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Exact Spatially Homogeneous Cosmologies

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Abstract

We consider perfect fluid spatially homogeneous cosmological models. Starting with a new exact solution of Bianchi type VIII, we study generalizations which lead to new classes of exact solutions. These new solutions are discussed and classified in several ways. In the original type VIII solution, the ratio of matter shear to expansion is constant, and we present a theorem which delimits those space-times for which this condition holds.

§(1): *Introduction*

Recent studies of classical general relativity theory have tended to focus on two areas: those involving "local" questions (e.g., stellar models), and those involving "global" phenomena (e.g., cosmology and black holes). For both mathematical and physical reasons, it is customary in either situation to postulate the existence of an r-parameter continuous isometry group, G_r. In stellar and black hole problems, stationary axisymmetric systems are frequently studied (in which $r \geqslant 2$ and the orbits are timelike), whereas in cosmology most attention is given to spatially homogeneous models (in which $r \geqslant 3$ and the orbits are spacelike). Normally, *either* local *or* global questions (but not both) are considered. In the present work, models originally local in nature (i.e., stellar end-states) are transformed into cosmological space-times. This is achieved by starting with a metric

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which is stationary and axisymmetric, and then employing a complex (i.e., imaginary) transformation. In Section 2, the complex transformation yields a metric (2.1) which is spatially homogeneous (in fact, there is a G_4 isometry group whose spacelike orbits are orthogonal to the flow lines of a perfect fluid matter congruence). The fluid congruence possesses the intriguing feature that the ratio of the rate of shear, σ, to the volume expansion, θ, is constant. This fact, together with the form of (2.1), suggests two generalizations, which we explore at some length. In Section 3, we generalize the form of metric (2.1) by inclusion of arbitrary functions in place of specific terms. This leads to a (surprisingly) large number of exact solutions which appear to be new, and with σ/θ generally not constant. In Section 4, we examine a large subclass of spatially homogeneous models in which σ/θ is constant. The complete specification of the subclass is rather complicated, and is given in the statement of the theorem in Section 4. However, it is sufficiently broad to allow for the existence of solutions for all Bianchi-Behr types except IV and VII_h (for details of the Bianchi-Behr classification, see [1]). This leads to even further exact solutions. The isotropy of the microwave background radiation limits the magnitude of σ/θ in all these models.

In most of the analysis, acquaintance with the usual techniques and conventions will be assumed. Details of the orthonormal tetrad formalism are given in [1] and [2], where this is applied in particular to the study of "orthogonal" spatially homogeneous models (in which the matter flow is orthogonal to the spatially homogeneous hypersurfaces). Locally rotationally symmetric (L.R.S.) models are considered in general in [3] and [4], and Ellis and MacCallum [1] discuss orthogonal spatially homogeneous models which are L.R.S. We use the notation and conventions of [1] throughout.[1] The equations most relevant to our discussion in Section 4 are the Jacobi identities {(2.11) and (2.12) of [1]}, the (0ν) field equations {(3.3) of [1]}, and the trace-free parts of the $(\beta\,\delta)$ field equations {(3.4) of [1]}. For convenience, these are written in the Appendix [equations (A.1)-(A.4)].

§(2): *An Exact Solution*

In this section,[2] an explicit family of spatially homogeneous cosmological space-times is presented. The cosmological matter is modelled by a perfect fluid with energy-momentum tensor

$$T_{ab} = (\mu + p)u_a u_b + p g_{ab}$$

[1] All indices which appear are tetrad indices. Green indices have values 1, 2, and 3, and Latin indices have values 0, 1, 2, and 3. The conventions for the Riemann and Ricci tensors are given by $2u^b{}_{;[cd]} = R_a{}^b{}_{cd}u^a$ and $R_{ab} = R^c{}_{acb}$. The Einstein field equations are $R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$; the cosmological constant, Λ, is assumed to be zero throughout, although the theorem in Section 4 remains valid if $\Lambda \neq 0$.

[2] The content of Section 2 is taken from [5].

and line element

$$ds^2 = -a^2t^{10}(1 + c^2t^2)^{-1}dt^2 + b^2t^{10}\,\Sigma^{-2}y^2dy^2 + \tfrac{9}{8}c^2t^{10}\,\Sigma^2dx^2 + t^8(dz - k_2y^2dx)^2 \quad (2.1a)$$

where $\Sigma := k_1 + k_2y^2$, $c^2 = (\frac{2}{3}k_2ab^{-1})^2$, and a,b,c,k_1, and k_2 are constants (without loss of generality, a,b,c, and k_2 are positive).

The solution of Einstein's field equations given by (2.1a) arises from a search for stationary, axisymmetric "interior" solutions to model relativistic stellar end states. Such space-times have two commuting Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$ and a metric of the form [6]

$$ds^2 = -\alpha^{-2}(dt - \Omega_0y^2d\phi)^2 + \alpha^2V^2d\phi^2 + \Sigma^{-2}(y^2dy^2 + \alpha^8V^2dz^2)$$

for a rigidly rotating perfect fluid with unit flow vector $\mathbf{u} = \alpha(\partial/\partial t)$, and where the 2-space of Killing orbits has coordinates chosen in the vorticity gauge [6].

Assuming multiplicative separability for the metric functions $\alpha(y,z)$, $V(y,z)$, and $\Sigma(y,z)$ leads to a class of unphysical solutions with the property that $\alpha = \alpha(z), p = p(z)$ and $\mu = \mu(z)$. These unphysical solutions have a parameter which appears in the metric to even powers. Allowing this parameter to become purely imaginary and then relabeling $(t,z,\phi) \to (z,t,x)$ yields a metric of the form given by (2.1a). Substituting back into the Einstein field equations then establishes the restrictions on the parameters necessary to satisfy the requirement that the source is a perfect fluid. The change of variables

$$t = \alpha t', \quad x = x' + \frac{b\alpha^5}{2k_2}, \quad k_1 + k_2y^2 = \frac{1}{\alpha^4}e^{y'}, \quad z = \frac{1}{\alpha^4}z' - k_1x'$$

where

$$\alpha^8 = \frac{2k_2{}^4a^2}{b^4}$$

reduces the metric to the form

$$ds^2 = -\frac{a^2t^{10}}{1 + c^2t^2}dt^2 + b^2t^{10}(dy^2 + e^{2y}dx^2) + t^8(dz - e^ydx)^2 \quad (2.1b)$$

where $c^2 = \frac{8}{9}b^2$ and $a^2 = 8b^4$, and we have dropped primes.

