GRG Vol.4, No.2 (1973), pp. 105 - 129.

RESEARCH ARTICLES

ON SHEAR FREE NORMAL FLOWS OF A PERFECT FLUID

A. BARNES†

Department of Mathematics,
Imperial College of Science and Technology, London SW7¶

Received 25 *August* 1971

ABSTRACT

Flows of a perfect fluid in which the flow-lines form a time-like shear-free normal congruence are investigated. The space-time is quite severely restricted by this condition on the flow: it must be of Petrov Type I and is either static or degenerate. All the degenerate fields are classified and the field equations solved completely, except in one class where one ordinary differential equation remains to be solved. This class contains the spherically symmetric non-uniform density fields and their analogues with planar or hyperbolic symmetry. The type D fields admit at least a one-parameter group of local isometries with space-like trajectories. All vacuum fields which admit a time-like shear-free normal congruence are shown to be static. Finally, shear-free perfect fluid flows which possess spherical or a related symmetry are considered, and all uniform density solutions and a few non-uniform density solutions are found. The exact solutions are tabulated in section 7.

§(1): *INTRODUCTION*

Non-degenerate vacuum fields admitting a shear-free normal congruence of time-like curves were considered by Trümper [*1*] and were shown to be static. It is interesting, therefore, to investigate whether this result can be generalised. It is found that this is

† Supported by a Science Research Council Research Studentship and by a Turner and Newall Research Fellowship.

¶ Present address: Department of Applied Mathematics, The University, Newcastle upon Tyne, Northumberland, England.

the case for degenerate vacuum fields and non-degenerate perfect fluid fields. In the latter case the generalisation depends essentially on the fact that the stress tensor is spatially isotropic.

If the mean dynamical velocity, energy-density, and pressure are denoted by u^a, μ, and p respectively, the energy-momentum tensor can be written as

$$T_{ab} = \mu u_a u_b + p h_{ab}, \tag{1.1}$$

where h_{ab} is the projection tensor into the infinitesimal three-space orthogonal to u^a, defined by

$$h_{ab} = g_{ab} + u_a u_b. \tag{1.2}$$

Equation (1.1) represents a vacuum field if $p = -\mu = \Lambda$, where Λ is the cosmological constant. The field equations, $R_{ab} - \frac{1}{2}Rg_{ab} = T_{ab}$, become

$$R_{ab} = \tfrac{1}{2}(\mu + 3p)u_a u_b + \tfrac{1}{2}(\mu - p)h_{ab}, \tag{1.3}$$

where units are chosen so that $8\pi G = c = 1$.

The tensor $u_{a;b}$ may be split up as follows [2]

$$u_{a;b} = \omega_{ab} + \sigma_{ab} + \frac{1}{3}\theta h_{ab} - \dot{u}_a u_b, \tag{1.4}$$

where

$$\omega_{(ab)} = \sigma_{[ab]} = \omega_{ab}u^b = \sigma_{ab}u^b = \dot{u}_b u^b = \sigma^b{}_b = 0.$$

The kinematic quantities ω_{ab}, σ_{ab}, θ, and $\dot{u}_a$ represent respectively the vorticity, shear, volume expansion and acceleration of the flow. The conservation equations, $T_a{}^b{}_{;b} = 0$, are

$$h_a{}^b p_{,b} + \dot{u}_a(p + \mu) = 0, \tag{1.5}$$

$$\dot{\mu} + (p + \mu)\theta = 0. \tag{1.6}$$

The following propagation equations for the kinematic quantities defined above may be obtained [3-5] from the Ricci identity,

$$u_{a;[bc]} = \tfrac{1}{2}u^d R_{dabc},$$

and the field equations

$$\dot{\theta} + \frac{1}{3}\theta^2 - \dot{u}^a{}_{;a} + 2(\sigma^2 - \omega^2) + \tfrac{1}{2}(\mu + 3p) = 0, \tag{1.7}$$

$$h_a{}^c h_b{}^d \dot{\omega}_{cd} - h_a{}^c h_b{}^d \dot{u}_{[c;d]} + 2\sigma_{d[a}\omega^d{}_{b]} + \frac{2}{3}\theta\omega_{ab} = 0, \qquad (1.8)$$

$$h_a{}^c h_b{}^d \dot{\sigma}_{cd} - h_a{}^c h_b{}^d \dot{u}_{(c;d)} - \dot{u}_a \dot{u}_b + \omega_{ad}\omega^d{}_b + \sigma_{ad}\sigma^d{}_b$$

$$+ \frac{2}{3}\theta\sigma_{ab} + \frac{1}{3}h_{ab}(2(\omega^2 - \sigma^2) + \dot{u}^c{}_{;c}) + E_{ab} = 0, \qquad (1.9)$$

$$\omega_{[ab;c]} - \dot{u}_{[a;c}u_{b]} - \dot{u}_{[a}\omega_{bc]} = 0, \qquad (1.10)$$

$$h^a{}_b(\omega^{bc}{}_{;c} - \sigma^{bc}{}_{;c} + \frac{2}{3}\theta^{,b}) + (\omega^a{}_b + \sigma^a{}_b)\dot{u}^b = 0, \qquad (1.11)$$

$$2\dot{u}_{(a}\omega_{b)} + h^c{}_a h^d{}_b[\omega_{(c}{}^{e;f} + \sigma_{(c}{}^{e;f}]\eta_{d)gef}u^g = H_{ab}, \qquad (1.12)$$

where E_{ab} and H_{ab} are respectively the 'electric' and 'magnetic' parts of the Weyl tensor with respect to u^a, and are defined by

$$E_{ab} = C_{acbd}u^c u^d,$$

and

$$H_{ab} = \tfrac{1}{2}\eta_{acef}C^{ef}{}_{bd}u^c u^d,$$

the Weyl tensor C_{abcd} being defined by the equation

$$C^{ab}{}_{cd} = R^{ab}{}_{cd} - 2\delta^{[a}{}_{[c}R^{b]}{}_{d]} + \frac{1}{3}\delta^a{}_{[c}\delta^b{}_{d]}R.$$

The Bianchi identities, $R_{ab[cd;e]} = 0$, are equivalent to the equations

$$C^{abcd}{}_{;d} = R^{c[a;b]} - \frac{1}{6}g^{c[a}R^{,b]}. \qquad (1.13)$$

Equations (1.13) are equivalent to the conservation equations (1.5,6) and the sixteen integrability conditions [3-5]:

$$h^b{}_a E_{bc;d}h^{cd} + 3H_{ab}\omega^b + \eta_{abcd}u^b\sigma^{ce}H^d{}_e = \frac{1}{3}\mu_{,b}h^b{}_a, \qquad (1.14)$$

$$h^b{}_a H_{bc;d}h^{cd} - 3E_{ab}\omega^b + \eta_{abcd}u^b\sigma^{ce}E^d{}_e = (\mu + p)\omega_a, \qquad (1.15)$$

$$h^c{}_a h^d{}_b \dot{E}_{cd} + h^f{}_{(a}\eta_{b)cde}u^c H_f{}^{d;e} + E_{ab}\theta - 3E^c{}_{(a}\sigma_{b)c} + \ldots \quad \text{(Cont.)}$$

$$+ h_{ab}E_{cd}\sigma^{cd} - E^{c}{}_{(a}\omega_{b)c} + 2H^{d}{}_{(a}\eta_{b)cde}u^{c}\dot{u}^{e} \qquad \text{(Cont.)}$$

$$= -\tfrac{1}{2}(\mu + p)\sigma_{ab}, \tag{1.16}$$

$$h^{c}{}_{a}h^{d}{}_{b}\dot{H}_{cd} - h^{f}{}_{(a}\eta_{b),cde}u^{c}E_{f}{}^{d;e} + H_{ab}\theta - 3H^{c}{}_{(a}\sigma_{b)c}$$

$$+ h_{ab}H_{cd}\sigma^{cd} - H^{c}{}_{(a}\omega_{b)c} - 2E^{d}{}_{(a}\eta_{b)cde}u^{c}\dot{u}^{e} = 0. \tag{1.17}$$

§(2): *SPACE-TIMES ADMITTING A SHEAR-FREE NORMAL CONGRUENCE*

If the vorticity vanishes a congruence is called normal and the vector u^a is hypersurface-orthogonal [*6*]. If in addition the congruence is shear-free, coordinates may be chosen [*1*] so that

$$h_{\mu\nu} = P^{-2}(x^a)\gamma_{\mu\nu}(x^\lambda), \tag{2.1}$$

and consequently the metric may be written as

$$G = P^{-2}\gamma_{\mu\nu}dx^\mu dx^\nu - V^2(x^a)dt^2, \tag{2.2}$$

where Greek and Latin indices run from 1 to 3 and 1 to 4 respectively. The expansion and acceleration of the flow are given by

$$\theta = -3(PV)^{-1}P', \qquad \dot{u}_\mu = V^{-1}V_{|\mu}, \tag{2.3}$$

where a prime and a single stroke represent respectively differentiation with respect to t and x^μ.

