General Relativity and Gravitation, Vol. 25, No. 11, 1993

Limiting Configurations Allowed by the Energy Conditions

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Received October 10, 1992

The energy conditions of general relativity are satisfied by all experimentally detected fields. We discuss their interpretation and application to charged spheres. It is found that they prevent the existence of naked singularities, and demand that the effective gravitational mass be everywhere non-negative. We focus on the emergence of limiting configurations—sources of the Reissner–Nordström field that have vanishing effective mass everywhere within the sphere. These configurations have a number of interesting features. Among them we find that, near the center, the limiting form of the equation of state is $\rho + 3p = 0$. Notably this is the only equation of state consistent with the existence of zero-point electromagnetic field, and it has been considered in different contexts, in discussions of cosmic strings and in derivations of (3+1) properties of matter from (4+1) geometry. The consistency of these configurations with the Einstein–Maxwell equations is shown by means of explicit examples. These configurations can be interpreted as due to self-interacting gravitational effects of the zero-point electromagnetic field.

1. INTRODUCTION

General relativity is a theory originally designed to account for macroscopic phenomena, where the curvature of space-time might play a significant role. However, in principle, it can be extrapolated to the microscopic scale, without internal logical contradictions. Such extrapolation is pursued to

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0001-7701/93/1100-1123$07.00/0

extract physical predictions and to understand some important aspects in the theory. There are several questions in this regard.

i) Is general relativity applicable to small structures such as the electron?

ii) If not, in extrapolating its domain of applicability to the microscopic scale, how far can one go and yet have physically plausible results?

iii) What are the features exhibited by the limiting configurations, that stand at the edge of this domain? These questions are the concern of this paper.

Some authors assume a positive answer to the first question, and apply the Einstein–Maxwell equations to the electron, which is considered to have finite size [1–3]. In particular, Bonnor and Cooperstock [3] apply them to discuss the question of how the validity of singularity theorems of general relativity may be affected by the experimental evidence that the diameter of the electron is no larger than 10^{-16}cm. Apart from details, they show that the weak and the strong energy conditions, whose fulfillment is mandatory for the validity of these theorems, must be violated within the electron. Bonnor and Cooperstock argue that these results are a consequence of the extremely small electron's size, and not of the particular model used to represent its exterior field.[2]

However, *all experimentally detected fields satisfy the energy conditions* (Ref. 4, p.85). In addition, it is not obvious whether general relativity is appropriate for the description of microscopic objects. Therefore it is natural to ask whether the violation of the energy conditions[3] (and the corresponding emerging negative mass) is not an indication that the theory fails, and does not provide a correct description when it is applied to such small particles as an electron.

Accepting this point of view one immediately arrives at the second question above. In other words, suppose the applicability of the theory can be extrapolated to the microscopic scale, where do we stop? At the nuclear scale? At the electron scale? The answer to this question is probably not unique and depends on the criterion of applicability used.

In this paper the criterion adopted is suggested by the experimental evidence that all detected fields satisfy the energy conditions. Conse-

[2] It has not yet been tested experimentally whether or not the zone near the core of an electron exhibits the character of a negative energy density.

[3] All forms of experimentally detected known matter satisfy the energy conditions and are described by energy-momentum tensors of type I, except in some special cases of radiation when they are of type II. Energy-momentum tensors of type III or IV describe fields that violate the energy conditions, which have not yet been experimentally detected (Ref. 4, p.85).

quently, it is assumed that the domain of applicability of Einstein–Maxwell theory should be restricted to regions of space-time where the weak and strong energy conditions are satisfied.

The purpose of this paper is to find the consequences of assuming the universal validity of the energy conditions for sources of the Reissner–Nordström field. We will consider an arbitrary charged sphere in static equilibrium (not necessarily an electron model), and show that the energy conditions lead to two sets of consequences, which will be discussed separately.