In order to discuss the properties of the space-times given by (2.1), an orthonormal tetrad is chosen by defining the 1-forms

$$\mathbf{e}^0 = -at^5(1 + c^2t^2)^{-1/2}\,\mathbf{dt} = \mathbf{u}$$

$$\mathbf{e}^1 = t^4(\mathbf{dz} - e^y\mathbf{dx})$$

$$\mathbf{e}^2 = bt^5\,\mathbf{dy}$$

$$\mathbf{e}^3 = bt^5e^y\,\mathbf{dx}$$

Kinematic Properties. The matter flow is geodesic and irrotational, and the rate of expansion is given by

$$\theta = 14a^{-1}t^{-6}(1 + c^2t^2)^{1/2}$$

The space-times are anisotropic with rate of shear tensor given in the orthonormal frame by

$$\sigma_{\alpha\beta} = (\sigma/3^{1/2})\mathrm{diag}(-2, 1, 1)$$

where $\sigma = a^{-1}3^{-1/2}t^{-6}(1 + c^2t^2)^{1/2}$.

Symmetries. The space-times admit a four-parameter group of motions with three-parameter subgroups, each acting simply transitively on the homogeneous hypersurfaces (t = const). Hence the space-times are L.R.S. The Killing vectors of the G_4 are given by

$$\begin{aligned} \xi_1 &= \frac{\partial}{\partial x} \\ \xi_2 &= \frac{\partial}{\partial z} \\ \xi_3 &= x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} \\ \xi_4 &= (-x^2 + e^{-2y})\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} + 2e^{-y}\frac{\partial}{\partial z} \end{aligned} \tag{2.2}$$

The Lie algebra of the G_4 is given by

$$\begin{aligned} [\xi_1, \xi_2] &= [\xi_2, \xi_3] = [\xi_2, \xi_4] = 0 \\ [\xi_1, \xi_3] &= \xi_1 \\ [\xi_1, \xi_4] &= -2\xi_3 \\ [\xi_3, \xi_4] &= \xi_4 \end{aligned} \tag{2.3}$$

There is a three-parameter subgroup of Bianchi type VIII, generated by $\{\xi_1, \xi_3, \xi_4\}$, and since the space-time is L.R.S., there is a family of three-parameter subgroups of Bianchi type III [1]. For example, $\{\xi_1, \xi_2, \xi_3\}$ and $\{\xi_2, \xi_3, \xi_4\}$ generate groups of Bianchi type III ($n^{\beta}{}_{\beta} = 0$).

This solution belongs to class IIIb in the Stewart and Ellis classification of L.R.S. models [4].

Petrov Type. It is clear that since the eigenvectors of the Weyl tensor are preserved by the local rotational symmetry, the Weyl tensor either belongs to Petrov type D or vanishes (Petrov type O). Stephani has shown [7] that, for perfect fluids, conformal flatness implies shear-free flow. Thus (2.1) is of Petrov type D.

Physical Properties. The pressure, p, and energy density, μ, are given by

$$p = -a^{-2}t^{-12}(9 + 16c^2t^2)$$

$$\mu = 7a^{-2}t^{-12}(9 + 8c^2t^2)$$

These relations yield an equation of state:

$$a^2(p/8 + \mu/56)^6 = c^{12}(p/9 + 2\mu/63)^5$$

Although the pressure is negative, the mass-energy is the dominating quantity and the strong energy condition

$$\mu + p \geqslant 0, \qquad \mu + 3p \geqslant 0$$

the dominant energy condition

$$\mu \geqslant 0, \qquad -\mu \leqslant p \leqslant \mu$$

and hence the weak energy condition are all satisfied. Since these energy conditions are all satisfied, the homogeneous, anisotropic space-time of metric (2.1) is a reasonable candidate for a model of the early universe. In fact, a recent study [8, 9] of superdense nucleons interacting through scalar (attractive), vector, and spin-2 f^0 mesons (attractive) allows negative pressures, and requires only $\mu + 3p$ to be positive.

The ratio of the shear to expansion scalar is constant, with

$$\sigma/\theta = \tfrac{1}{14}\, 3^{-1/2} \simeq 0.04$$

This value is larger than the present-day upper limit $\sim 10^{-3}$ obtained from indirect arguments concerning the isotropy of the primordial blackbody radiation. This difference suggests the early universe as the regime best modelled by the space-times of metric (2.1).

Generalizations. The metric given in (2.1) and its associated properties suggest two separate generalizations:

(a) Replace the explicit functions of t and y in the metric (2.1) by arbitrary functions. This leads to an unexpectedly rich source of new solutions, which are discussed in Section 3. A particularly simple solution with equation of state $p = \mu/3$ is given.

(b) Start with σ/θ = const and search for spatially homogeneous cosmologies possessing that feature. This is investigated in Section 4.

§(3): *Generalization of (2.1)*

In this Section the metric (2.1) is generalized to

$$ds^2 = -dt^2 + F^2(t)\,dy^2 + H^2(t)\,\alpha^2(y)\,dx^2 + G^2(t)\,[dz - \beta^2(y)\,dx]^2 \quad (3.1)$$

where $F(t), H(t), G(t), \alpha(y)$, and $\beta(y)$ $(FHG\alpha \neq 0)$ are arbitrary functions which must satisfy the Einstein field equations. The general metric form (3.1) will be shown to contain solutions belonging to four distinct classes, all of which are spatially homogeneous, and with σ/θ generally not constant.

In all the above cases it is possible to decouple the field equations into two first-order ordinary differential equations determining the spatial dependence of the metric and two second-order nonlinear ordinary differential equations governing the time evolution of the metric. Essentially, the solutions of the spatial equations determine the symmetries of the space-time, while the solutions to the time equations lead to expressions for the pressure and the density.

In each case the field equations are exhibited and algorithms for finding exact solutions are presented. The kinematic quantities of the fluid are calculated, together with the nonzero components of the Weyl tensor. In addition, isometry groups are investigated and each case is classified.