It follows immediately from equation (1.12) with $\omega_{ab} = \sigma_{ab} = 0$, that $H_{ab} = 0$, and consequently that the space-time is of type I with vanishing eigenpseudoscalars. If $(e_A{}^a, u^a)$ is a tetrad of Weyl principal vectors, we may write

$$E^{ab} = \sum_{A=1}^{3} \alpha_A e_A{}^a e_A{}^b. \tag{2.4}$$

This result, which does not depend on any assumption concerning the energy-momentum tensor, was proved by Trümper [*16*], using different methods.

Attention will be confined from now on to shear-free normal flows of a perfect fluid. With the aid of equation (2.3), Raychaudhuri's equation (1.7) and equation (1.11) become

$$V^{-1}V^{|\mu}{}_{\|\mu} - 3\left(\frac{P'}{PV}\right)^2 + 3V^{-1}\left(\frac{P'}{PV}\right)' = \tfrac{1}{2}(\mu + 3p) \tag{2.5}$$

and

$$\theta_{|\mu} = 0, \quad \text{i.e.} \quad \theta = \theta(t), \tag{2.6}$$

where a double stroke denotes a covariant derivative taken with the metric $h_{\mu\nu}$. The remaining field equations are [1]

$$\tilde{R}^{\mu}{}_{\nu} - V^{-1}V^{|\mu}{}_{\|\nu} + \left[3\left(\frac{P'}{PV}\right)^{2} - V^{-1}\left(\frac{P'}{PV}\right)'\right]\delta^{\mu}{}_{\nu} = \tfrac{1}{2}(\mu - p)\delta^{\mu}{}_{\nu}, \tag{2.7}$$

where $\tilde{R}_{\mu\nu}$ is the Ricci tensor of $h_{\mu\nu}$. The equations for the propagation of vorticity (1.8) and shear (1.9) are respectively

$$\dot{u}_{[a;b]} + \ddot{u}_{[a}u_{b]} + \frac{1}{3}\theta\dot{u}_{[a}u_{b]} = 0, \tag{2.8}$$

$$E_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{2}{3}(\mu - \frac{1}{3}\theta^{2})h_{\mu\nu} = P_{\mu\nu}, \tag{2.9}$$

where $P_{\mu\nu}$ is the trace-free part of $\tilde{R}_{\mu\nu}$. If the acceleration is Fermi-propagated along u^{a}, i.e. if $\ddot{u}_{[a}u_{b]} = 0$, it follows that $(\dot{u}_{[a} - \frac{1}{3}\theta u_{[a});_{b]} = 0$, and the space-time admits a conformal Killing vector, or, if $\theta = 0$, a Killing vector parallel to u^{a} [2]. From equation (2.9) it follows that the Weyl principal vectors are eigenvectors of $\tilde{R}_{\mu\nu}$ with eigenvalues

$$\alpha_{A} + \frac{2}{3}(\mu - \frac{1}{3}\theta^{2}).$$

If tetrad components of equations (1.14-17) are taken with respect to a Weyl principal tetrad $(e_{A}{}^{a}, u^{a})$ and the summation convention is suspended, the following equations are obtained:

$$\alpha_{A.A} + \sum_{D}(\alpha_{A} - \alpha_{D})\gamma_{ADD} - \frac{1}{3}\mu_{.A} = 0, \tag{2.10}$$

$$\alpha_{A.4}\delta_{AC} + (\alpha_{C} - \alpha_{A})\gamma_{CA4} + \theta\alpha_{A}\delta_{AC} = 0, \tag{2.11}$$

$$\gamma_{123}(\alpha_{1} - \alpha_{2}) = \gamma_{231}(\alpha_{2} - \alpha_{3}) = \gamma_{312}(\alpha_{3} - \alpha_{1}) = E, \quad \text{say}, \tag{2.12}$$

$$\alpha_{A.C} - \alpha_{B.C} + (\alpha_{C} - \alpha_{B})\gamma_{CBB} - (\alpha_{C} - \alpha_{A})\gamma_{CAA} + 2V^{-1}V_{.C}(\alpha_{A} - \alpha_{B}) = 0, \tag{2.13}$$

where a dot signifies a tetrad component of a covariant derivative, upper case Latin indices run from 1 to 3, and the γ's are the Ricci rotation coefficients of the tetrad. Equation (2.11) is equivalent to the equation $(P^{-1}P_{\mu\nu})' = 0$, and with the aid of equation (2.3) we obtain

$$\alpha_A = P^3 \beta_A(x^\lambda), \tag{2.14}$$

$$(\alpha_C - \alpha_A)\gamma_{CA4} = 0, \qquad \text{for} \quad A \neq C. \tag{2.15}$$

Since $\sigma_{ab} = \omega_{ab} = 0$, it follows from equation (1.4) that

$$\gamma_{4AB} = 0, \qquad \gamma_{4AA} = \gamma_{4BB}, \qquad \text{for} \quad A \neq B.$$

For non-degenerate fields it follows from equations (2.12,15) that the Ricci eigenvectors are Fermi-propagated along u^a and that either all or none of the eigenvectors are hypersurface orthogonal. For degenerate fields the following theorem is valid.

Theor. 1. A degenerate field contains a Weyl principal tetrad which is hypersurface orthogonal.

Proof: For type D fields we may choose $\alpha_1 = \alpha_2$, without loss of generality and from equations (2.12,13,15) it follows that

$$\gamma_{314} = \gamma_{234} = \gamma_{312} = \gamma_{231} = \gamma_{411} - \gamma_{422} = \gamma_{311} - \gamma_{322} = 0. \tag{2.16}$$

The vectors $e_1{}^a$ and $e_2{}^a$ are only determined up to a rotation of the form

$$\tilde{e}_1{}^a = e_1{}^a \cos\phi + e_2{}^a \sin\phi, \quad \tilde{e}_2{}^a = -e_1{}^a \sin\phi + e_2{}^a \cos\phi. \tag{2.17}$$

Equation (2.16) is unaltered by this rotation, but since

$$\tilde{\gamma}_{124} = \gamma_{124} + \phi_{.4}, \qquad \tilde{\gamma}_{123} = \gamma_{123} + \phi_{.3},$$

γ_{124} may be set equal to zero by means of a suitable rotation. If $\gamma_{123} \neq 0$, it may be made so by means of a time-independent rotation of the form (2.17). This can be seen as follows: the orthonormal triad $(\eta_A{}^\mu)$ of the auxiliary metric $\gamma_{\mu\nu}(x^\lambda)$ defined by $\eta_A{}^\mu = P^{-1} e_A{}^\mu$, $\eta_{A\mu} = P e_{A\mu}$, is time-independent (since $\gamma_{CA4} = 0$) and hence so are its rotation coefficients Γ_{ABC}. The rotation coefficients are related to those of $(e_A{}^\mu)$ by the equation

$$\gamma_{ABC} = P\Gamma_{ABC} + 2\delta_{C[A} P_{:B]}, \tag{2.18}$$

where $P_{:B} = P_{|\lambda} n_B{}^\lambda$. Hence $\Gamma_{312} = \Gamma_{231} = 0$, and $\gamma_{123} = P\Gamma_{123}$, and since under the rotation (2.16) $\tilde{\Gamma}_{123} = \Gamma_{123} + \phi_{:3}$, γ_{123} may be set equal to zero by a suitable time-independent rotation. For the conformally flat case it follows from equation (2.8) that $P_{\mu\nu} = 0$, and consequently the space cross-sections are of uniform curvature

$K(t)$ [7], and coordinates may be chosen so that [8]

$$P = \alpha + \beta r^2 - 2\mathbf{r}.\mathbf{a}(t), \qquad \gamma_{\mu\nu} = \delta_{\mu\nu}, \qquad K = 4(\alpha\beta - \mathbf{a}^2), \quad (2.19)$$

where $\mathbf{r} = x^1\mathbf{i} + x^2\mathbf{j} + x^3\mathbf{k}$, and α, β, and $\mathbf{a}$ are functions of t. The coordinate vectors, $P^{-1}\delta_A{}^a$, and the velocity vector u^a are hypersurface orthogonal.