The first set imposes geometrical restrictions on the source. In Section 3, we will show that the fulfillment of the energy conditions in the Reissner–Nordström field (a) ensures the non-negativeness of the matter distribution, and (b) prevents the existence of naked singularities.

The second set concerns the third question above. It allows us to deduce some interesting aspects of the internal structure of limiting configurations, that is, of configurations that stand at the junction between those that do satisfy the energy conditions and those that do not. In Section 4, we will show that they have the following properties: (i) the effective gravitational mass is zero within the body; (ii) the principal pressures are unequal, except at the center; (iii) the equation of state is a generalization of $(\rho + 3p) = 0$, which has been discussed in different contexts [5–10].

The above conclusions make no use of the Einstein–Maxwell equations in their explicit form. They use the boundary conditions only. Therefore in Section 5 we show, by means of explicit examples, that the limiting configurations discussed here are consistent with the field equations. In particular, a simple model is constructed, for which the equation of state is similar to the one that appears in discussions of cosmic strings. Section 6 contains a summary of the results.

2. FIELD EQUATIONS AND ENERGY CONDITIONS

To facilitate the discussion, we start by reviewing the basic equations that describe a static, spherically symmetric charge distribution in curved space-time. We consider here anisotropic matter, which is crucial to understanding the properties of limiting configurations, as we will see in Section 4.

We choose the line element in curvature coordinates

$$ds^2 = e^{\nu} dt^2 - e^{\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{1}$$

where ν and λ are functions of r alone. In these coordinates the energy-momentum tensor $T_{\mu\nu}$ is diagonal, viz.,

$$T^{\mu}_{\nu} = \text{diag}\left(M^0_0 + \frac{E^2}{8\pi}, M^1_1 + \frac{E^2}{8\pi}, M^2_2 - \frac{E^2}{8\pi}, M^3_3 - \frac{E^2}{8\pi}\right), \qquad (2)$$

where $(0,1,2,3) \equiv (t,r,\theta,\phi)$, E is the usual electric field intensity, $M_{\mu\nu}$ represents the energy-momentum tensor associated with the matter contribution, and $M^2_2 = M^3_3$ because of the spherical symmetry. Note that M^1_1, in general, does not have to be equal to M^2_2.

We assume that the energy-momentum tensor satisfies the weak and the strong energy conditions, viz., $T_{\mu\nu}\xi^\mu\xi^\nu \geq 0$, and $[T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\,T]\xi^\mu\xi^\nu \geq 0$, respectively, for an arbitrary non-spacelike vector ξ^μ. The weak energy condition, for (2), demands

$$\left(\rho + \frac{E^2}{8\pi}\right) \geq 0, \quad (\rho + p_r) \geq 0, \quad \left(\rho + p_\perp + \frac{E^2}{4\pi}\right) \geq 0. \qquad (3)$$

The strong energy condition requires, in addition,

$$(T^0_0 - T^1_1 - T^2_2 - T^3_3) = \left(\rho + p_r + 2p_\perp + \frac{E^2}{4\pi}\right) \geq 0, \qquad (4)$$

where $\rho \equiv M^0_0$, $p_r = -M^1_1$ and $p_\perp = -M^2_2 = -M^3_3$ denote the rest energy density and the principal pressures of the matter present, respectively.[4]

In geometric units ($G = c = 1$) the Einstein–Maxwell equations, corresponding to (1) and (2), are

$$8\pi\rho + E^2 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}, \qquad (5)$$

$$-8\pi p_r + E^2 = -e^{-\lambda}\left(\frac{1}{r^2} + \frac{\nu'}{r}\right) + \frac{1}{r^2}, \qquad (6)$$

$$-8\pi p_\perp - E^2 = -\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right), \qquad (7)$$

where the primes denote differentiation with respect to r. The electric field is

$$E(r) = \frac{q(r)}{r^2}, \qquad q(r) \equiv 4\pi\int_0^r \rho_e r^2 dr, \qquad (8)$$

4 The so-called dominant energy condition provides two additional requirements, namely, $[\rho - p_r + (E^2/4\pi)] \geq 0$ and $(\rho - p_\perp) \geq 0$. We will see, in subsection 4.2, that they are automatically satisfied.

where $q(r)$ is the charge inside a sphere of radius r and ρ_e is the charge density which is related to the proper charge density $\hat{\rho}_e$ by $\rho_e = e^{\lambda/2}\hat{\rho}_e$ (for details see, e.g., Refs. 11,12).