Field equations. From (3.1) we choose an orthonormal tetrad:

$$\begin{aligned} \mathbf{e}_0 &= \mathbf{u} = \frac{\partial}{\partial t} \\ \mathbf{e}_1 &= G^{-1} \frac{\partial}{\partial z} \\ \mathbf{e}_2 &= F^{-1} \frac{\partial}{\partial y} \\ \mathbf{e}_3 &= (H\alpha)^{-1} \left(\beta^2 \frac{\partial}{\partial z} + \frac{\partial}{\partial x}\right) \end{aligned} \tag{3.2}$$

The commutation coefficients, $\gamma^c{}_{ab}$, are defined from the commutators by

$$[\mathbf{e}_a, \mathbf{e}_b] =: \gamma^c{}_{ab}\, \mathbf{e}_c$$

and satisfy

$$\gamma^c{}_{ab} = -\gamma^c{}_{ba}$$

The nonzero $\gamma^c{}_{ab}\,(a < b)$ are

$$\begin{aligned} &\gamma^1{}_{01} = -G^{-1}\dot{G}, \qquad \gamma^2{}_{02} = -F^{-1}\dot{F}, \qquad \gamma^3{}_{03} = -H^{-1}\dot{H} \\ &\gamma^1{}_{23} = 2(FH)^{-1}G\lambda, \qquad \gamma^3{}_{23} = -F^{-1}\nu \end{aligned} \tag{3.3}$$

where we have introduced the functions

$$\lambda := \alpha^{-1}\beta\beta' \qquad \text{and} \qquad \nu := \alpha^{-1}\alpha' \tag{3.4}$$

Here the primes denote $\partial/\partial y$ and dots denote $\partial/\partial t$. The field equations relative to the basis (3.2) can now be written as follows:

$$R_{00} = -F^{-1}\ddot{F} - H^{-1}\ddot{H} - G^{-1}\ddot{G} = \tfrac{1}{2}(\mu + 3p)$$

$$\begin{aligned} R_{11} &= G^{-1}\ddot{G} + G^{-1}\dot{G}(H^{-1}\dot{H} + F^{-1}\dot{F}) + 2\lambda^2F^{-2}H^{-2}G^2 \\ &= \tfrac{1}{2}(\mu - p) \end{aligned}$$

$$R_{22} = F^{-1}\ddot{F} + F^{-1}\dot{F}(H^{-1}\dot{H} + G^{-1}\dot{G}) - 2\lambda^2 F^{-2}H^{-2}G^2 - F^{-2}\nu^2 - F^{-2}\nu'$$
$$= \tfrac{1}{2}(\mu - p)$$
$$R_{33} = H^{-1}\ddot{H} + H^{-1}\dot{H}(F^{-1}\dot{F} + G^{-1}\dot{G}) - 2\lambda^2 F^{-2}H^{-2}G^2 - F^{-2}\nu^2 - F^{-2}\nu'$$
$$= \tfrac{1}{2}(\mu - p)$$
$$R_{31} = F^{-2}H^{-1}G\lambda' = 0 \tag{3.5}$$
$$R_{02} = F^{-1}\nu(H^{-1}\dot{H} - F^{-1}\dot{F}) = 0 \tag{3.6}$$

Equation (3.5) implies

$$\lambda' = 0, \qquad \text{i.e.,} \qquad \lambda = \text{const} = \lambda_0 \tag{3.7}$$

Substituting (3.7) into the remaining field equations (using $R_{11} = R_{22} = R_{33}$) leads to

$$F^{-1}\ddot{F} - H^{-1}\ddot{H} + G^{-1}\dot{G}(F^{-1}\dot{F} - H^{-1}\dot{H}) = 0 \tag{3.8}$$
$$H^{-1}\ddot{H} - G^{-1}\ddot{G} - F^{-1}\dot{F}(G^{-1}\dot{G} - H^{-1}\dot{H}) - 4\lambda_0^2 F^{-2}H^{-2}G^2 - \epsilon\nu_0^2 F^{-2} = 0 \tag{3.9}$$
$$\nu' + \nu^2 = \epsilon\nu_0^2 \tag{3.10}$$

where $\epsilon = \pm 1$ and $\epsilon\nu_0^2$ is a separation constant, together with equation (3.6). The pressure, p, and the density, μ, are given by

$$p = -F^{-1}\ddot{F} - H^{-1}\ddot{H} - 2G^{-1}\ddot{G} - G^{-1}\dot{G}(H^{-1}\dot{H} + F^{-1}\dot{F}) - 2\lambda_0^2 F^{-2}H^{-2}G^2 \tag{3.11}$$

and

$$\mu = -F^{-1}\ddot{F} - H^{-1}\ddot{H} + 2G^{-1}\ddot{G} + 3G^{-1}\dot{G}(H^{-1}\dot{H} + F^{-1}\dot{F}) + 6\lambda_0^2 F^{-2}H^{-2}G^2 \tag{3.12}$$

Four distinct classes of solutions can be distinguished:

(1) either $\nu = 0$, $\lambda_0 = 0$ or $\nu\nu' \neq 0, \nu_0 = 0, \lambda_0 = 0$

(2) $\nu \neq 0$, and either $\nu' = 0$ or $\nu_0\nu' \neq 0$

(3) $\nu = 0$, $\lambda_0 \neq 0$, $\dot{F}/F \neq \dot{H}/H$

(4) either $\nu = 0$, $\lambda_0 \neq 0$, $\dot{F}/F = \dot{H}/H$ or $\nu\nu' \neq 0$, $\nu_0 = 0$, $\lambda_0 \neq 0$

When ν_0 and λ_0 are nonzero they can be scaled to unity and $\frac{1}{2}$, respectively, and will be so taken in what follows.

Class (1): either $\nu = 0, \lambda_0 = 0$, or $\nu\nu' \neq 0, \nu_0 = 0, \lambda_0 = 0$.

In this case the metric (3.1) reduces to the general form of a type I space-time. Many authors [1] have investigated these solutions and they will not be discussed further here.

Class (2): $\nu \neq 0$; either $\nu' = 0$ or $\nu_0 \nu' \neq 0$.

(i) Suppose first that ν is a nonzero constant.

Equation (3.10) is satisfied identically with $\epsilon = 1$ and $\nu = \pm\nu_0$, and (3.6) can be integrated to give

$$F = kH, \qquad k = \text{const}$$

where without loss of generality we may take $k = 1$. Equation (3.8) is now satisfied identically and (3.9) becomes ($\nu_0 = 1, \lambda_0 = 1/2$)

$$F^{-1}\ddot{F} - G^{-1}\ddot{G} + F^{-2}\dot{F}^2 - F^{-1}\dot{F}G^{-1}\dot{G} - F^{-4}G^2 - F^{-2} = 0 \qquad (3.13)$$

The spatial dependence of the metric is determined using (3.4), from which we get

$$\alpha = e^{\pm y}$$

and

$$\beta^2 = \pm e^{\pm y} + k_0$$

where k_0 is a constant of integration which can be transformed to zero.