Since the vectors are hypersurface orthogonal, coordinates may be chosen so that

$$h_{\mu\nu}dx^\mu dx^\nu = P^{-2}\{(e^{\gamma_1}dx^1)^2 + (e^{\gamma_2}dx^2)^2 + (e^{\gamma_3}dx^3)^2\}, \quad (2.20)$$

where $\gamma_A = \gamma_A(x^\lambda)$. The coordinate vectors are Ricci eigenvectors and they and the rotation coefficients are given by

$$e_A{}^a = Pe^{-\gamma_A}\delta_A{}^a, \quad \gamma_{ABC} = 0,$$
$$\text{for } A,B,C \neq. \quad (2.21)$$
$$\gamma_{ABB} = Pe^{-\gamma_A}(\gamma_B - \log P)_{,a}\delta_A{}^a,$$

From equations (2.13,14,18) it follows that

$$(\beta_A - \beta_B)_{:C} + (\beta_C - \beta_B)\Gamma_{CBB} - (\beta_C - \beta_A)\Gamma_{CAA}$$
$$+ 2(\beta_A - \beta_B)(\log PV)_{:C} = 0. \quad (2.22)$$

The only term involving t in this equation is $\log PV$ and consequently if the space is non-degenerate, it may be deduced that $PV = f(x^\lambda)$ $g(t)$. After a suitable redefinition of $\gamma_{\mu\nu}$, t, and V, we see that the metric (2.2) is of the form

$$G = V^2\{\gamma_{\mu\nu}(x^\lambda)dx^\mu dx^\nu - dt^2\}. \quad (2.23)$$

The metric admits a conformal Killing vector and from equation (2.8) it follows that $\dot{u}_{[a}u_{b]} = 0$, and hence that $V^{-1} = A(x^\lambda) + B(t)$. Consequently we have proved that a non-degenerate perfect fluid field admits a conformal Killing vector parallel to the flow. A theorem of Trümper [1], which states that a vacuum space-time admitting a conformal Killing vector is either conformally flat or static, may be generalised immediately to include the case of a perfect fluid field in which the flow-lines are parallel to a conformal Killing vector field (since the only additional term in the field equations is proportional to $\gamma_{\mu\nu}$). Hence the following theorem is valid.

Theor. 2. A non-degenerate perfect fluid field in which the flow-lines form a normal shear-free congruence is static.

Degenerate fields will now be considered, the conformally flat ones in section 3 and those of type D in sections 4-6.

§(3): *CONFORMALLY FLAT SPACE-TIMES*

Since $E_{\mu\nu} = 0$, it follows from equations (2.7,8) that

$$V'^{\mu}{}_{\|\nu} - \frac{1}{3} V'^{\lambda}{}_{\|\lambda}\delta^{\mu}{}_{\nu} = 0. \tag{3.1}$$

It may be deduced from equations (2.3,19) that the metric can be written in the form

$$G = (\alpha + \beta r^2 - 2\mathbf{r}.\mathbf{a})^{-2}\{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\} - V^2 dt^2, \tag{3.2}$$

where

$$V = -\frac{3}{\theta}\frac{\{\alpha' + \beta' r^2 - 2\mathbf{r}.\mathbf{a}'\}}{(\alpha + \beta r^2 - 2\mathbf{r}.\mathbf{a})},$$

where α, β and $\mathbf{a}$ are functions of t only and where three-dimensional vector notation has been used for conciseness. V satisfies equation (3.1) identically and from equations (1.6) and (2.8) it follows that the energy-density and pressure are given by

$$\mu = 3K + \frac{1}{3}\theta^2 = 12(\alpha\beta - \mathbf{a}^2) + \frac{1}{3}\theta^2, \tag{3.3}$$

$$p = -(3K + \frac{1}{3}\theta^2) - \frac{1}{\theta V}(3K + \frac{1}{3}\theta^2)'. \tag{3.4}$$

The metric (3.2) admits no Killing vectors in general since four functionally independent invariants may be constructed from μ, p and their derivatives if the rank of the five functions α, β, $\mathbf{a}(t)$ over the real constants is four or five. If the rank is three, only three invariants may be so constructed and there exists a space-like Killing vector. If the rank is two, the isometry group is of dimension three with two-dimensional space-like trajectories. The spherically symmetrical members of this class were obtained by Thompson and Whitrow [*9*]. The metric may be written in the form IB (see Table) with y given by equation (6.5). If the rank is one, $V = V(t)$ and the flow is geodesic and the solution is one of the Friedmann models [*3*].

If $\theta = 0$, the above analysis is not valid. However, coordinates may be chosen so that $P = 1 + \frac{1}{4}Kr^2$ where K is a constant. V is given by equation (3.1) and on integration it follows that

$$V = \frac{\{\alpha(t) + \beta(t)r^2 + \mathbf{r}.\mathbf{a}(t)\}}{(1 + \frac{1}{4}Kr^2)}. \tag{3.5}$$

The energy density and pressure are given by

$$\mu = 3K, \qquad p = -\ 3K + \frac{1}{V}(\alpha K + 4\beta). \tag{3.6}$$

If the rank of the functions α, β and a is greater than two, the dimension of the isometry group is the same as in the corresponding case with $\theta \neq 0$. If the rank is two, the metric admits a three-dimensional isometry group acting on two-dimensional space-like trajectories of constant curvature. The spherically symmetrical members of this class are the interior Schwarzschild metric with time-dependent pressure [*10*,*11*]. If the metric is static (i.e. if the rank is one) the metric is equivalent to that obtained by Stepanyuk [*12*] and admits a four-dimensional isometry group [*13*]. If $p \neq$ constant, this group is complete. If, however, p = constant $\neq -\mu$ the solution is the Einstein universe and admits a complete seven-dimensional group. If $p + \mu = 0$ the space is an Einstein space and being conformally flat is of constant curvature [*7*] (i.e. it is one of the de Sitter universes).

If the rank of the functions is one or two, the metric may be written in one of the following forms

$$G = (k - Kr^2)^{-1}dr^2 + r^2 d\Omega^2$$

$$- \{D(t) + E(t)(k - Kr^2)^{\frac{1}{2}}\}^2 dt^2 \qquad \text{for} \quad K \neq 0, \tag{3.7}$$

$$G = dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$- \{E(t) + D(t)r^2\}dt^2 \qquad \text{for} \quad K = 0, \tag{3.8}$$

where $d\Omega^2 = d\theta^2 + f^2(\theta)d\phi^2$ with $f(\theta) = \sin\theta$, θ, $\sinh\theta$, for $k = +1$, 0, -1, respectively. The energy-density and pressure are given by

$$\mu = 3K, \qquad p = -\ 3K + \frac{2DK}{V}, \qquad \text{for} \quad K \neq 0, \tag{3.9}$$

$$\mu = 0 \qquad p = \frac{4D}{V}, \qquad \text{for} \quad K = 0. \tag{3.10}$$

Finally, it is noted from equations (3.7-10) that all static conformally flat perfect fluid fields with non-negative density are spherically symmetrical.

§(4): *SPACE-TIMES OF TYPE D*

With the aid of equations (2.20,21) it follows that the integrability conditions are equivalent to the equations:

$$(\alpha - \frac{1}{3}\mu)_{.3} + 3\alpha(\gamma_C - \log P)_{.3} = 0, \tag{4.1}$$

$$\left.\begin{aligned}(\alpha + \tfrac{1}{6}\,\mu)_{.C} + \tfrac{3}{2}\,\alpha(\gamma_3 + \log[\tfrac{V}{P}])_{.C} &= 0,\\ (\alpha - \tfrac{1}{3}\,\mu)_{.C} + 3\alpha(\log V)_{.C} &= 0,\end{aligned}\right\} \tag{4.1}$$

where $C = 1,2$, and $\alpha_1 = \alpha_2 = -2\alpha_3 = -2\alpha$. It follows from equation (2.14) that P can be chosen equal to $\alpha^{1/3}$ in these equations and hence it may be deduced that

$$G = \alpha^{-2/3}(e^{2\gamma(x^\lambda)}d\sigma^2 + e^{2\gamma_3(x^\lambda)}(dx^3)^2 - e^{-\gamma_3+2Z(x^3,t)}dt^2), \tag{4.2}$$

where $d\sigma^2$ is a two-dimensional metric involving x^1 and x^2 only and

$$\gamma_{|3} = \frac{1}{9}\,\alpha^{-1}\mu_{|3}, \tag{4.3}$$

$$(\log\alpha^{1/3}V)_{|C} = \frac{1}{9}\,\alpha^{-1}\mu_{|C} = -\frac{1}{2}\,\gamma_{3|C}. \tag{4.4}$$