The effective gravitational mass inside a sphere of radius r is given by the Tolman–Whittaker formula [13], viz.,

$$M_{\rm G}(r) = 4\pi \int_0^r (T_0^0 - T_1^1 - T_2^2 - T_3^3) r^2 e^{(\nu+\lambda)/2} dr. \tag{9}$$

There is a notable likeness between the definitions of charge (8) and effective mass (9). Therefore, by analogy with (8), the quantity

$$\mu \equiv [(T_0^0 - T_1^1 - T_2^2 - T_3^3) e^{(\nu+\lambda)/2}] \tag{10}$$

can be interpreted as an effective mass density. A much simpler expression can be obtained from (9) if one substitutes eqs. (5)–(7) into it, namely,

$$M_{\rm G}(r) = \tfrac{1}{2} r^2 e^{(\nu-\lambda)/2} \nu'. \tag{11}$$

It follows from eqs. (5) and (9)–(11) that the energy conditions (3), (4) require[5]

$$e^{-\lambda} \leq 1, \qquad \nu' \geq 0. \tag{12}$$

We note that the opposite is in general not true. In particular, the positiveness of the effective gravitational mass $M_{\rm G}(r)$, or $\nu' \geq 0$, does not guarantee $\mu \geq 0$ everywhere. Situations of this kind could appear in viscous fluid [14], in the interior of charged spheres [15], and, in general, when the purely gravitational field energy, as described by the Weyl tensor, is large and negative [16].

To find the implications of (12), we have to use the boundary conditions with the exterior electrovacuum region. This region is described by the Reissner–Nordström field, which, in curvature coordinates, has the form

$$ds^2 = \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right) dt^2 - \left(1 - \frac{2M}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{13}$$

[5] The first inequality follows from (3) after (5) has been integrated as $e^{-\lambda} = 1 - (8\pi/r)\int_0^r (\rho + E^2/8\pi) r^2 dr$.

where M and q represent the total mass and charge, viz., $M \equiv M_{\rm G}(\infty)$, and $q \equiv q(\infty)$. In the absence of surface concentration of charge at the boundary surface, say $r = r_{\rm b}$, necessary and sufficient conditions for matching the metrics (1) and (13) are given by the continuity of E, ν, λ, and ν', viz.,

$$E^2(r_{\rm b}) = \frac{q^2}{{r_{\rm b}}^4}, \tag{14}$$

$$e^{\nu(r_{\rm b})} = e^{-\lambda(r_{\rm b})} = \left(1 - \frac{2M}{r_{\rm b}} + \frac{q^2}{{r_{\rm b}}^2}\right), \tag{15}$$

$$\nu'(r_{\rm b}) = \frac{2M}{{r_{\rm b}}^3}\left(r_{\rm b} - \frac{q^2}{M}\right)\left(1 - \frac{2M}{r_{\rm b}} + \frac{q^2}{{r_{\rm b}}^2}\right)^{-1}. \tag{16}$$

It then follows from (6) and (14)–(16) that

$$p_r(r_{\rm b}) = 0. \tag{17}$$

We note that $p_\perp$ *need not vanish at the boundary*. We are now ready to discuss the implications of (12).

3. IMPLICATIONS OF THE ENERGY CONDITIONS ON THE CHARGE AND SIZE

Condition $e^{-\lambda} \leq 1$ requires $r_{\rm b} \geq (q^2/2M)$, while $\nu' \geq 0$ demands $r_{\rm b} \geq (q^2/M)$. These relations can be interpreted in two possible ways.