In order to simplify equation (3.13), the function

$$R := GF^{-1} \qquad (3.14)$$

is introduced and substituted into (3.13), yielding

$$3(F^2)^{\cdot}\dot{R} + 2F^2\ddot{R} + 2R^3 + 2R = 0 \qquad (3.15)$$

where $\dot{R} \not\equiv 0$. Considered as a first-order linear differential equation for F^2, (3.15) yields the formal first integral

$$F^2 = \frac{-2}{3\dot{R}^{2/3}} \int \frac{R^3 + \epsilon R}{\dot{R}^{1/3}}\, dt \qquad (3.16)$$

where $R(t)$ is arbitrary and $\epsilon = +1$. The explicit form of the line element is then

$$ds^2 = -dt^2 + F^2(t)\,\{dy^2 + e^{\pm 2y}dx^2 + R^2(t)\,[dz \mp e^{\pm y}dx]^2\} \qquad (3.17)$$

Note that (2.1) is a special case of (3.17).

Pressure and density. An algorithm for generating exact solutions of this class (2i) is as follows. Specify the function $R(t)$ and integrate (3.16), thus obtaining $F(t)$. The function $G(t)$ can then be determined from (3.14) and the pressure and density are given by (3.11) and (3.12). It should be noted, however, that the solutions generated by this procedure are not guaranteed to satisfy any physically reasonable energy conditions (cf. the remarks in Section 4).

Kinematic properties. It is clear from (3.1) and (3.2) that the acceleration, $\dot{\mathbf{u}}$, and the vorticity vector, $\boldsymbol{\omega}$, are both zero (this will be true for all four classes). The expansion and the shear are given by

$$\theta = 3F^{-1}\dot{F} + R^{-1}\dot{R} \qquad (3.18)$$

and

$$\sigma_{\alpha\beta} = (\sigma/3^{1/2})\ \mathrm{diag}\,(-2, 1, 1) \tag{3.19}$$

where

$$\sigma = 3^{-1/2}\ |R^{-1}\dot{R}|$$

Symmetries. Direct calculation using Killing's equation yields the following Killing vector fields:

$$\xi_1 = \frac{\partial}{\partial x}, \qquad \xi_2 = \frac{\partial}{\partial z}$$

$$\xi_3 = x\frac{\partial}{\partial x} - \frac{\partial}{\partial y}$$

$$\xi_4 = (-x^2 + e^{-y})\frac{\partial}{\partial x} + 2x\frac{\partial}{\partial y} + 2e^{-y}\frac{\partial}{\partial z}$$

[cf. (2.2)], with the Lie algebra

$$[\xi_1, \xi_2] = [\xi_2, \xi_3] = [\xi_2, \xi_4] = 0$$

$$[\xi_1, \xi_3] = \xi_1, \qquad [\xi_1, \xi_4] = -2\xi_3, \qquad [\xi_3, \xi_4] = \xi_4$$

[cf. (2.3)]. There is a three-parameter subgroup of type VIII, generated by $\{\xi_1, \xi_3, \xi_4\}$. Since the space-time is L.R.S., there is a family of three-parameter subgroups of Bianchi type III [1]. For example $\{\xi_1, \xi_2, \xi_3\}$ and $\{\xi_2, \xi_3, \xi_4\}$ generate groups of Bianchi type III ($n^\alpha{}_\alpha = 0$).

Petrov type. By local rotational symmetry and the fact that the shear is non-zero, it follows that the solutions are of Petrov type D. The repeated principal null vectors are

$$\begin{aligned} \mathbf{l} &= 2^{-1/2}(\mathbf{e}_0 - \mathbf{e}_1) \\ \mathbf{n} &= 2^{-1/2}(\mathbf{e}_0 + \mathbf{e}_1) \end{aligned} \tag{3.20}$$

(ii) Now suppose that $\nu_0 \nu \neq 0$.

Without loss of generality, $\epsilon = -1$, since the case $\nu_0\nu \neq 0$, $\epsilon = +1$ yields a metric of form (3.17) with new functions $\alpha(y)$ and $\beta(y)$. The new metric is diffeomorphic to (3.17) under transformations of the form

$$x = -e^{-2\bar{x}} \tanh \bar{y}$$

$$y = 2\bar{x} + \ln(\cosh \bar{y})$$

$$z = \bar{z} - \tan^{-1}(\sinh \bar{y})$$

The general form of the metric is

$$ds^2 = -dt^2 + F^2(t)\ [dy^2 + \alpha^2 dx^2 + R^2(t)(dz - \beta^2\, dx)^2] \tag{3.21}$$

where

$$\alpha = a_1 \sin y + b_1 \cos y$$

and

$$\beta^2 = -a_1 \cos y + b_1 \sin y + c_1$$

where a_1, b_1, and c_1 are constants satisfying $a_1 b_1 \neq 0$. Without loss of generality $a_1 > 0$; moreover we may, without loss of generality, put $a_1 = 2^{-1/2}$, $b_1 = 2^{-1/2}$, and $c_1 = 0$ by means of a linear coordinate transformation. These results follow from (3.7) and (3.10) when $\nu_0 \nu' \neq 0$. The functions $F(t)$ and $R(t)$ are related via (3.16) with $\epsilon = -1$.

An algorithm for exact solutions proceeds as in class (2i) above.

The metric (3.21) allows the following Killing vectors:

$$\xi_1 = \frac{\partial}{\partial x}, \qquad \xi_2 = \frac{\partial}{\partial z}$$

$$\xi_3 = \nu \cos x \frac{\partial}{\partial x} + \sin x \frac{\partial}{\partial y} - (\alpha - \nu\beta^2) \cos x \frac{\partial}{\partial z}$$

$$\xi_4 = -\nu \sin x \frac{\partial}{\partial x} + \cos x \frac{\partial}{\partial y} + (\alpha - \nu\beta^2) \sin x \frac{\partial}{\partial z}$$

The Lie algebra is

$$[\xi_1, \xi_2] = [\xi_2, \xi_3] = [\xi_2, \xi_4] = 0$$

$$[\xi_1, \xi_3] = \xi_4$$

$$[\xi_1, \xi_4] = -\xi_3$$

$$[\xi_3, \xi_4] = \xi_1$$

There is a three-parameter subgroup generated by $\{\xi_1, \xi_3, \xi_4\}$, which is of type VIII. The space-time is L.R.S. and therefore admits a family of subgroups of Bianchi type III [1].

Class (2) solutions belong to class IIIb of Stewart and Ellis's classification of L.R.S. metrics [4].