The conservation equation (1.5) becomes

$$p_{|\lambda} = -V^{-1}V_{|\lambda}(p + \mu), \tag{4.5}$$

and it is easily seen that since $V \neq V(t)$, otherwise $E_{\mu\nu} = 0$,

$$p = p(V,t), \qquad \mu = \mu(V,t). \tag{4.6}$$

The volume expansion is given by

$$\theta(t) = -(\alpha V)^{-1}\alpha'. \tag{4.7}$$

It follows from this equation and (1.6) that if $\theta = 0$,

$$\alpha = \alpha(x^\lambda), \qquad \mu = \mu(x^\lambda).$$

The fields may be divided into four invariant classes:

I. $\mu = \mu(t)$; II. $\mu = \mu(x^3,t)$; III. $\mu = \mu(x^1,x^2,t)$; IV. $\mu = \mu(x^1,x^2,x^3,t)$.

Class I may be subdivided into four invariant subclasses defined as follows:

A. α = constant;
B. $\alpha = \alpha(x^3,t)$. i.e. the hypersurface, α = constant, contains the space-like eigenblade of the Weyl tensor;

C. $\alpha = \alpha(x^1,x^2)$, i.e. the hypersurface, α = constant, contains the time-like eigenblade of the Weyl tensor;

D. $\alpha = \alpha(x^1,x^2,x^3,t)$, i.e. the hypersurface contains neither eigenblade.

This classification generalises that of static perfect fluid and vacuum fields [*13,14*].

Class I

Since $\mu = \mu(t)$, the conservation equation (4.5) can be integrated to yield

$$p + \mu = \frac{2A(t)}{V} . \qquad (4.8)$$

From equations (4.2-4) it follows that the metric may be written as

$$G = \alpha^{-2/3}(d\sigma^2 + (dx^3)^2 - e^{2Z(x^3,t)}dt^2). \qquad (4.9)$$

Class IA

For this class it can be shown that $p + \mu = 0$, and that both the eigenblades of the Weyl tensor are two-dimensional spaces of constant curvature μ. Consequently the metric is [*13,15*]

$$G = I(\mu)(d\theta^2 + f^2(\theta)d\phi^2 + (dx^3)^2 - f^2(x^3)dt^2),$$

where $f(\theta) = \sin\theta$, θ, $\sinh\theta$, and $I(\mu) = \mu^{-1}$, 1, $-\mu^{-1}$, for $\mu > 0$, $=0$, <0, respectively. The invariant α is given by $\alpha = -2\mu/3$, and for $\mu = 0$ it is easily seen that the space is flat. For $\mu \neq 0$, the space is static and the complete isometry and isotropy groups are of dimension six and two respectively.

Class IB

As $\alpha = \alpha(x^3,t)$, the metric (4.9) can be expressed as

$$G = R^2(z,t)\{d\sigma^2 + z^{-2}dz^2\} - V^2(z,t)dt^2, \qquad (4.10)$$

by means of a coordinate transformation of the form $z = z(x^3)$. The field equations are

$$-\frac{z^2}{R^2}\left(\frac{1-k}{z^2} + \frac{R_{|33}}{R} + \frac{V_{|3}R_{|3}}{VR}\right) + V^{-2}\left(\frac{R''}{R} + \frac{2R'^2}{R^2} - \frac{V'R'}{VR}\right)$$

$$= \tfrac{1}{2}(\mu - p), \qquad (4.11)$$

$$- \frac{z^2}{R^2}\left(\frac{V_{,33}}{V} + \frac{V_{,3}}{Vz} + \frac{2R_{,33}}{R} + \frac{2R_{,3}}{Rz} - \frac{2R_{,3}{}^2}{R^2} - \frac{V_{,3}R_{,3}}{VR}\right)$$

$$+ V^{-2}\left(\frac{R''}{R} + \frac{2R'^2}{R^2} - \frac{V'R'}{VR}\right) = \tfrac{1}{2}(\mu - p), \quad (4.12)$$

$$\frac{z^2}{R^2}\left(\frac{V_{,33}}{V} + \frac{V_{,3}R_{,3}}{VR} + \frac{V_{,3}}{Vz}\right) + 3V^{-2}\left(\frac{R'V'}{RV} - \frac{R''}{R}\right) = \tfrac{1}{2}(\mu + 3p), \quad (4.13)$$

$$\left(\frac{3R'}{RV}\right)_{,3} = 0. \quad (4.14)$$

It can be deduced from equation (4.11) that the Gaussian curvature k of $d\sigma^2$ is constant and consequently coordinates can be chosen so that $d\sigma^2 = d\Theta^2 + f^2(\Theta)d\phi^2$, where $f(\Theta) = \sin\Theta$, Θ, $\sinh\Theta$, for $k = 1$, 0, -1, respectively. If $\theta(t) = 0$, i.e. if $\alpha = \alpha(z)$, it follows that in terms of curvature coordinates the metric can be written as

$$G = r^2 d\sigma^2 + W^2(r)dr^2 - V^2(r,t)dt^2. \quad (4.15)$$

Integration of the field equations yields [13]

$$W^{-2} = k - \frac{1}{3}\mu r^2 - 2mr^{-1}, \quad (4.16)$$

$$V = W(B(t) + A(t)\int r(k - \frac{1}{3}\mu r^2 - 2mr^{-1})^{-3/2}dr). \quad (4.17)$$

The pressure is given by equation (4.8) and $\alpha = -2mr^{-3}$. If $m = 0$ the space-time is conformally flat and the metric reduces to the form (3.7). If $A(t)$ is proportional to $B(t)$, the field is static, and, in particular, if $A(t) = 0$, the metric represents a static vacuum field, and it is the exterior Schwarzschild line-element with cosmological constant or a related solution possessing planar or hyperbolic symmetry [10,17]. If $\alpha = \alpha(t)$, it follows from equation (4.7) that either $V = V(t)$ or α = constant As $V \neq V(t)$, otherwise $E_{\mu\nu} = 0$, it follows that α = constant and that the fields belong to class IA. The cases with $\alpha = \alpha(z,t)$ will be considered in section 6.

Classes C and D

For these classes $\alpha_{,C} \neq 0$ for $C = 1,2$. The field equations (2.7) become

$$\bar{\bar{R}}_{\mu\nu} + 2\alpha^{-1/3}(\alpha^{1/3})_{\shortparallel\mu\nu} - e^{-Z}(e^{Z})_{\shortparallel\mu\nu} = \ldots \quad \text{(Cont.)}$$

$$- \left[2p\alpha^{-2/3} + 2\left\{\left(\frac{\alpha'}{3\alpha}\right)^2 - \frac{2\alpha' Z'}{3\alpha}\right\} - \alpha^{2/3}e^{-Z}(\alpha^{-2/3}e^{Z})^{|\gamma}{}_{\|\gamma}\right]\bar{\bar{g}}_{\mu\nu} = 0, \quad (4.18)$$

where $\bar{\bar{R}}_{\mu\nu}$ and $\|$ denote the Ricci tensor and covariant derivative of the metric $\bar{\bar{g}}_{\mu\nu}dx^\mu dx^\nu = d\sigma^2 + (dx^3)^2$. From equations (4.7,18) with $\mu = 3$, $\nu = 1,2$, it may be deduced that

$$\pm\, \alpha^{1/3} = X(x^1,x^2) + Y(x^3,t). \quad (4.19)$$

For $\mu,\nu = 1,2$, equation (4.18) is equivalent to the equation

$$X_{\|\mu\nu} - \tfrac{1}{2}X^{|\lambda}{}_{\|\lambda}\bar{\bar{g}}_{\mu\nu} = 0,$$

and consequently the metric $d\sigma^2$ is of the form [*14*]

$$d\sigma^2 = f^{-1}dX^2 + fd\phi^2, \quad (4.20)$$

where $f = f(X)$.