3.1. Constraints on the charge

The first possible interpretation is that *the demand that the energy conditions be satisfied leads to a maximum charge*. Specifically, for every given (positive) mass M, there exists an upper limit, dictated by the energy conditions, on the amount of charge that a sphere of radius $r_{\rm b}$ might carry, namely,

$$|q|_{\rm max.} = \sqrt{M r_{\rm b}}\,. \tag{18}$$

Were the charge larger than this value, then the energy conditions would be violated.

The existence of this limit is an interesting consequence of the energy conditions, because according to classical electrodynamics, a body *in vacuum* might, in principle, carry an arbitrary charge. In fact, Maxwell equations by themselves do not impose limitations on the amount of charge.

The limitations come from outside the theory, from considerations related to the properties of dielectrics and conductors.[6]

The question now is how restrictive this limit is. In principle, this limit can be applied to bodies of any mass and size. However, in practice, it seems that for all macroscopic bodies the charge is significantly less than in (18). For example, let us consider a dense object in vacuum, e.g., a neutron star of two solar masses and a radius of 10 km. According to (18), the upper value for the electrostatic potential near its surface is about 10^{26} volts. In real life exchange processes between the star and the surrounding medium appear to put a more stringent limit on it, namely, 10^8–10^{10} volts [17]. A similar situation occurs in the case of some microscopic charged objects, for example in atomic nuclei. For instance, the uranium nucleus contains ten times less charge than allowed by (18). An electron, however, does not satisfy (18); its charge is about $100 q_{\max}$.[7]

The conclusion is that, *although classical electrodynamics by itself does not impose formal limitations on the (maximum) amount of charge carried by a body in vacuum the energy conditions of general relativity do provide an upper limit on it*, in the sense that for charges larger than in (18) a negative effective mass density becomes necessary to support the gravitational field. This limit, however, does not appear to have practical consequences at the macroscopic level.

3.2. Constraints on the size

The second possible interpretation is that *the demand that the energy conditions be satisfied leads to a minimum size*. Specifically, for every *fixed* charge q and mass M, there exists a lower limit, dictated by the energy conditions, on the size of the corresponding sphere, namely,

$$r_{\mathrm{b}_{\min}} = \frac{q^2}{M}. \tag{19}$$

Were the size less than this value, then the energy conditions would be violated.

It should be noted that only for $M < |q|$ does this lower limit affect the structure of the sphere. This is because sources with $M \geq |q|$ have

6 This can be illustrated by the following well-known example. When a positively charged metal ball is placed in contact with the inside of an insulated hollow conductor, all the charge of the ball is transferred to the hollow conductor. Then, in principle, the charge on a hollow conductor and its potential can be increased without limit by repeating the process. In practice, however, the charge can be increased until electrical discharge occurs through the air.

7 This number follows from the experimental evidence that the diameter of the electron is no larger than 10^{-16} cm.

an event horizon at $r_+ = M + (M^2 - q^2)^{1/2}$, and a Cauchy horizon at $r_- = M - (M^2 - q^2)^{1/2}$, so that $r_+ > M > r_{b_{min}}$ ($r_+ = r_- = M$, if $M = |q|$). Therefore, as long as one is outside a source with $M \geq |q|$ and outside the event horizon, $r_b > r_{b_{min}}$. Consequently, the contention in (19) is that for every given value of q and M, with $M < |q|$, there is a limiting configuration for which its lower radius is $r_{b_{min}} = (q^2/M)$, dictated by the energy conditions.

It is interesting that (19) is *formally* equal to a quantity commonly known as the classical electron radius, which is a radius ascribed to the electron in order to avoid an infinite (classical) self-energy. It might seem, therefore, that $r_{b_{min}}$ must be very small, as in the electron that it is of the order of 10^{-13} cm. But this need not be so in general, because *in our discussion q and M are not restricted to any specific value* (besides, of course, $M < |q|$).