Class (3): $\nu = 0, \lambda_0 = 1/2, \dot{F}/F \neq \dot{H}/H$.

In this case the general metric can be written as

$$ds^2 = -dt^2 + F^2(t)\, dy^2 + H^2(t)\, dx^2 + G^2(t)\, (dz - y\, dx)^2 \qquad (3.22)$$

which follows from the definitions (3.4) of λ and ν upon setting $\nu = 0$ (we have chosen suitable coordinates to absorb integration constants).

The field equations (3.6) and (3.10) are satisfied identically and equations (3.8) and (3.9) remain to be solved. In order to simplify these equations, we introduce the functions

$$\begin{aligned} Q &:= FH^{-1} \\ S &:= H^2 G \\ T &:= FHG^{-1} \end{aligned} \tag{3.23}$$

Equations (3.8) and (3.9), respectively, now become

$$\dot{Q}S = \text{const} \tag{3.24}$$

and

$$S^{-1}\ddot{S} - 3T^{-1}\ddot{T} - Q^{-1}\ddot{Q} - 3T^{-1}\dot{T}(S^{-1}\dot{S} + Q^{-1}\dot{Q} - T^{-1}\dot{T}) + 4T^{-2} = 0 \tag{3.25}$$

Since equation (3.25) is a nonlinear equation in each of Q, S, and T, no direct algorithm exists in this case. One can, however, specify $Q(t)$ and then, upon using (3.24), a second-order nonlinear differential equation remains to be solved for $T(t)$. The pressure and density can then be determined from (3.11) and (3.12) with the use of (3.23). Here again, solutions generated by this procedure may not necessarily satisfy physically reasonable energy conditions.

Kinematic Properties. The expressions for the expansion and the shear are given by

$$\theta = (FHG)^{-1}\,(FHG)^{\cdot}$$

and

$$\sigma_{\alpha\beta} = \text{diag}\,[(3G^2)^{-1}FH(G^2H^{-1}F^{-1})^{\cdot},\; (3F^2)^{-1}GH(F^2G^{-1}H^{-1})^{\cdot}, (3H^2)^{-1}FG(H^2F^{-1}G^{-1})^{\cdot}]$$

The shear scalar is given by

$$\sigma = 3^{-1/2}\,|G^{-2}\dot{G}^2 + H^{-2}\dot{H}^2 + F^{-2}\dot{F}^2 - F^{-1}\dot{F}G^{-1}\dot{G} - H^{-1}\dot{H}F^{-1}\dot{F} - H^{-1}\dot{H}G^{-1}\dot{G}|$$

Petrov Type. With respect to the null tetrad formed from the vectors (3.20) and $\mathbf{m} = 2^{-1/2}(\mathbf{e}_2 + i\,\mathbf{e}_3)$, the nonzero components of the Weyl tensor are ψ_0, ψ_2, and ψ_4. The solutions of this class are therefore of Petrov type I.

Symmetries. The Killing vector fields allowed in this case for the metric (3.22) are

$$\xi_1 = \frac{\partial}{\partial x}$$

$$\xi_2 = \frac{\partial}{\partial z}$$

$$\xi_3 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

with Lie algebra

$$[\xi_1, \xi_2] = [\xi_2, \xi_3] = 0$$

$$[\xi_1, \xi_3] = \xi_2$$

The solutions of class (3) are therefore of type II; they are not, however, L.R.S.

Class (4): either $\nu = 0, \lambda_0 = 1/2, \dot{F}/F = \dot{H}/H$, or $\nu\nu' \neq 0, \nu_0 = 0, \lambda_0 = 1/2$.

(i) Suppose first that $\nu = 0, \lambda_0 \neq 0$, and $\dot{F}/F = \dot{H}/H$. In this case the metric takes the form

$$ds^2 = -dt^2 + F^2(t)\,[dy^2 + dx^2 + R^2(t)\,(dz - y\,dx)^2] \tag{3.26}$$

$$F^2 = \frac{-2}{3\dot{R}^{2/3}} \int \frac{R^3}{\dot{R}^{1/3}}\,dt$$

which follows from (3.9) and (3.14) upon setting $\nu_0 = 0$ (we have eliminated integration constants by a suitable choice of coordinates).

Kinematic Properties. The expressions for the expansion and the shear are identical with those of Class (2), viz., (3.18) and (3.19).

Pressure and Density. An algorithm for finding exact solutions and thus the pressure and density proceeds exactly as in Class (2).

Symmetries. With regard to the Bianchi classification the metric (3.26) allows four Killing vectors:

$$\xi_1 = \frac{\partial}{\partial x}, \quad \xi_2 = \frac{\partial}{\partial z}, \quad \xi_3 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

and

$$\xi_4 = x\frac{\partial}{\partial y} + \tfrac{1}{2}(x^2 - y^2)\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}$$

The Lie algebra is

$$[\xi_1, \xi_2] = [\xi_2, \xi_3] = [\xi_2, \xi_4] = 0$$

$$[\xi_1, \xi_3] = \xi_2, \quad [\xi_1, \xi_4] = \xi_3, \quad [\xi_3, \xi_4] = -\xi_1$$

The only three-parameter subgroup is generated by $\{\xi_1, \xi_2, \xi_3\}$ and is of type II. The solutions are thus type II L.R.S. solutions. They belong to class IIIb of Stewart and Ellis [4].

Petrov Type. The solutions are L.R.S. and hence belong to type D.

(ii) Now suppose that $\nu\nu' \neq 0, \nu_0 = 0$, and $\lambda_0 = \frac{1}{2}$. The metric for this case is diffeomorphic to (3.26) under transformations of the form

$$x = \bar{y}\sin\bar{x}$$

$$y = \bar{y}\cos\bar{x}$$

$$z = \bar{z} + \bar{y}^2 \sin\bar{x}\cos\bar{x}$$

Table I. A Summary of the Results Obtained in Section 3

Solution class	1	2	3	4
ν	0 $\nu \neq 0; \nu' \neq 0; \nu_0 = 0$	$\neq 0; \nu' \neq 0$ $\nu \neq 0; \nu_0\nu' \neq 0$	0	0 $\nu \neq 0; \nu' \neq 0; \nu_0 = 0$
λ_0	0	arbitrary $(\dot{F}/F = \dot{H}/H)$	$\lambda_0 \neq 0$ $(\dot{F}/F \neq \dot{H}/H)$	$\lambda_0 \neq 0$ $(\dot{F}/F = \dot{H}/H)$
Group type	I	III, VIII	II	II
L.R.S.	Some	Yes	No	Yes
Petrov type	I, *D*, O	*D*	I	*D*

An example of (3.26) with equation of state $p = \mu/3$ is given by $(t = 3^{1/2}T^4/4)$

$$ds^2 = -3T^6 dT^2 + T^5(dy^2 + dx^2) + T^2(dz - ydx)^2 \tag{3.27}$$

where $p = (7/6)T^{-8}$, and $\sigma/\theta = 3^{1/2}/12 \simeq 0.144$.