Class C

As $\alpha = \alpha(x^1,x^2)$, after a translation of the coordinate X, Y may be set equal to zero. It can be shown [*13*] that $p + \mu = 0$, and

$$G = (k - \tfrac{1}{3}\,\mu x^2 - 2mx^{-1})^{-1}dx^2 + (k - \tfrac{1}{3}\,\mu x^2 - 2mx^{-1})d\phi^2 + x^2(dz^2 - f^2(z)dt^2), \quad (4.21)$$

where $f(z) = \sin z$, z, $\sinh z$, for $k = 1, 0, -1$, respectively. The time-like eigenblade of the Weyl tensor is of constant curvature and the solution is static. The complete isometry and isotropy groups are of dimension four and one respectively.

Class D

Only three of the field equations are not identically satisfied: the (3.3) component and the trace of the equation (4.18) and Raychaudhuri's equation (2.5). These equations are equivalent to

$$(X + Y)(Z_{|33} + Z_{|3}{}^2 - \tfrac{1}{2}f''(X)) - 2Y_{|33} + f'(X) = 0, \quad (4.22)$$

$$(X + Y)^2(Z_{|33} + Z_{|3}{}^2) - 2(X + Y)(Y_{|33} + Y_{|3}Z_{|3} + \tfrac{1}{2}f') + \ldots \text{(Cont.)}$$

$$\ldots + 3(Y_{|3}{}^2 + f) = p + (Y_{|4}{}^2 + 2(X + Y)Y_{|4}Z_{|4})e^{-2Z}, \quad (4.23)$$

$$(Y_{|33} + Y_{|3}Z_{|3} - Y_{|4}Z_{|4}e^{-2Z})(X + Y) = \tfrac{1}{2}(\mu + p) - Y_{|4}{}^2e^{-2Z}. \quad (4.24)$$

From equations (4.8,24) and the equation

$$V = (X + Y)^{-1}e^{Z},$$

it may be deduced by equating coefficients of X that

$$Y_{|4} = 0, \qquad Y_{|33} - Y_{|3}Z_{|3} = A(t)e^{-Z}.$$

Consequently the volume expansion is zero and the analysis of [*13*] shows that

$$G = (X + Y)^{-2}\Big\{f^{-1}\mathrm{d}X^2 + f\mathrm{d}\phi^2 + g^{-1}(Y)\mathrm{d}Y^2$$

$$- g(Y)(B(t) + A(t)\int g^{-3/2}\mathrm{d}Y)^2\mathrm{d}t^2\Big\}, \quad (4.25)$$

where

$$f(X) = \pm\, X^3 + cX + d, \qquad g(X) = -\, f(-X) - \frac{1}{3}\,\mu,$$

with $X + Y > 0$, and $\alpha = \pm(X + Y)^3$. The field is static if $A(t)$ is proportional to $B(t)$, and in particular if $A(t) = 0$, the metric represents a static vacuum field. The complete isometry group is Abelian and of dimension one or two.

It has been shown that all fields of classes A, C and D with $p + \mu = 0$, i.e. vacuum solutions with a cosmological constant, are static. This result is also valid for class B since the proof of Birkhoff's theorem [*18*] generalises immediately to include the cases with planar or hyperbolic symmetry. Consequently the following theorem is valid.

> *Theor.* 3. A vacuum space-time admitting a shear-free normal congruence of time-like curves is static.

§(5): *CLASSES II-IV (VARIABLE DENSITY)*

Class II

In this class $\mu = \mu(x^3,t)$ and consequently it is easily seen from equations (4.3,4,6) that $V = V(x^3,t)$, $p = p(x^3,t)$, $\alpha = \alpha(x^3,t)$, $\gamma = \gamma(x^3)$, and $\gamma_3 = \gamma_3(x^3)$, and by means of a coordinate transformation of the form $Z = Z(x^3)$, the metric may be put in a form identical with equation (4.10).

If $\theta = 0$, it follows that $\alpha = \alpha(z)$, and that the metric may be written in the form (4.11) with

$$W^{-2} = k - \frac{1}{r}\int_0^r \mu(r) r^2 dr - 2mr^{-1},$$

$$\alpha = \frac{1}{3} r^{-3}\int_0^r \mu'(r) r^3 dr - 2mr^{-3}.$$

The remaining field equation for the function V cannot be integrated in closed form except in special cases. This class contains the static spherically symmetric fields which have been considered extensively [*19–21*], and their generalisations with either time-dependent pressure or planar or hyperbolic symmetry. The general case with $\alpha = \alpha(z,t)$ will be considered in section 6.

Classes III and IV

Since $\mu_{,C} \neq 0$, for $C = 1,2$, it follows that $V_{,C} \neq 0$. As the coordinate vectors in equation (4.2) are eigenvectors of the spatial Ricci tensor $\tilde{R}_{\mu\nu}$, it follows that $\tilde{R}^3{}_C = 0$, and from equation (2.7) that $V'^3{}_{\|C} = 0$, i.e.

$$(V_{,3}e^{-\delta})_{,C} - \beta_{,3}V_{,C}e^{-\delta} = 0, \tag{5.1}$$

$$(\beta_{,3}e^{-\delta})_{,C} = 0, \tag{5.2}$$

where for simplicity we have written

$$e^{\beta} = \alpha^{-1/3}e^{\gamma}, \quad e^{\delta} = \alpha^{-1/3}e^{\gamma_3}, \quad V = \alpha^{-1/3}e^{-\frac{1}{2}\gamma_3 + Z}.$$

Integration of equations (5.1,2) yields

$$\beta_{,3}e^{-\delta} = h(z,t), \tag{5.3}$$

$$V_{,3}e^{-\delta} = h(z,t)V + \ell(z,t), \tag{5.4}$$

where h and ℓ are arbitrary functions of integration.

From equations (4.4,6) it follows that

$$\gamma_{3,[c}V_{,b]} = \alpha_{,[c}V_{,b]} = 0, \qquad \text{for } b,c = 1,2,$$

and consequently from equation (5.1) that

$$V_{,3[1}V_{,2]} = \gamma_{3,3[1}\gamma_{3,2]} = 0,$$

where the last index 3 is not included in the antisymmetrisation. On integrating the last equation it follows that $\gamma_3 = \gamma_3(f(x^1,x^2), x^3)$, and on making the coordinate transformations, $\tilde{x}^1 = X = f(x^1, x^2)$, $z = x^3$, that $\gamma_3 = \gamma_3(X,z)$. It is easily seen from equations (4.4,6) that

$$V = V(X,z,t), \quad \gamma = \gamma(X,z), \quad \alpha = \alpha(X,z,t), \quad \mu = \mu(X,z,t),$$

If the coordinate $x^2 = \phi$ is chosen so that the x^1- and x^2-lines are orthogonal, the metric can be written in the form

$$G = e^{2\beta}(A^2(X,\phi)dX^2 + B^2(X,\phi)d\phi^2) + e^{2\delta}dz^2 - V^2dt^2.$$

Since the space is of type D it follows that $\tilde{R}^\mu{}_\nu$ and consequently $V^{|\mu}{}_{\|\nu}$ are proportional to $\delta^\mu{}_\nu$ for $\mu,\nu = 1,2$. Hence

$$A^{-3}A_{|2}V_{|1}e^{-2\beta} = 0,$$

$$(V_{|1}e^{-2\beta})_{|1} - (A^{-1}A_{|1} + B^{-1}B_{|1})V_{|1}e^{-2\beta} = 0.$$

Consequently, $A_{|2} = 0$, and $B = \tilde{B}(X)g(\phi)$. By means of coordinate transformations of the form $\tilde{X} = \tilde{X}(X)$, $\tilde{\phi} = \tilde{\phi}(\phi)$, $A(X)$ may be set equal to $B(X)$, and on redefining β as $\beta + \log A$, the metric becomes

$$G = e^{2\beta(X,z,t)}(dX^2 + d\phi^2) + e^{2\delta(X,z,t)}dz^2 - V^2(X,z,t)dt^2. \tag{5.5}$$

The equations $\tilde{R}^1{}_1 = \tilde{R}^2{}_2$ and $V^{|1}{}_{\|1} = V^{|2}{}_{\|2}$ become, on integration,

$$V_{|1}e^{-2\beta} = f(z,t), \qquad \delta_{|1}e^{\delta-2\beta} = g(z,t), \tag{5.6}$$

where f and g are arbitrary functions of integration and $f \neq 0$, otherwise $V_{|1} = 0$. It follows from equation (5.6) that

$$e^\delta = m(z,t)V + k(z,t), \tag{5.7}$$

where $m = g/f$ and k is a function of integration.