Equation (19) implies that the matter density ρ is bounded above, viz., $\rho_{max} \sim (M^4/q^6)$, and it cannot be increased without limit. This leads to the following conclusion: *A naked singularity does not happen in Reissner–Nordström, unless the energy conditions are violated.* This is very interesting, because it is precisely the opposite to what happens in the singularity theorems, which require the validity of energy conditions.

From the above discussion, we conclude that the fulfillment of the energy conditions in the Reissner–Nordström field (a) ensures the non-negativeness of the matter distribution, and (b) prevents the existence of naked singularities.

4. PROPERTIES OF THE LIMITING CONFIGURATION

We have seen that the energy conditions will be satisfied if and only if $|q| \leq |q|_{max}$ or equivalently $r_b \geq r_{b_{min}}$. Although these requirements on q and/or r_b seem to be well satisfied in all practical cases, one may still ask, as a question of principle, what the properties of the *limiting configuration* are where $|q| \to |q|_{max}$, $r_b \to r_{b_{min}}$. The specific question we want to answer is the following. In such a configuration, what relations must be satisfied by the matter variables in order for the energy conditions to be satisfied everywhere within the source?

4.1. Effective mass

Outside a sphere with radius $r_b > r_{b_{min}}$ (charge $|q| < |q|_{max}$), the gravitational mass (9)–(11) is positive. Therefore, its interior structure can always be modeled as having $\mu > 0$ everywhere. However, the situation is different in charged sources that have $r_b = r_{b_{min}}$ ($|q| = |q|_{max}$). Indeed, in such sources M_G vanishes not only at the boundary, but, in the absence of singularities, it also vanishes at the center $r = 0$. This means μ cannot be positive at every point because the slope $(dM_G/dr) = 4\pi r^2\mu$ changes its sign somewhere inside the structure. The consequence of this is that the strong energy condition will be violated within such a sphere unless $\mu = 0$ at every interior point; viz.,

$$\left(\rho + p_r + 2p_\perp + \frac{E^2}{4\pi}\right) = 0. \tag{20}$$

The conclusion, therefore, is that when $r_b = r_{b_{min}}$ ($|q| = |q|_{max}$) the energy conditions will be satisfied everywhere if and only if $M_G(r) = 0$ within the source. Since the force per unit mass, or acceleration of gravity, $g(r)$, with which the gravitational field acts on a neutral test particle, at rest at a point r, is given by $g(r) = -[M_G(r)/r^2]e^{-\nu/2}$, it follows that this is equivalent to the requirement that *the material content of the sphere has no effect on gravitational interactions*.

This conclusion may at first seem odd, but matter with such properties has been considered in different contexts by several authors. In fact, in the case of uncharged perfect fluid ($E(r) = 0$, $p_r = p_\perp$), eq. (20) reduces to the equation of state $\rho = -3p$, which appears in discussions of the premature recollapse problem [5], in coasting cosmologies [6], in cosmic strings [7,8], and in derivations of (3+1) properties of matter from (4+1) geometry [9,10].

The question is how to understand the vanishing of the gravitational mass. Indeed, if the contribution from the matter is positive, where does the negative contribution, that cancels it out, come from? One could ask whether this cancellation is not a consequence of the electrical field. We will answer this question in terms of the purely gravitational field energy, which is represented by the Weyl tensor.

