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Geodesic Deviation at Null Infinity and the Physical Effects of Very Long Wave Gravitational Radiation

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Given the news function $\dot{\sigma}$ of a radiating space-time describing an isolated source, one can construct two physically important functions on the infinite celestial sphere surrounding the source:

$$\int_{-\infty}^{\infty} \dot{\sigma}\dot{\bar{\sigma}}\, d\tau \qquad \text{and} \qquad \int_{-\infty}^{\infty} \dot{\sigma}\, d\tau \ (\tau = \text{Bondi parameter})$$

The first describes the energy flux of radiation through the sphere and is the dominant function for high-frequency radiation. The second function contains information about the very low-frequency radiation and dominates at such frequencies. The physical effects of this function are investigated, and it is shown that, even for an arbitrarily small energy flux, it can cause a finite amount of geodesic deviation in the radiation zone. An explicit formula for this deviation is obtained in the case of a bifurcating star in the low-frequency approximation where the energy flux can be neglected.

1. INTRODUCTION

It is well known that the energy flux of a radiating space-time can be described by a single weighted function, $\dot{\sigma}$ (the news function), defined at future null infinity I^+. Although no rigorous proof exists, there are arguments which indicate that the quantity

$$[\sigma] = \int_{-\infty}^{\infty} \dot{\sigma}\, d\tau \qquad (\tau = \text{Bondi parameter})$$

is in general nonvanishing for all physically reasonable radiating systems

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with no incoming radiation. These arguments are based on the behavior of linearized scattering systems [2, 3] (for such a system, nontrivial scattering implies $[\sigma]$ is nonzero) and the semiclassical limit of a quantized nonlinear gravitation [4] (the infrared modes of the gravitation make $[\sigma]$ nonzero). Furthermore, it is precisely the nonvanishing of this quantity which prevents a canonical reduction of the BMS group to the Poincaré group, and hence the associated difficulties of defining angular momentum for radiating systems. Therefore, it is of interest to know what actual physical effects the nonvanishing of $[\sigma]$ can have, and it is the purpose of this paper to describe one such effect: total geodesic deviation near null infinity.

Imagine an inertial observer in the radiation zone of a star that emits a pulse of radiation. He places a stationary test particle at a distance d from himself before the pulse, and observes the same particle again after the pulse. We shall show that if $[\sigma] \neq 0$ the particle will again appear stationary but, in general, its distance will be different. Only if $[\sigma] = 0$ will the total deviation $[d]$ (the change in distance) be zero in all directions. Furthermore, we shall show that $[d] \neq 0$ only in lateral directions (i.e., space-like directions orthogonal to both the world line of the observer and the radius direction linking the observer to the star) and that

$$[z] = \sqrt{2}\, r^{-1} [\sigma]\, \bar{z} + 0(r^{-2})$$

where z is a complex number specifying the particle's initial position in the complex plane of lateral directions, and $[z]$ is the total deviation $z' - z$.

In the final section of this paper we derive an explicit formula for the total deviation $[z]$ in the case of a star which emits radiation by exploding into two halves with relative velocity, v. In order to obtain this formula, we employ a low-frequency approximation in that we assume $\int_{-\infty}^{\infty} \dot{\sigma}\dot{\bar{\sigma}}\, d\tau$ to be negligible. (It is important to note that $\dot{\sigma}$ can be chosen such that $\int_{-\infty}^{\infty} \dot{\sigma}\dot{\bar{\sigma}}\, d\tau$ is arbitrarily small for any given $[\sigma]$). Physically, this corresponds to the assumption that the division of the star occurs at a sufficiently leisurely rate for the total energy flux of radiation to be negligible. It is clear that, by allowing sufficient time for the division to take place, the rate of change (or rather, acceleration) of the system can be made arbitrarily small for any given relative velocity between the components of the star. It thus appears reasonable to suppose that this low-frequency approximation is valid for a wide range of physically interesting processes.

