaussian coordinates with S given by $x^4 = 0, \bar{x}^4 = 0$ form such a system. To see this we note that the differences between L and O are

(i) that in L $\partial g_{4i}/\partial x^4$ must be continuous whereas in O this is not required.
(ii) that $T^4{}_k$ is to be continuous in O.

Regarding (i) it is clear that in Gaussian coordinates $\partial g_{4i}/\partial x^4$ are necessarily continuous because g_{4i} are constants; and if g_{ik} and $\partial g_{ik}/\partial x^4$ are continuous so is $T^4{}_k$ [5] so (ii) is automatically satisfied. Hence in Gaussian coordinates both L and O reduce to the same requirement, namely, the continuity of g_{ik} and $\partial g_{\alpha\beta}/\partial x^4$.

Since O are not covariantly stated it need occasion no surprise that they are equivalent to L (and therefore to D) in some coordinate systems but not in others. The difficulty with O is to know whether the coordinate system one is using is suitable for O or not: in the example of Section 4 the use of O with comoving coordinates led to an unphysical result. With L too there is a difficulty, namely, to transform the metric into an admissible system of coordinates, which is often very hard. This leaves D as the most convenient and reliable formulation of junction condition in general relativity.

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