In a spherically symmetric space-time all the components of the Weyl tensor are proportional to the quantity W, defined by [15,16]

$$W = \frac{r}{6} - \frac{r^3e^{-\lambda}}{6}\left(\frac{\nu''}{2} + \frac{\nu'^2}{4} - \frac{\nu' - \lambda'}{2r} - \frac{\nu'\lambda'}{4} + \frac{1}{r^2}\right). \tag{21}$$

Now using the field equations (5)–(7) and (11) we obtain

$$M_G = \left[W + \frac{4\pi r^3}{3}(\rho + 2p_r + p_\perp)\right]e^{(\nu+\lambda)/2}. \tag{22}$$

This expression is interesting because it *does* give the effective mass as the sum of two parts only; W and $(\rho + 2p_r + p_\perp)$, for the purely gravitational field and matter contribution,[8] respectively. But it *does not* contain the electrical field at all.

In the case under consideration, one can conclude from (22) that *the vanishing of the effective gravitational mass throughout the source is produced not by the electrical field, but by a large negative gravitational energy* W, *which exactly balances the positive contribution from the matter.* (Note that this contribution in the case of perfect fluid reduces to the familiar $\rho + 3p$.) This conclusion is completely general, because (22) follows solely from the field equations, and not from any particular assumption.

Another interesting feature, in the case under consideration, is that the total mass, measured by an observer at infinity, is completely of electromagnetic origin, in the sense that the only nonvanishing contribution to the effective mass in (9) comes from the electrical field outside the sphere, $\mu = (q^2/r^4)$, while the content inside the sphere does not contribute at all.

4.2. Internal structure

A charged body with radius $r_b = r_{b_{min}}$ ($|q| = |q|_{max}$) cannot be constructed of perfect fluid but it necessarily consists of anisotropic matter. This follows from the fact that (20) should take place not only in the interior of such a body, but also at its outer surface. In addition, the weak energy condition (3) and the boundary condition (17) require $\rho(r_b) \geq 0$, and $p_r(r_b) = 0$, respectively. Therefore, (20) cannot be satisfied unless $p_\perp$ is different from zero, and negative on the boundary surface. This implies that *the limiting configuration with* $r_b = r_{b_{min}}$ ($|q| = |q|_{max}$) *must have unequal principal pressures.*

In order to understand the physics behind this behavior, let us consider for a moment the (generalized) Tolman–Oppenheimer–Volkov equation of hydrostatic equilibrium, corresponding to (2). It is [18]

$$\frac{M_G(\rho + p_r)}{r^2}\, e^{(\lambda-\nu)/2} = -\frac{dp_r}{dr} + \frac{q\rho_e}{r^2} + \frac{2}{r}(p_\perp - p_r). \tag{23}$$

This equation uncovers a number of interesting features. In particular, we see that there is an additional force, namely, $2(p_\perp - p_r)/r$, which points outward when $p_\perp > p_r$ and inward when $p_\perp < p_r$. We will see that it is precisely the existence of this force that allows the energy conditions to be satisfied throughout limiting configurations.

First, note that the regularity conditions require $q(r) = 0$, and $p_r = p_\perp$ at the origin $r = 0$. Then it follows from (20) that the limiting form

[8] Equation (22) suggests that the quantity $(\rho + 2p_r + p_\perp)e^{(\nu+\lambda)/2}$ can be interpreted as an "average" effective matter density inside a sphere of radius r.

of the equation of state at the center is $\rho = -3p$, which is the equation of state mentioned above [5–10].

This, in particular, means that p_r is negative at the center.[9] In addition, (dp_r/dr) is positive, because p_r vanishes at the boundary. This indicates that the force associated with the pressure gradient points inward. Consequently, in the case where $M_{\rm G}(r) > 0$ inside the source, the pressure acts in conjunction with gravitation to counteract the electrical repulsion and maintain the equilibrium. This shows that the hydrostatic force is not strong enough to counteract, by itself, the electrostatic repulsion. Therefore, in the case where $M_{\rm G}(r) = 0$, in the absence of gravitational attraction in the sphere, the equilibrium requires the presence of a substitute force acting toward the center, at every point. This force is provided by the last term in (23).

The conclusion is that the energy conditions are satisfied in spheres that have $|q| = |q|_{\rm max}$, $r_{\rm b} = r_{\rm b_{min}}$ if and only if $(p_\perp - p_r) \leq 0$, where the equality takes place at the $r = 0$ only.