It is assumed that the reader is familiar with the properties of asymptotically flat space-times and the asymptotic spin-coefficient formalism based on future null infinity I^+. The metric of physical space-time, $\hat{M}$, is denoted by $\hat{g}$ and that of the conformally related space, M, by $g(=\Omega^2 \hat{g})$. Indices of hatted quantities are raised and lowered by $\hat{g}$ and those of unhatted by g. We use units such that $c = G = 1$.

2. GEODESIC DEVIATION NEAR I^+

The description of total geodesic deviation given in the introduction depends on four physical concepts:

1. An isolated gravitating system (e.g., a star).
2. Asymptotic parallelism.
3. The space of lateral directions for an asymptotic inertial observer.
4. The radial distance r of the observer from the star.

Our first task is therefore to capture these physical concepts in the form of definitions applicable to general relativity.

We begin by assuming that an isolated system can be adequately described by an asymptotically simple space-time $(\hat{M}, \hat{g})$. The definition of asymptotic simplicity [1] provides us with a conformally rescaled space-time $(M, g = \Omega^2\hat{g})$ with a null boundary I^+ on which $\Omega = 0$ and $\nabla_a \Omega = n_a \neq 0$.

In the presence of radiation, which has a well-defined meaning in the context of asymptotically simple space-times, the definition of an asymptotically parallel vector field, $\hat{v}^a$, is not as straightforward as one might expect. This is because the obvious definition, namely

$$\lim_{\Omega \to 0} (\Omega^{-1}\hat{\nabla}_a \hat{v}_b) = 0$$

is incompatible with the nonvanishing of the news function. Therefore, we require a weaker definition that allows radiation but is sufficiently strong to capture the physical idea of asymptotic parallelism.

The definition given below satisfies these conditions in the sense that it imposes no restrictions on the physical space-time and restricts the vector field to have the expected number of asymptotic degrees of freedom. It also reduces to the "obvious" definition when the news function vanishes.

Definition. A vector field $\hat{v}^a$ in $\hat{M}$, normalized for convenience such that $\hat{v}^a\hat{v}_a = 2$, is said to be asymptotically constant if

1. $v^a = \hat{v}^a$ admits a smooth extension to I^+.
2. The condition, $n_a = v_a$ on I^+ determines a Bondi scaling on I^+ (i.e., I^+ is divergence-free and the Gaussian curvature of its cross sections is given by $K = 1/2$).
3. The condition $\dot{\Omega} = 0$ implies $\Omega^{-1}\dot{n}_a = 0$ on I^+ ("$\cdot$" $= v^a\nabla_a$).

Note that, since $\hat{v}_a\hat{v}^a = 2$, $v_a v^a = 2\Omega^2$, and hence v^a is null on I^+. The requirement that the scaling be of Bondi type restricts the vector field to be, in a sense, laterally parallel. It imposes no restriction on the radial behavior of the field. The condition $\dot{\Omega}=0$ can easily be seen to imply that $\dot{n}^a=0$ on I^+, and hence that $\Omega^{-1}\dot{n}^a$ is well defined on I^+. The requirement that this quantity vanishes on I^+ restricts the vector field to be asymptotically parallel in a radial sense.

The freedom in the choice of Bondi scaling on I^+ is given by $\Omega'=v^{-1}\Omega$ where:

1. $n^a\nabla_a v=0$ on I^+. This implies that I^+ is divergence-free in the new scaling.
2. $\eth^2 v=0$ on I^+. This implies that the new Gaussian curvature, K', is constant.
3. $1=v^2+v\eth\bar{\eth}v-\eth v\bar{\eth}v$. This implies $K'=1/2$. Here "$\eth$" is the edth operation based on any cross section of I^+ (in the original scaling).