If β, $V_{|3}$, μ and δ are eliminated from equation (5.1) with the aid of equations (4.3,4) and (5.3,4,7) it follows that

$$V_{|1}\left[\frac{1}{3}V\left\{2Z_{|3}m - \overset{*}{m}_{|3} - 3\ell^2m\right\}\right.$$

$$\left. + \frac{1}{3}\left\{2Z_{|3}k - k_{|3} - 6m\ell k\right\} - \ell k^2V^{-1}\right] = 0.$$

As $V_{|1} \neq 0$, it may be deduced that either

$$k = 0, \qquad 3\ell^2 m - 2Z_{|3}m - m_{|3} = 0.$$

or

$$\ell = 0, \qquad m_{|3} = 2Z_{|3}m, \qquad k_{|3} = 2Z_{|3}k. \tag{5.8}$$

In the former case it may be seen from equations (4.4,5,7) that $\gamma_{3|C} = \mu_{|C} = 0$, $C = 1,2$, and consequently such fields do not belong to classes III or IV, but to classes I or II. Hence equation (5.8) is valid. It follows from equations (5.3,4) that $\beta_{|3} = V^{-1}V_{|3}$, and hence $V^2 = H(X,t)e^{2\beta}$, or equivalently

$$e^{-\gamma_3 - 2\gamma} = H(X,t)e^{-2Z(z,t)}.$$

Since the left-hand side of this equation is independent of t it can be deduced that $e^Z = C(t)e^{\tilde{Z}(z)}$. Consequently the metric (4.2) admits a conformal Killing vector and is therefore static. All the functions in equation (5.5) may be made independent of t and the analysis of [13] applied to obtain the following solutions.

Class III

$$G = (n + mx)^{-2}\{F^{-1}dx^2 + Fd\phi^2 + dz^2 - x^2dt^2\},$$

with

$$F = F(x) = b + c(n^2\log x + 2mnx + \tfrac{1}{2}m^2x^2),$$

where $x > 0$, $m = \pm 1, 0$, and n, b, and c are constants. The pressure, energy-density, and the invariant α are given by

$$2p = cx^{-2}(n + mx)^3(n - mx) + 2m(m - \frac{2n}{x})F,$$

$$2\mu = cx^{-2}(n + mx)^3(n + 3mx) - 6m^2F,$$

$$\alpha = -\frac{1}{3}ncx^{-2}(n + mx)^3.$$

If $nc = 0$, the space-time is conformally flat. If $nc \neq 0$, it admits a complete three-dimensional Abelian isometry group.

Class IV

$$G = n^{-2}(z)\{F^{-1}dx^2 + Fd\phi^2 + dz^2 - x^2dt^2\},$$

with $F = F(x) = ax^2 + b\log x + c$, and $n = A\sin\sqrt{a}z$, Az, $A\sinh\sqrt{(-a)}z$, for $a > 0$, $=0$, <0, respectively, and where a, b, c and A are constants. The pressure, energy-density and α are given by

$$2p = 6A^2I(a) + bn^2x^{-2}, \quad 2\mu = -6A^2I(a) + bn^2x^{-2},$$

$$\alpha = -\frac{1}{3}bn^2x^{-2},$$

where $I(a) = 1$, $|a|$, for $a = 0$, $a \neq 0$, respectively. The complete isometry group is two-dimensional and Abelian.

§(6): *CLASSES IB AND II*

We recall that the space-time possesses spherical, planar, or hyperbolic symmetry and that for class IB $\mu = \mu(t)$, whereas for class II $\mu = \mu(z,t)$, with $\mu_{;3} \neq 0$. On equating the left-hand sides of equations (4.11,12) and integrating, it can be seen that

$$z^2R_{zz} - 2z^2R^{-1}R_z^{\,2} + zR_z + kR + b(z) = 0,$$

where b is an arbitrary function of integration, or equivalently, writing $y = R^{-1}$,

$$z^2y_{zz} + zy_z - ky = b(z)y^2 \tag{6.1}$$

From the field equations (4.13,14) it is easily seen that

$$\mu_z = \mu_{;3} = 2b'(z)R^{-3} \tag{6.2}$$

$$\alpha = \frac{2}{3}b(z)R^{-3} \tag{6.4}$$

$$V = 3(\theta R)^{-1}R'. \tag{6.4}$$

If $b = 0$, the space-time is conformally flat and equation (6.1) is easily integrated to give

$$\left.\begin{array}{ll} y = A(t)z^{-1} + B(t)z, & \text{for } k = +1, \\ y = A(t) + B(t)\log z, & \text{for } k = 0, \\ y = A(t)\cos\log z + B(t)\sin\log z, & \text{for } k = -1, \end{array}\right\} \tag{6.5}$$

where $A(t)$ and $B(t)$ are abitrary functions of integration. The energy-density is given by the equations

$$\mu - \frac{1}{3}\theta^2(t) = 12AB,\ -3B^2,\ -3(A^2 + B^2),$$

for $k = +1$, 0, -1, respectively. These solutions are special cases of those obtained in section 3, but those with $k = 0$ and $k = -1$ are expressed in a different coordinate system. The spherically symmetric solutions (i.e. $k = +1$) have been used by a number of authors [9,22,23] to investigate the motion of spheres of uniform density.

The fields of class I are characterised by the condition $b(z) = b \neq 0$, where b is a constant. The integral of equation (6.1) is

$$y = (2b)^{-1}\left\{\frac{1}{3}\wp(B(t) + \log z) - 1\right\}, \tag{6.6}$$

where $\wp$ is the Weierstrassian elliptic function with invariants g_2 and g_3 given by [24]

$$g_2 = \frac{1}{12}k^2, \qquad g_3 = -\frac{1}{72}(2C(t)b^2 + \frac{1}{3}k^3),$$

and where $B(t)$ and $C(t)$ are arbitrary functions of integration. The density is given by $\mu = \frac{1}{3}\theta^2(t) - 3C(t)$, and the mass function [25] by $m = (b/3) + \mu R^3/6$. Since $m_z = \frac{1}{2}\mu R^2 R_z$, it follows that $b/3$ may be interpreted as a point-mass at $R = 0$.

The elliptic function reduces to an elementary function in certain special circumstances. For $k = +1$ and $C = 0$,

$$y = 6b^{-1}\beta z(1 - \beta z)^{-2}, \tag{6.7}$$

and for $k = 1$ and $C = -(3b^2)^{-1}$,

$$y = (2b)^{-1}\{1 + 3\tan^2(\tfrac{1}{2}\log\beta z)\}, \tag{6.8}$$

where $\beta = \beta(t)$ is an arbitrary function of integration. For $k = 0$ and $C = 0$, equation (6.6) becomes

$$y = 6b^{-1}(\beta - \log z)^{-2}, \tag{6.9}$$

whereas for $k = -1$ and $C = 0$,

$$y = \frac{3}{2}b^{-1}\sec^2(\tfrac{1}{2}\log\beta z), \tag{6.10}$$

and for $k = -1$ and $C = (3b^2)^{-1}$,

$$y = b^{-1}(1 + 4\beta z + \beta^2 z^2)(1 - \beta z)^{-2}. \tag{6.11}$$

These metrics are not regular at $z = 0$ nor $z = \infty$, and for this reason they have usually been ruled out on physical grounds. However, the solution could represent an annular region, $z_1 < z < z_2$, surrounding a core with a different density distribution.

Space-times of class II are characterised by the condition $b'(z) \neq 0$. As both the metric and the left-hand side of equation (6.1) are invariant under the coordinate transformations $\tilde{z} = az^{\pm 1}$, where a is a constant, fields with $b = f(z)$ and $b = f(az^{\pm 1})$ are equivalent. If the transformations

$$v = \frac{y}{u}, \qquad x = \pm \int (zu^2)^{-1} dz, \tag{6.12}$$

are made, equation (6.1) becomes

$$v_{xx} = A(x)v^2, \tag{6.13}$$

where $A(x) = b(z)u^5$ and u is a solution of the linear equation

$$z^2 u_{zz} + zu_z - ku = 0.$$

If $A(x)$ is a constant ($= 3A/2$) the general integral of equation (6.13) is

$$v = + \left\{\frac{-C(t)}{A}\right\}^{1/3} \wp(B(t) + \tfrac{1}{2}(A^2C(t))^{1/6}x), \tag{6.14}$$

where B and C are arbitrary functions of integration and the invariants g_2 and g_3 are given by $g_2 = 0$, $g_3 = -4$. If $C = 0$, v may be written in terms of elementary functions, i.e.

$$v = (B(t) - \tfrac{1}{2}(-A)^{\frac{1}{2}}x)^{-2}. \tag{6.15}$$

From equation (6.12) for $k = 1$, it follows that $u = az + bz^{-1}$, $x = \pm(2a)^{-1}(az^2 + b)^{-1}$, for $a \neq 0$, or $x = \frac{1}{2}b^{-2}z^2$, for $a = 0$. Three distinct solutions of equation (6.1) with $k = 1$, corresponding to $b = 3Az^5/2$, $(3A/2)(z + z^{-1})^{-5}$, $(3A/2)(z - z^{-1})^{-5}$, may be obtained from equation (6.14) or (6.15) by means of the substitutions above with $a = 0$, $b = 1$; $a = 1$, $b = 1$; and $a = 1$ $b = -1$, respectively. The solution corresponding to $b = 3Az^5/2$ and $C = 0$ has previously been found by Faulkes [26]. Solutions corresponding to $k = 0$, $b = (3A/2)(\log z)^{-5}$, and $k = -1$, $b = (3A/2)(\cos\log z)^{-5}$, may be obtained in a similar manner by means of the substitutions $u = \log z$, $x = (\log z)^{-1}$, and $u = \cos\log z$, $x = \tan\log z$, respectively.