We summarize the results of this section in Table I.

§(4): *σ/θ Constant*

We have shown in Section 2 that the metric (2.1) is L.R.S., spatially homogeneous (of a particular type), and such that the normal congruence to the homogeneous hypersurfaces satisfies the condition

$$\sigma/\theta = \text{const} \tag{4.1}$$

We observed that the matter content was a perfect fluid flowing along the normals and that the shear eigenframe could be chosen to be Fermi-propagated along the fluid flow. Since the metric is L.R.S., the shear tensor, σ_{ij}, possesses two equal eigenvalues, and, since σ_{ij} is trace-free, there is therefore (at most) one algebraically independent component (viz., one of the eigenvalues) of σ_{ij}. Thus we can express our restriction (4.1) in one of two ways: *either* (i) the eigenvalues of the shear tensor, σ_{ij}, of the normal congruence are in constant proportion to the expansion scalar, θ, *or* (ii) the shear scalar, σ, is in constant proportion to the expansion scalar, θ.

In a *general* space-time, condition (i) is stronger than (ii) [i.e., (i) ⇒ (ii) but not necessarily conversely], but whenever σ_{ij} possesses two equal eigenvalues, conditions (i) and (ii) are equivalent.

In order to obtain a simple, tractable generalization of this situation, which encompasses the entire class of spatially homogeneous models, we could relax the condition of local rotational symmetry, while maintaining the condition that σ_{ij} has two equal eigenvalues and at the same time adopting either of the equivalent conditions (i) or (ii) above as an "ansatz." In fact, we can improve upon this by relaxing also the condition on the equality of the eigenvalues, and then demanding that (i) [and hence (ii)] holds. The case where (ii) holds, but (i) does not, does not appear to be very tractable.

Thus we shall investigate the consequences of imposing the condition (i) above on the class of all[3] spatially homogeneous models, together with the requirements that the source of the gravitational field be a perfect fluid flowing along the normals to the spatially homogeneous hypersurfaces, and that the shear eigenframe is, or can be chosen to be, Fermi-propagated along the fluid flow. Condition (4.1) is sufficiently strong to allow the construction of explicit solutions in all types except IV and VII_h.

It turns out that in "most" models, these conditions in fact force the shear tensor to have two equal eigenvalues; however, there are certain exceptional cases, viz., types I, V, and VI_h ($n^\alpha{}_\alpha = 0$). This is understandable to some extent, since one might expect more freedom for this to happen in the most special types (see the specialization diagram in [2]).

Theorem. Let M be a space-time which is spatially homogeneous, and in which the matter source is a perfect fluid flowing orthogonally to the homogeneous hypersurfaces. If M is of class B, suppose that the shear eigenframe is (or can be chosen to be) Fermi-propagated along the fluid congruence, and further that M is not of class Bbii (type VI_h with $h = -\frac{1}{9}$). Suppose also that the fluid has the property that all shear eigenvalues are proportional to the volume expansion rate and that $\sigma\theta \neq 0$. Then M belongs to one of the following types:

Class A: I, II, VI_0 ($n^\alpha{}_\alpha = 0$), VII_0, VIII, and IX

Class B: V, VI_h (and $h = -1$ if $n^\alpha{}_\alpha \neq 0$)

Moreover, for each type listed, there exist space-times with the above properties, and the fluid shear tensor necessarily has two equal eigenvalues in all types except I, V, and VI_h ($n^\alpha{}_\alpha = 0$).

Proof. It is shown in [1] that for class A space-times there is an orthonormal tetrad $\{\mathbf{e}_a\}$, with $\mathbf{e}_0 = \mathbf{u}$, such that the vectors $\{\mathbf{e}_\beta\}$ are eigenvectors of $n_{\alpha\beta}$, and moreover that they are Fermi-propagated shear eigenvectors of the fluid flow. Thus, according to the conditions of the theorem, the shear eigenframe will always be Fermi-propagated. The proof will now divide into two parts, since we shall first discuss class A space-times, and later we shall examine class B space-times (in which there is in general no tetrad of eigenvectors common to $\sigma_{\alpha\beta}$

[3] Except for one case, viz., Case Bbii of [1], which is of type VI_h with $h = -1/9$.

and $n_{\alpha\beta}$). In all cases, we will express the components of $\sigma_{\alpha\beta}$ in a frame of shear eigenvectors as follows:

$$\sigma_{\alpha\beta} = \mathrm{diag}\left(\tfrac{2}{3}k\theta, (\tfrac{1}{2}l - \tfrac{1}{3}k)\theta, (-\tfrac{1}{2}l - \tfrac{1}{3}k)\theta\right)$$

where k and l are constants $[(k^2 + l^2)\theta \neq 0]$. This represents, in general, two constraints on the time-evolution of $\sigma_{\alpha\beta}$. These constraints are compatible if and only if a certain algebraic relation holds [viz., (4.4) below in class A, and (4.9) below in class B]. The method of proof then lies in demanding that this relation holds *in an open set*; i.e., it must remain valid when propagated along $\mathbf{e}_0$.

Class A. The shear propagation equations (A.4) can be written in the form

$$\begin{aligned} k(\partial_0\theta + \theta^2) &= \partial_0[\sigma_{11} - \tfrac{1}{2}(\sigma_{22} + \sigma_{33})] + \theta[\sigma_{11} - \tfrac{1}{2}(\sigma_{22} + \sigma_{33})] \\ &= -2n_{11}^2 + n_{22}^2 + n_{33}^2 + [n_{11} - \tfrac{1}{2}(n_{22} + n_{33})](n_{11} + n_{22} + n_{33}) \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} l(\partial_0\theta + \theta^2) &= \partial_0(\sigma_{22} - \sigma_{33}) + \theta(\sigma_{22} - \sigma_{33}) \\ &= -2(n_{22}^2 - n_{33}^2) + (n_{22} - n_{33})(n_{11} + n_{22} + n_{33}) \end{aligned} \tag{4.3}$$

Without imposing any further restrictions, such as a specific equation of state, there are no more constraints that must be satisfied. It therefore remains to determine under what conditions (4.2) and (4.3) are compatible. This is the case if and only if

$$\begin{aligned} l\{-2n_{11}^2 + n_{22}^2 + n_{33}^2 + [n_{11} - \tfrac{1}{2}(n_{22} + n_{33})](n_{11} + n_{22} + n_{33})\} \\ = k\{-2(n_{22}^2 - n_{33}^2) + (n_{22} - n_{33})(n_{11} + n_{22} + n_{33})\} \end{aligned} \tag{4.4}$$

for some constants k and l satisfying $k^2 + l^2 \neq 0$.