Since the equation $\bar{\eth}^2 v=0$ is linear and has four linearly independent solutions (e.g., the first four spherical harmonics), the space of solutions V forms a four-dimensional vector space. Furthermore, V possesses a natural Lorentz metric given by

$$v\cdot w=2vw+v\eth\bar{\eth}w+w\eth\bar{\eth}v-\eth v\bar{\eth}w-\bar{\eth}v\eth w$$

(note that, since $\eth^2 w=\eth^2 v=0$, $\eth(v\cdot w)=0$, and hence $v\cdot w$ is a constant). We thus see that each Bondi scaling, and hence the asymptotic components of each asymptotically parallel vector field, is in one-to-one correspondence with a vector $v\in V$ normalized, such that $v\cdot v=2$. Our definition therefore gives the expected number of degrees of freedom for an asymptotically parallel vector field.

We now consider the problem of defining radial distance r and the space of lateral directions for an asymptotic inertial observer. We make essential use of the fact that v_a determines $n_a(=\nabla_a\Omega)$ uniquely on I^+, and hence that Ω is defined up to $\Omega\to\theta\Omega$ where $\theta=1$ on I^+.

Since we are concerned purely with the radiation zone where all quantities which fall off faster than Ω can be neglected, it is necessary to define r^{-1} only up to this order. Thus r may adequately be defined by $\sqrt{2}r=\Omega^{-1}$. (The factor $\sqrt{2}$ is due to our normalization $v^a v_a=2$.)

Since $v_a-n_a=0$ on I^+, the quantity $(v_a-n_a)\Omega^{-1}$ is well defined on I^+. Using this fact, together with $v^a v_a=2\Omega^2$ and $\dot{\Omega}=v^a\nabla_a\Omega=v^a n_a=0$, it is easy to see that $n_a n^a\Omega^{-2}=-2$ on I^+. Thus n^a defines an asymptotically unique vector in $\hat{M}$ according to $\hat{n}^a=n^a$, which satisfies $\lim_{\Omega\to 0}\hat{n}^a\hat{n}_a=-2$ and $\hat{v}^a\hat{n}_a=0$. This vector defines the radial direction for an asymptotic observer with four velocity $\hat{v}^a$. The space of lateral directions can now be defined as those which are orthogonal to both $\hat{v}^a$ and $\hat{n}^a$. This space is spanned by a complex vector $\hat{m}^a$ defined by $\hat{n}^a\hat{m}_a = \hat{v}^a\hat{m}_a = 0$, $\hat{m}^a\hat{\bar{m}}_a = -1$ and $\dot{\hat{m}}^a = 0$. This vector is defined up to $\hat{m}^a \to e^{i\lambda}\hat{m}^a$ where $\dot{\lambda} = 0$ and can be used to define a basis (n^a, m^a) on I^+ by setting $\hat{m}^a = \Omega m^a$ (note that this implies $n^a m_a = 0$ $m^a\bar{m}_a = -1$ on I^+).

Using the fact that $K = 1/2$, $n_a = v_a$ on I^+ and $\dot{\Omega} = 0$ in M, one can now show that the following relations hold on I^+.

$$\left.\begin{aligned} \Omega^{-1} n^a n^b \nabla_a v_b = 0,\ \Omega^{-1} n^a m^b \nabla_a v_b = 0 \\ \Omega^{-1} m^a n^b \nabla_a v_b = 0,\ \Omega_{-1} m^a \bar{m}^b \nabla_a v_b = 0 \\ \Omega^{-1} m^a m^b \nabla_a v_b = \Omega^{-1} m^a m^b \nabla_a n_b = \dot{\sigma} \end{aligned}\right\} \tag{1}$$

where $\dot{\sigma}$ is the news function on I^+.

We now consider the problem of geodesic deviation near I^+ for an asymptotically parallel, time-like, geodesic congruence with tangent vector $\hat{v}^a$.