It is also possible to obtain solutions of equation (6.13) corresponding to other forms of the function $A(x)$ by applying a transformation of the form (6.12) to equation (6.1) with $b(z) = $ constant. In this way the following solutions of equation (6.13) are obtained:

$$\begin{aligned} v &= (2bc^3)^{-1}(1 + c^2x^2)^{\frac{1}{2}}\{\tfrac{1}{3}\wp(B(t) + \tan^{-1}(cx)) - 1\} \\ A(x) &= bc^{5/2}(1 + c^2x^2)^{-5/2}, \end{aligned} \tag{6.16}$$

where

$$g_2 = \frac{1}{12}, \qquad \text{and} \qquad g_3 = -\frac{1}{72}(2C(t)b^2c^5 - \frac{1}{3});$$

$$v = \frac{1}{2A}[2x(1 - 2abx)]^{\frac{1}{2}}\{\frac{1}{3}\wp(B(t) + \frac{1}{2}\log(x^{-1} - 2ab)) - 1\},$$
$$A(x) = A(2x)^{-5/2}(1 - 2abx)^{-5/2}, \tag{6.17}$$

where

$$g_2 = \frac{1}{12}, \qquad \text{and} \qquad g_3 = -\frac{1}{72}(2C(t)A^2 + \frac{1}{3}).$$

Four special cases arise where the elliptic function in equation (6.16) or (6.17) degenerates into an elementary function. These cases correspond to equations (6.7,8,10,11).

From equation (6.16), with the aid of the substitutions $u = az + bz^{-1}$, $x = (2a)^{-1}(az^2 + b)^{-1}$, and $u = 1$, $x = \log z$, it is possible to obtain the solutions of equation (6.1) corresponding to $k = 1$, $b(z) = bc^{5/2}z^5\{(az^2 + b)^2 + c^2\}^{-5/2}$, and $k = 0$, $b(z) = bc^{5/2}(1 + c^2\log^2 z)^{-5/2}$, respectively.

Similarly, from equation (6.17) by means of the substitutions, $u = az + bz^{-1}$, $x = (2a)^{-1}(az^2 + b)^{-1}$; $u = 1$, $x = \log z$; $u = \log z$, $x = (\log z)^{-1}$; and $u = \cos\log z$, $x = \tan\log z$; the solutions of equation (6.1) corresponding to $k = 1$, $b(z) = A(1 - abz^2)^{-5/2}$; $k = 0$, $b(z) = 2^{-5/2}A(\log z)^{-5/2}(1 - 2ab\log z)^{-5/2}$; $k = 0$, $b(z) = 2^{-5/2}A(\log z - 2ab)^{-5/2}$; and $k = -1$, $b(z) = A(2\sin\log z)^{-5/2}(\cos\log z - 2ab\sin\log z)^{-5/2}$; respectively can be obtained. If $C = 0$, in equation (6.17) it may be deduced from equation (6.7) that

$$v = 6\alpha\beta A^{-1}(1 - 2abx)\{a - \beta(\tfrac{1}{2}x^{-1} - ab)^{\frac{1}{2}}\}^{-2}.$$

If the substitutions $x = (2z^2)^{-1}$, and $u = z$, are made, the solution of equation (6.1) with $k = 1$, $b(z) = Az^5(z^2 - ab)^{-5/2}$, namely

$$y = 6\alpha\beta(Az)^{-1}(z^2 - ab)(a - \beta(z^2 - ab)^{\frac{1}{2}})^{-2}$$

is obtained. This solution was found by Nariai [27]. As far as the author is aware, all solutions in this section have not previously been obtained, except where it is explicitly stated to the contrary.

§(7): *SUMMARY*

All solutions of Einstein equations representing a degenerate shear-free and twist-free flow of a perfect fluid have been found and are displayed in the Table. The type D solutions all admit at least one space-like Killing vector field. In general the conformally flat solutions admit no Killing vector fields.

The expanding flows of class IB are the uniform density spherically symmetrical shear-free flows considered by Thompson and

TABLE

Solutions of Einstein's Equations Representing a Degenerate Shear-free and Twist-free Flow of a Perfect Fluid

Class	Flow	Metric	α	Density and pressure	r	S
0	E	$G = \{\alpha(t) + \beta(t)r^2 - 2\mathbf{r}.\mathbf{a}(t)\}^{-2}[dr^2 + r^2(d\Theta^2 + \sin^2\Theta d\phi^2)] - V^2dt^2$ where $V = -\frac{3}{\theta}\{\alpha' + \beta' r^2 - 2\mathbf{r}.\mathbf{a}'\}(\alpha + \beta r^2 - 2\mathbf{r}.\mathbf{a})^{-1}$.	0	$\mu = 12(\alpha\beta - \mathbf{a}^2) + \frac{1}{3}\theta^2$ $p = -\mu - (\theta V)^{-1}\mu'$	0 1 3 6	0 0 1S 3S
0	R	$G = (1 + \frac{1}{4}Kr^2)^{-2}\{dr^2 + r^2(d\Theta^2 + \sin^2\Theta d\phi^2)\} - (\alpha(t) + \beta(t)r^2 + \mathbf{r}.\mathbf{a}(t))^2dt^2$	0	$\mu = 3K, \quad p = -3K + (\alpha K + 4\beta)V^{-1}$	0 1 3	0 0 1S
0	S	$G = (k - Kr^2)^{-1}\{dr^2 + r^2(d\Theta^2 + f^2(\Theta)d\phi^2)\} - (D + E(k - Kr^2)^{\frac{1}{2}})^2dt^2$ for $K \neq 0$ where $f(\Theta) = \sin\Theta,\ \Theta,\ \sinh\Theta$ for $k = +1, 0, -1$ respectively	0	$\mu = 3K, \quad p = -3K + 2DKV^{-1}$ for $K \neq 0$	4 7	1S ($E \neq 0$) 3S ($E = 0$)
0	S	$G = dr^2 + r^2(d\Theta^2 + \sin^2\Theta d\phi^2) - (E + Dr^2)^2dt^2$ for $K = 0$	0	$\mu = 0, \quad p = 4D(E + Dr^2)^{-1}$ for $K = 0$	4	1S
IA	S	$G = I(\mu)\{d\Theta^2 + f^2(\Theta)d\phi^2 + dz^2 - f^2(z)dt^2\}$ where $f(\Theta) = \sin\Theta, \sinh\Theta$, and $I(\mu) = \mu^{-1}, -\mu^{-1}$, for $\mu > 0$, < 0 respectively	$-\frac{2}{3}\mu$	$\mu =$ constant, $\quad p = -\mu$	6	2 (1S,1T)
IB	E	$G = y^{-2}(r,t)\left\{r^{-2}dr^2 + d\Theta^2 + f^2(\Theta)d\phi^2 - \left[\frac{y'}{3\theta}\right]^2dt^2\right\}$ $f(\Theta)$ as in class 0S where $y = (2b)^{-1}\{\frac{1}{3}\wp(B(t) + \log(r)) - 1\}$ with $g_2 = \frac{1}{12}k^2$, $g_3 = -\frac{1}{72}(2C(t)b^2 + \frac{1}{3}k^3)$	$\frac{2}{3}by^3$	$\mu = \frac{1}{3}\theta^2 - 3C(t)$ $p = -\mu + \frac{y'}{3y}\mu'$	3	1S
IB	R	$G = W^2(r)\{dr^2 + r^2(d\Theta^2 + f^2(\Theta)d\phi^2)\} - V^2(r,t)dt^2$, $f(\Theta)$ as in class 0S with $W^{-2}(r) = k - \frac{1}{3}\mu r^2 - 2mr^{-1}$ and $V = W^{-1}(B(t) + A(t)\int rW^{-3}(r)dr)$	$-2mr^{-3}$	$\mu =$ constant $p = -\mu + \frac{2A(t)}{V(r,t)}$	3	1S
IB	S	As in class IB R but with A and B constant	$-2mr^{-3}$	$\mu =$ constant, $\quad p = -\mu + \frac{2A}{V(r)}$	4	1S
IC	S	$G = F^{-1}(x)dx^2 + F(x)d\phi^2 + x^2(dz^2 - f^2(z)dt^2)$ $F(x) = k - \frac{1}{3}\mu x^2 - 2mx^{-1}$, $f(z) = \sin z,\ z,\ \sinh z$, for $k = +1, 0, -1$ respectively	$-2mx^{-3}$	$\mu =$ constant, $\quad p = -\mu$	4	1T