5. SOME EXPLICIT MODELS OF LIMITING CONFIGURATIONS

So far, we have mainly used the consistency between the boundary and the energy conditions. But, the field equations have not yet been used. Therefore, one should ask the question of whether the limiting configurations are consistent with the field equations. Now we will show, by means of explicit examples, that the answer to this question is positive.

Since $M_{\rm G}(r) = 0$ inside the source, it follows from (11) that $\nu' = 0$. Einstein equations (5)–(7) can be combined to get

$$8\pi(\rho + 2p_r + p_\perp) = e^{-\lambda}\left(\frac{1}{r^2} + \frac{\lambda'}{2r}\right) - \frac{1}{r^2}\,. \tag{24}$$

This equation can be easily integrated as

$$e^{-\lambda} = 1 + Cr^2 - 16\pi r^2 \int_0^r \frac{(\rho + 2p_r + p_\perp)}{r}\, dr, \tag{25}$$

where C is a constant of integration.

5.1. Equation of state $\rho + 2p_r + p_\perp = 0$

We have seen that the configurations under discussion exhibit the following features: (i) the effective gravitational mass is zero within the

[9] Also, one finds that the dominant energy condition holds everywhere within the sphere.

source, (ii) the principal pressures are anisotropic, and (iii) the limiting form of the equation of state at the center is $\rho + 3p = 0$. The latter motivates one to assume that the equation $\rho + 2p_r + p_\perp = 0$, which is the anisotropic generalization of $\rho + 3p = 0$, takes place everywhere within the source. The interior metric is then

$$ds^2 = \left(1 - \frac{M}{r_b}\right) dt^2 - \left(1 - \frac{Mr^2}{r_b^3}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (26)$$

The corresponding equations of state are

$$\rho = -p_r + \frac{M}{4\pi r_b{}^3}, \quad (27)$$

$$\rho = p_\perp + \frac{M}{2\pi r_b{}^3}. \quad (28)$$

Note that $|dp_r/d\rho| = |dp_\perp/d\rho| = 1$, which means that the distribution is consistent with the causality condition $|dp/d\rho| \le 1$ (see for example Ref. 19). In the case where the charge is uniformly distributed throughout the sphere, the interior matter distribution is

$$\rho(r) = -p_r(r) + \frac{M}{4\pi r_b{}^3} = \frac{3M}{8\pi r_b{}^3}\left(1 - \frac{r^2}{3r_b{}^2}\right), \quad (29)$$

$$E^2(r) = \frac{M}{r_b{}^5} r^2. \quad (30)$$

5.2. Uniform density

Another simple model arises if one assumes that ρ and ρ_e are constants throughout the matter. In this case

$$e^{-\lambda} = 1 - \frac{8\pi}{3}\rho r^2 - \frac{16\pi^2}{45}\rho_e^2 r^4, \quad (31)$$

where the constant of integration has been set equal to zero to avoid singularities at $r = 0$. From the boundary conditions we find $\rho = (3M/10\pi r_b{}^3)$, and $\rho_e^2 = (9M/16\pi^2 r_b{}^5)$. The interior line element is

$$ds^2 = \left(1 - \frac{M}{r_b}\right) dt^2 - \left(1 - \frac{4Mr^2}{5r_b{}^3} - \frac{Mr^4}{5r_b{}^5}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (32)$$

Note that the matter contribution to the effective mass in (22) is positive everywhere, viz.,

$$(\rho + 2p_r + p_\perp) = \frac{Mr^2}{40\pi {r_b}^5}\,. \tag{33}$$

The radial and tangential pressures are given by

$$p_r = -\frac{\rho}{3}\left[1 - \frac{r^2}{r^2}\right], \qquad {r_b}^2 = \frac{15\rho}{8\pi\rho_e^2}, \tag{34}$$

$$p_\perp = -\frac{\rho}{3}\left[1 + \frac{14}{8}\frac{r^2}{{r_b}^2}\right]. \tag{35}$$