Let $\hat{s}^a$ be an orthogonal connecting vector satisfying the deviation equation $\mathscr{L}_v \hat{s} = 0$, i.e.,

$$\dot{\hat{s}}^a = (\hat{\nabla}_b \hat{v}^a)\, \hat{s}^b \tag{2}$$

By writing

$$\hat{s}^a = a\hat{n}^a + z\hat{m}^a + z m^a \tag{3}$$

we see that, in the asymptotic region near I^+, $\dot{a}$ and $\dot{z}$ describe the observed radial and lateral velocities of a neighboring particle with position vector (a, z) with respect to the frame $(\hat{n}^a, \hat{m}^b)$. Furthermore, by expressing Eq. (2) in terms of the conformally related quantities n^a, m^a, and v^a, and using Eq. (1), we obtain

$$\left.\begin{aligned} \lim_{\Omega \to 0} (\dot{a}\Omega^{-1}) = 0 \\ \lim_{\Omega \to 0} (\dot{z}\Omega^{-1}) = \dot{\sigma}\bar{z} \end{aligned}\right\}$$

or

$$\dot{a} = O(r^{-2}) \tag{4}$$

$$\dot{z} = (\sqrt{2}r)^{-1}\, \dot{\sigma}\bar{z} + O(r^{-2}) \tag{5}$$

Thus, in the radiation zone, geodesic deviation takes place only in lateral directions and is determined by the news function $\dot{\sigma}$. Assuming that $\dot{\sigma}$ has compact support (or else falls off sufficiently rapidly) the formula for total deviation follows directly from (5) and is given by

$$[z]=\sqrt{2}r^{-1}[\sigma]\bar{z}+O(r^{-2}) \tag{6}$$

3. TOTAL DEVIATION PRODUCED BY A BIFURCATING STAR

In this section we derive an explicit formula for the total deviation produced by a star which, initially in a stationary state, explodes into two halves, each of which eventually settles down to a stationary state. Our first task is to develop a relativistic model of such a physical situation.

In terms of the standard spin-coefficient formalism based on I^+ [1], the "mass aspect" of a solution of the Einstein equations is given by a weighted function Ψ_2 on I^+. In the case of a stationary solution, Ψ_2 has the general form

$$\Psi_2 = -Mw^{-3} \tag{7}$$

where $w \in V$ and is normalized such that $w \cdot w = 2$, and M is the Bondi mass.

Also, in the stationary case, Ψ_2 transforms according to $\Psi_2' = v^3\Psi_2$ under a change of Bondi scaling. Thus, by setting $v = w$, we obtain a center of mass scaling in terms of which Ψ_2' assumes the simple form $\Psi_2 = -M$.

Since w may be considered as a time-like vector (normalized such that $w \cdot w = 2$) in the abstract vector space V, it is natural to interpret w as the four velocity of the center of mass world line of the solution.

From these considerations, the most natural relativistic model of the physical situation described above appears to be an asymptotically simple space-time with mass aspect Ψ_2, satisfying

$$\left.\begin{array}{ll} 1. & \lim_{\tau \to -\infty} \Psi_2 = -Mw^{-3} \\ \text{and} & \\ 2. & \lim_{\tau \to \infty} \Psi_2 = -m(v^{-3} + v'^{-3}) \end{array}\right\} \tag{8}$$

where M and w are the total mass and center of mass velocity of the star, and m, v, v' are the mass and velocities of its component halves. From the symmetry of the situation, we also demand that $w \cdot v = w \cdot v'$ and that w, v, and v' lie in the same plane.

Using the spin-coefficient equation [1]

$$\dot{\Psi}_2 = -\eth^2\dot{\bar{\sigma}} - \sigma\ddot{\bar{\sigma}}$$

together with (8) we obtain

$$\eth^2[\bar{\sigma}] = -\{Mw^{-3} - m(v^{-3} + v'^{-3})\} + \int_{-\infty}^{\infty} \dot{\sigma}\dot{\bar{\sigma}}\, d\tau \tag{9}$$

The quantity $\int \dot{\sigma}\dot{\bar{\sigma}}\, d\tau$ describes the energy flux of radiation through I^+ and depends on the detailed nature of the physical process. It is, however, a second-order quantity depending, essentially, on the rate of change of the process and may therefore be neglected if we restrict attention to processes which occurs at a sufficiently leisurely rate. Under this approximation, Eq. (9) reduces to the simple form

$$\eth^2[\bar{\sigma}] = -\{Mw^{-3} - m(v^{-3} + v'^{-3})\} \tag{10}$$

which allows us to determine $[\bar{\sigma}]$ and hence the total deviation.