ID	R	$G = (X+Y)^{-2}\{(f(X))^{-1}dX^2 + f(X)d\phi^2 + (g(Y))^{-1}dY^2 - g(Y)[B(t) + A(t)\int g^{-3/2}dY]^2 dt^2\}$ with $f(X) = \pm X^3 + cX + d$, $g(X) = -f(-X) - \frac{1}{3}\mu$, $X + Y > 0$	$\pm (X+Y)^3$	$\mu =$ constant $p = -\mu + 2A(X+Y)g^{-\frac{1}{2}}(Y)[B + A\int g^{-3/2}dY]^{-1}$	1	0
ID	S	As in class ID R but with A and B constant	$\pm (X+Y)^3$	As in class ID R	2	0
II	E	As in class IB E but with y given by $r^2 y_{rr} + r y_r - ky = b(r)y^2$	$\frac{2}{3} b(r) y^3$	$\mu = \mu(r,t)$ $p = p(r,t)$ with $\mu_r = 2b_r y^3$	3	1S
II	R	Metric as in class IB R with $W^{-2} = k - r^{-1}\int_0^r \mu(\zeta)\zeta^2 d\zeta - 2mr^{-1}$ and $V = V(r,t)$	$\frac{r^{-3}}{3}\int_0^r s^3\mu'(s)ds - 2mr^{-3}$	$\mu = \mu(r)$ $p = p(r,t)$	3	1S
II	S	As in class II R but with $V = V(r)$	As in class II R	$\mu = \mu(r)$ $p = p(r)$	4	1S
III	S	$G = (n + mX)^{-2}\{(F(X))^{-1}dX^2 + F(X)d\phi^2 + dz^2 - X^2dt^2\}$ with $F(X) = b + c(n^2\log(X) + 2mnX + \frac{1}{2}m^2X^2)$ where $m = \pm 1, 0,$ $x > 0$	$-\frac{1}{3} ncX^{-2}(n + mX)^3$	$2\mu = cX^{-2}(n + mX)^3(n + 3mX) - 6m^2F(X)$ $2p = cX^{-2}(n + mX)^3(n - mX) + 2m[m - \frac{2n}{x}]F(X)$	3	0
IV	S	$G = n^{-2}(z)\{(F(X))^{-1}dX^2 + F(X)d\phi^2 + dz^2 - X^2dt^2\}$ $F(X) = aX^2 + b.\log(X) + c$ $n(z) = A\sin(a^{\frac{1}{2}}z)$, Az, $A\sinh((-a)^{\frac{1}{2}}z)$ for $a > 0, = 0, < 0$ respectively	$-\frac{1}{3} bn^2(z)X^{-2}$	$2\mu = -6A^2I(a) + bn^2X^{-2}$ $p = \mu + 6A^2I(a)$, $I(a) = \|a\|, 1$ for $a \neq 0, = 0$ respectively	2	0

Whitrow [*9*] and their analogues with planar and hyperbolic symmetry. Those in class II are their generalisations with non-uniform denssity. The rigid flows of class IB are the interior Schwarzschild solution and its analogues with a point mass at the centre and a time-dependent pressure. Such solutions could represent a motionless annular region onto which matter is accreting and which surrounds a core of a different density. The rigid flows of class II are generalisations of those of class IB with variable density.

Class IA fields are Einstein spaces and are the direct product of two two-dimensional spaces of constant curvature. Classes IC and ID are generalisations of the vacuum B and C metrics of [*14*] and as far as I am aware no physical interpretation is known for even the vacuum metrics. Classes III and IV have no analogue in the vacuum case. Class III solutions possess cylindrical symmetry. Class IV have axial symmetry but are unrealistic physically since $p > \mu$.

In column two of the table E, R or S signify that the flow is expanding, rigid or static, respectively. In columns six and seven the dimensions of the isometry and isotropy groups are given and the S and T in column seven denote whether the isotropy group has a space-like or time-like trajectory, respectively.

ACKNOWLEDGMENTS

I am indebted to Dr. G.J. Whitrow for suggesting the problem of shear-free flows of a perfect fluid with spherical symmetry and would like to thank him for much helpful advice during the preparation of the paper. I am also grateful to the referee for comments on the improvement of the presentation.

REFERENCES

1. Trümper, M. (1962). *Z. Phys.*, **168**, 55.
2. Ehlers, J. (1961). *Akad. Wiss. Mainz, Math-Nat. Kl.*, No. 11.
3. Kundt, W. and Trümper, M. (1962). *Akad. Wiss. Mainz, Math-Nat. Kl.*, No. 12.
4. Hawking, S.W. (1966). *Singularities and the Geometry of Space-Time*, (Essay submitted for the Adams Prize, 1966).
5. Hawking, S.W. (1966). *Astrophys. J.*, **145**, 544.
6. Synge, J.L. (1937). *Proc. Lond. Math. Soc.*, **43**, 376.
7. Eisenhart, L.P. (1949). *Riemannian Geometry*, (Princeton University Press, Princeton, N.J.), p. 92.
8. Robertson H.T. and Noonan, T.W. (1968). *Relativity and Cosmology*, (W.B. Saunders Co., Philadelphia, Pa.), p. 219, 325.
9. Thompson, I.H. and Whitrow, G.J. (1967). *M. Not. R. Astr. Soc.*, **136**, 207; *ibid*, **139**, 499.
10. Schwarzschild, K. (1916). *Sitzber. Preuss. Akad. Wiss.*, p.189 and 424.
11. Estabrook, F. and Wahlquist, H. (1967). *Phys. Rev.*, **156**, 1359.
12. Stepanyuk, N.M. (1968). *Sov. Phys. JETP*, **26**, 369.

13. Barnes, A. (1972). *J. Phys. A*, **5**, 374.
14. Jordan, P., Ehlers, J. and Kundt, W. (1960). *Akad. Wiss. Mainz. Math-Nat. Kl.*, No. 2.
15. Barnes, A. (1971). Thesis, (University of London).
16. Trümper, M. (1965). *J. Math. Phys.*, **6**, 584.
17. Levi-Civita, T. (1917-19). *Rend. Acc. Lincei*, **26-28**, (nine articles).
18. Birkhoff, G.D. (1927). *Relativity and Modern Physics*, (Cambridge University Press).
19. Tolman, R.C. (1939). *Phys. Rev.*, **55**, 364.
20. Bondi, H. (1964). *Proc. Roy. Soc. A*, **282**, 303.
21. Harrison, B.K., Thorne, K.S., Wakano, M. and Wheeler, J.A. (1965). *Gravitational Theory and Gravitational Collapse*, (Chicago University Press).
22. Bonnor, W.B. and Faulkes, M.C. (1967). *Mon. Not. R. Astr. Soc.*, **137**, 239.
23. Bondi, H. (1969). *Mon. Not. R. Astr. Soc.*, **142**, 333.
24. Whittaker, E.T. and Watson, G.N. (1969). *A Course in Modern Analysis*, (Cambridge University Press), pp. 429-461.
25. Cahill, M.E. and McVittie, G.C. (1970). *J. Math. Phys.*, **11**, 1382.
26. Faulkes, M.C. (1970). *Prog. Theor. Phys.*, **42**, 1139.
27. Nariai, H. (1969). *Prog. Theor. Phys.*, **40**, 1013.