Case 1. Suppose first that $\sigma_{\alpha\beta}$ possesses two equal eigenvalues. By renumbering if necessary, we can arrange for this to imply $l = 0$. Then we obtain, from (4.4), $(n_{22} - n_{33})(n_{22} + n_{33} - n_{11}) = 0$. Thus either $n_{22} = n_{33}$, or $n_{11} = n_{22} + n_{33}$, $n_{22} \neq n_{33}$. If $n_{22} = n_{33}$, the possibilities that arise are $n_{11} = 0, n_{22} = n_{33} = 0$ (type I), $n_{11} \neq 0, n_{22} = n_{33} = 0$ (type II), $n_{11} = 0, n_{22} = n_{33} \neq 0$ (type VII_0), and $n_{11} \neq 0, n_{22} = n_{33} \neq 0$ (types VIII or IX, according as $n_{11}n_{22} < 0$ or $n_{11}n_{22} > 0$). All such models are L.R.S. [1]. The space-time of metric (2.1) is an example of a type VIII L.R.S. space-time. If $n_{11} = n_{22} + n_{33}$ and $n_{22} \neq n_{33}$, there is only one possibility: $n_{11} = 0, n_{22} = -n_{33} \neq 0$ (type VI_0, $n^\alpha{}_\alpha = 0$). These space-times are *not* L.R.S. (and in fact the only class A space-times in which the shear tensor possesses two equal eigenvalues but which are *not* L.R.S. are those of type VI_0 with $n^\alpha{}_\alpha = 0$) [1].

Case 2. The eigenvalues of $\sigma_{\alpha\beta}$ are distinct. Thus $l(l - 2k)(l + 2k) \neq 0$, and we must investigate the preservation of (4.4) in general. We may write (4.4) as the vanishing of a homogeneous polynomial of degree 2 in n_{11}, n_{22}, and n_{33}:

$$An_{11}^2 + Bn_{22}^2 + Cn_{33}^2 + 2Dn_{22}n_{33} + 2En_{33}n_{11} + 2Fn_{11}n_{22} = 0 \quad (4.5)$$

where

$$A = 2D = -l, \quad B = 2E = \tfrac{1}{2}l + k, \quad \text{and} \quad C = 2F = \tfrac{1}{2}l - k$$

One possibility is $n_{11} = n_{22} = n_{33} = 0$ (type I), in which case (4.5) is identically satisfied, and there is no more to prove. The case where two eigenvalues of $n_{\alpha\beta}$ are zero, and the third is not, is inadmissible, since $ABC \neq 0$. We now show that the remaining case (in which there is at most one zero eigenvalue) is inadmissible.

Demanding that (4.5) be preserved yields, using (A.2), a second condition:

$$A'n_{11}^2 + B'n_{22}^2 + C'n_{33}^2 + 2D'n_{22}n_{33} + 2E'n_{33}n_{11} + 2F'n_{11}n_{22} = 0 \quad (4.6)$$

where $A' = A(1 - 4k)$, $B' = B(1 - 3l + 2k)$, $C' = C(1 + 3l + 2k)$, $D' = D(1 + 2k)$, $E' = E(1 + 3l/2 - k)$, and $F' = F(1 - 3l/2 - k)$. If we suppose without loss of generality that n_{11} is a nonzero eigenvalue of $n_{\alpha\beta}$, we may divide (4.5) and (4.6) by n_{11}^2. We thus obtain two equations for the quantities n_{22}/n_{11} and n_{33}/n_{11}. Each equation can be considered geometrically as defining a conic section in the $(n_{22}/n_{11}, n_{33}/n_{11})$ plane. Either these conic sections intersect in a finite number of points, or they are identical, or they are line pairs, with one line in common. In the first case, n_{22}/n_{11} and n_{33}/n_{11} are constants, and then equation (A.2) shows that $n_{22} = n_{33} = 0$. This is impossible, as we have already observed. If the conic sections were identical, the ratio of coefficients in (4.5) and (4.6) would have to be the same, and this is impossible. Finally, there is the case of each conic section being a line pair, and the two line pairs having one line in common. In this case there is a linear relationship connecting n_{11}, n_{22}, and n_{33}, of the form

$$\alpha n_{11} + \beta n_{22} + \gamma n_{33} = 0 \quad (\alpha^2 + \beta^2 + \gamma^2 \neq 0) \quad (4.7)$$

The requirement that (4.8) be preserved yields, using (A.2), the constraint

$$\alpha(1 - 4k)n_{11} + \beta(1 - 3l + 2k)n_{22} + \gamma(1 + 3l + 2k)n_{33} = 0 \quad (4.8)$$

As before, we can divide both equations (4.7) and (4.8) by n_{11} and regard them as specifying two straight lines in the $(n_{22}/n_{11}, n_{33}/n_{11})$ plane. These lines cannot be distinct, since then n_{22}/n_{11} and n_{33}/n_{11} would be constants, which, as we have seen, is not admissible. Therefore equations (4.7) and (4.8) are equivalent. Suppose first that $\alpha \neq 0$. Then either $\beta \neq 0$ or $\gamma \neq 0$ (or both), since $\beta = \gamma = 0 \Rightarrow n_{11} = 0$, a contradiction. Without loss of generality, $\beta \neq 0$, and so, by equating coefficients, we obtain $l = 2k$, a contradiction. Now suppose that $\alpha = 0$. Then at least one of β or γ is nonzero. Suppose, without loss of generality, that $\beta \neq 0$. If $\gamma \neq 0$, then, again by equating coefficients in (4.7) and (4.8), we have $l = 0$, a contradiction. Finally, if $\alpha = \gamma = 0$ and $\beta \neq 0$, then $n_{22} = 0$, and

(4.5) yields $An_{11}^2 + Cn_{33}^2 + 2En_{11}n_{33} = 0$, which implies either n_{33}/n_{11} is constant or $A = C = E = 0$, both of which are contradictions. Therefore, the only additional possibility in case 2 is $n_{11} = n_{22} = n_{33} = 0$, which is a type I space-time in which the shear eigenvalues are distinct.