More general solutions can be obtained if, instead of (27) and (28), one assumes the equations of state $p_r = A_1\rho + A_2$, $p_\perp = B_1\rho + B_2$, where A and B are constants. It is not difficult to show that the resulting models are physically reasonable, in the sense that, for a wide choice of A and B, the models are causal, free of singularities and satisfy the usual energy conditions, namely, $\rho > 0$, $\rho + \sum_{i=1}^{3} p_i \geq 0$ and $\rho \geq |p_i|$.

Consequently, one can conclude that *there are physically reasonable distributions of charged matter that have no effect on gravitational interactions.*

6. SUMMARY AND CONCLUSIONS

This work was motivated by the fact that all experimentally detected fields satisfy the energy conditions. Our aim was to discuss the answer to the following questions: If, in the Einstein–Maxwell theory, the energy conditions have to be satisfied,

(A) What can one deduce about the general behavior of charged spheres?

(B) What are the consequences on their internal structure?

We have seen that there are two sets of consequences. The first one concerns the question (A). In Section 3, using the Reissner–Nordström metric to represent the exterior field we found that the demand that the energy conditions be satisfied (i) ensures the non-negativeness of the matter distribution, and (ii) prevents the existence of naked singularities. This is interesting because the singularity theorems establish that a singularity is inevitable if the strong energy condition is satisfied (given some other technical conditions).

It is important to keep in mind that these conclusions follow solely from the boundary and energy conditions, not from the modeling. In particular, the source does not have to be assumed static. Therefore, they are valid also in the case of charged spheres in dynamical evolution.

The second set of consequences concerns the question (B) noted above. The purpose here was to describe the properties of what we have called limiting configurations, which are configurations that stand at the junction between those that do satisfy the energy conditions and those that do not. Their basic features are the following.

(i) The effective gravitational mass is zero within the source.

(ii) The principal pressures are anisotropic.

(iii) The limiting form of the equation of state at the center is $\rho+3p = 0$.

To the best of our knowledge, configurations with such properties have never been discussed in the literature.

These configurations are interesting because the matter does not gravitate. This property also appears in discussions of cosmic strings [7,8] and in a derivation of (3+1) properties of the matter from (4+1) geometry by Davidson and Owen [9], and by the present author and P. S. Wesson [10].

How can one interpret these configurations? As far as the physical origin is concerned, one may imagine these configurations as due to self-interacting gravitational effects of the zero-point electromagnetic field. This interpretation is motivated by two particular features of the models, which were discussed in subsection 4.1. One of them is that the mass is completely of electromagnetic origin, in the sense that the only nonvanishing contribution to the effective mass in (9) comes from the electrical field outside the sphere—the content inside the sphere does not contribute at all. The other feature is that the equation of state within the source is the anisotropic generalization of $\rho+3p = 0$, which is the *only* equation of state compatible with the existence of zero-point electromagnetic field [20]. The question of how these configurations relate to the real universe remains, however, open.

There is a notable likeness between the properties (i), (ii) and (iii) mentioned above and the properties of the effective energy-momentum tensor discussed by Davidson and Owen [9] in their four-dimensional interpretation of a five-dimensional universe. Besides being spherically symmetric, it has unequal principal pressures and contains a parameter that can be identified with the electrical charge. More importantly, the effective equation of state is $\rho + \sum_{i=1}^{3} p_i = 0$, as in the case discussed here.

Is there any physical meaning behind this likeness, or it is casual? The answer to this question is not yet clear. But, it clearly suggests that an interesting example of configurations of the kind discussed here might emerge from Davidson and Owen's effective energy-momentum tensor, if it is interpreted as a combination of matter and radial electric field. The demonstration of this assertion requires a more detailed investigation.

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