By conservation of momentum, which is actually implied by (10), we have

$$Mw = m(v + v') \tag{11}$$

and hence (since $w \cdot v = w \cdot v'$, and $w \cdot w = 2$)

$$M = m(w \cdot v) \tag{12}$$

By choosing a center of mass scaling (i.e., $w = 1$) we see that (10) becomes

$$\eth^2[\bar{\sigma}] = -\frac{M}{(w \cdot v)}(v + v' - v^{-3} - v'^{-3}) \tag{13}$$

Furthermore, since $w = 1$, we obtain

$$\left.\begin{aligned} &\eth v + \eth v' = 0 \\ &\text{and} \\ &v + \eth\bar{\eth} v = v' \end{aligned}\right\} \tag{14}$$

By direct substitution into (13) and using Eqs. (14) it can be now shown that a particular solution of (13) is given by

$$[\bar{\sigma}] = -\frac{M}{2(w \cdot v)}\left\{\frac{(\bar{\eth} v)^2}{v} + \frac{(\bar{\eth} v')^2}{v'}\right\} \tag{15}$$

That this is actually the only solution can be seen by considering the general properties of spin-weighted functions [1]. By Eq. (6) the formula for total deviation is therefore given by

$$[z] = \frac{\sqrt{2}}{(v \cdot u) r} M \left\{\frac{(\eth v)^2}{v} + \frac{(\eth v')^2}{v}\right\} \bar{z} + O(r^{-2}) \tag{16}$$

By introducing spherical coordinates θ and φ adapted to the axisymmetry of the system, we have

$$\left.\begin{array}{ll} w=\gamma(1+V\cos\theta), & \eth v=\gamma V\sin\theta \\ v'=\gamma(1-V\cos\theta), & \eth v'=-\gamma V\sin\theta \\ w\cdot v=\sqrt{2}\gamma & \end{array}\right\} \tag{17}$$

where $\gamma=\sqrt{2}/\sqrt{1-V^2}$ and V is the speed of the components of the star relative to the center of mass line. We therefore have

$$[z]=\frac{V^2\sin^2\theta}{r(1-V^2\cos^2\theta)}\bar{z}+O(r^{-2})=\frac{f(\theta)\bar{z}}{r}+O(r^{-2}) \tag{18}$$

Finally, by writing $m^a=1/\sqrt{2}(\hat{\theta}+i\hat{\varphi})$, and $z=1/\sqrt{2}(x+iy)$, we obtain

$$\left.\begin{array}{l} [x]=\dfrac{f(\theta)}{r}x+O(r^{-2}) \\ [y]=-\dfrac{f(\theta)}{r}y \quad O(r^{-2}) \end{array}\right\} \tag{19}$$

where $[x]$, $[y]$ are deviations in the directions of longitude and latitude. The major effects of total deviation are therefore

1. Zero deviation at the poles.
2. Maximum deviation at the equator with a contraction in the direction of latitude and an expansion in the direction of longitude.

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REFERENCES

1. Penrose, R., and Rindler, W. (1984). *Spinors and Space-Time*, vols. 1 and 2 (Cambridge University Press, Cambridge).
2. Ludvigsen, M. (1981). *Gen. Rel. Grav.* **13**, 105.
3. Ludvigsen, M. (1981). *J. Phys.*, **A15**, 1519.
4. Ashtekar, A. (1987). *Asymptotic Quantization* (Bibliopolis, Napoli).