Class B. Since we are not considering class B models which belong to case Bbii, i.e., which are type VI_h with $h = -\frac{1}{9}$, it follows that a^β is a shear eigenvector [1]. Thus, in a shear eigenframe,

$$a_\beta = (a, 0, 0) \qquad \text{with } a \neq 0, \qquad \text{and } n_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & n_{22} & n_{23} \\ 0 & n_{23} & n_{33} \end{pmatrix}$$

We shall assume that the shear eigenframe is (or can be chosen to be) Fermi-propagated, and use the same technique as before. The shear propagation equations (A.4) can be written in the form

$$k(\partial_0\theta + \theta^2) = \tfrac{1}{2}\,[n_{22} - n_{33})^2 + 4n_{23}^2]$$

and

$$l(\partial_0\theta + \theta^2) = 4an_{23} - n_{22}^2 + n_{33}^2$$

These equations are compatible if and only if

$$k\,[4an_{23} - n_{22}^2 + n_{33}^2] = \frac{l}{2}\,[(n_{22} - n_{33})^2 + 4n_{23}^2] \tag{4.9}$$

for constants k and l satisfying $k^2 + l^2 \neq 0$.

The (01) field equation (A.3) shows that, in a shear eigenframe,

$$3a\sigma_{11} = n_{23}(\sigma_{22} - \sigma_{33})$$

which, with our present specialization, becomes

$$2ak = ln_{23}. \tag{4.10}$$

It is clear from (A.1) and (A.2) that (4.10) is automatically preserved in time.

Substituting (4.10) into (4.9), we obtain

$$(n_{22} - n_{33})\left[k(n_{22} + n_{33}) + \frac{l}{2}(n_{22} - n_{33})\right] = 0 \tag{4.11}$$

If we demand that (4.11) holds in an open set, we obtain, renumbering if necessary, the following possibilities:

(i) $n_{22} = n_{33} = 0; n_{23} = 0$. The space-time is of type V. Equation (4.10) requires that $k = 0$, which is equivalent to $\sigma_{11} = 0$; thus the model is not L.R.S.

(ii) $n_{22} = n_{33} = 0; n_{23} \neq 0$. The space-time is of type VI_h with $n^\alpha{}_\alpha = 0$. This is L.R.S. if and only if $n_{23} = \pm a \Leftrightarrow h = -1$ (and there is a one-parameter family of Bianchi type III).

(iii) $n_{22} \neq n_{33}$ and $k = l/2 \neq 0, n_{22} = 0, n_{33} \neq 0$. The space-time is of type VI_h with $h = -1$ and $n^{\alpha}{}_{\alpha} \neq 0$. This is L.R.S. (and the space-time is invariant under a one-parameter family of groups of Bianchi type III, and a group of type VIII). This case was obtained in our discussion of class A, case 1a above, and is exemplified by metric (2.1).

This completes the proof. □

Despite the apparent freedom in being able to impose the condition (i) concerning the eigenvalues of σ_{ij}, it should be recognized that in general this may give rise to models in which the fluid variables are unreasonable in some sense (for instance, the usual energy conditions might be violated in some epoch of the evolution; cf. the remarks in Section 3). We have not attempted an exhaustive treatment of this question, but indications are that, at least for some of the simpler types of models, the energy conditions are satisfied. For instance, there are certain cases in which the energy density, μ, and the pressure, p, are such that $p = (\gamma - 1)\mu$ $(1 \leqslant \gamma \leqslant 2)$ and μ/θ^2 is constant. Some of these solutions have previously been obtained: for type I with $\gamma = 2$ [10], for type II with $1 \leqslant \gamma < 2$ [11], for type VI_0 with $\gamma = 1$ [1] and with $1 < \gamma < 2$ [11], and for type VI_h with $1 \leqslant \gamma < 2, h \neq \frac{1}{9}$, and $-(2-\gamma)/(3\gamma - 2) < h < 0$ [11]. Conditions under which $\mu \propto \theta^2$ implies $\sigma \propto \theta$ are given in a theorem in [12]. There are also cases in our analysis in which μ/θ^2 is not constant. These, together with the Bianchi type VIII and IX solutions in [13], provide further examples, in addition to those listed in [12], which show that the converse result (that $\sigma \propto \theta$ implies $\mu \propto \theta^2$) is false.

Because the only field equations to be used in this section are the (0ν) and trace-free $(\beta\delta)$ equations, the above theorem is also valid when the cosmological term is included.

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Appendix

The following equations are relevant to the discussion in Section 4, and are taken from [1]. The quantity Ω^{α} is the angular velocity of the shear eigenframe, relative to a Fermi-propagated frame; in the discussion in Section 4, $\Omega^{\alpha} = 0$.

Jacobi Identities

$$\partial_0 a_{\alpha} + \sigma_{\alpha\beta} a^{\beta} + \tfrac{1}{3}\theta a_{\alpha} + \epsilon_{\alpha\beta\gamma} a^{\beta} \Omega^{\gamma} = 0 \tag{A.1}$$

$$\partial_0 n_{\alpha\beta} + 2n^{\gamma}{}_{(\alpha}\epsilon_{\beta)\gamma\pi}\Omega^{\pi} - 2n_{\gamma\,(\alpha}\sigma_{\beta)}{}^{\gamma} + \tfrac{1}{3} n_{\alpha\beta}\theta = 0 \tag{A.2}$$

Field Equations

$$(0\nu):\ 3a_\kappa \sigma^\kappa{}_\nu - \epsilon_{\nu\kappa\tau} n^{\tau\mu} \sigma_\mu{}^\kappa = 0 \tag{A.3}$$

Trace-free $(\beta\delta)$:

$$\partial_0 \sigma_{\beta\delta} = -\theta\sigma_{\beta\delta} + 2\sigma^\kappa{}_{(\beta}\epsilon_{\delta)\tau\kappa}\Omega^\tau + 2\epsilon_{\tau\sigma(\beta}n_{\delta)}{}^\tau a^\sigma - 2n_{\delta\mu}n^\mu{}_\beta + nn_{\beta\delta} + \tfrac{1}{3}\delta_{\beta\delta}(2n^{\kappa\tau}n_{\kappa\tau} - n^2) \tag{A.4}$$

Here $n = n^\alpha{}_\alpha$.

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