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On the Fundamental Representation of Borcherds Algebras with One Imaginary Simple Root

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Abstract. Borcherds algebras represent a new class of Lie algebras which have almost all the properties that ordinary Kac–Moody algebras have, but the only major difference is that these generalized Kac–Moody algebras are allowed to have imaginary simple roots. The simplest nontrivial examples one can think of are those where one adds 'by hand' one imaginary simple root to an ordinary Kac–Moody algebra. We study the fundamental representation of this class of examples and prove that an irreducible module is given by the full tensor algebra over some integrable highest weight module of the underlying Kac–Moody algebra. We also comment on possible realizations of these Lie algebras in physics as symmetry algebras in quantum field theory.

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1. Introduction

Studying the Lie algebra of physical states for the 26-dimensional bosonic string compactified on a torus, Borcherds discovered his celebrated fake Monster Lie algebra as the first generic example of a generalized Kac–Moody algebra (cf. [1, 2] or the review [3] for physicists). Up to this point, it was only known that the tachyonic groundstates give rise to an infinite rank Lie algebra L_∞ with a set of simple roots isometric to the Leech lattice and with certain bounds on the dimension of the root spaces coming from the 'no-ghost' theorem (cf. [4–7]). It was Borcherds' great achievement to observe that this upper bound can be satisfied by adding a certain set of photonic physical states as additional generators to the set of generators for L_∞ [8]. Mathematically speaking, he adjoined a set of *imaginary simple roots* (where 'imaginary' means 'negative norm') to the set of real simple roots for the Kac–Moody algebra L_∞. In the sequel Borcherds was able to axiomatize his ideas and he developed a theory of generalized Kac–Moody algebras in terms of generators and relations (cf. [9, 11]).

To get a grasp of these new Lie algebras Slansky [12] investigated the Borcherds extensions of the Lie algebras $\mathfrak{su}(2)$, $\mathfrak{su}(3)$, and affine $\mathfrak{su}(2)$ by a single lightlike ($\equiv$ norm zero) simple root. Computer calculations of the first few weight multiplicities

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of the basic representations suggested that the latter might be written as the tensor algebra over some module for the underlying nonextended Kac–Moody algebras. In the following, we shall prove that this is true for any Kac–Moody algebra extended by an arbitrary imaginary simple root.

2. Definitions

Let us begin with a review of the definition of Borcherds algebras. As already mentioned, the original references on the subject are [2], [9] and [11].

DEFINITION 1. Let $\hat{A} = (a_{ij})$ be a real symmetric $n \times n$ matrix satisfying the following properties:

(i) either $a_{ii} = 2$ or $a_{ii} \leqslant 0$,
(ii) $a_{ij} \leqslant 0$ if $i \neq j$,
(iii) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$.

Then the *Borcherds algebra* (*generalized Kac–Moody algebra*) associated to $\hat{A}$ is defined to be the Lie algebra $\hat{\mathfrak{g}}(\hat{A})$ given by the following generators and relations:

Generators: *Elements* e_i, f_i, h_i for every i;
Relations:

(0) $[h_i, h_j] = 0$,
(1) $[e_i, f_j] = \delta_{ij} h_i$,
(2) $[h_i, e_j] = a_{ij} e_j$, $\quad [h_i, f_j] = -a_{ij} f_j$,
(3) $e_{ij} := (\operatorname{ad} e_i)^{1-a_{ij}} e_j = 0$, $\quad f_{ij} := (\operatorname{ad} f_i)^{1-a_{ij}} f_j = 0$ if $a_{ii} = 2$ and $i \neq j$,
(4) $e_{ij} := [e_i, e_j] = 0$, $\quad f_{ij} := [f_i, f_j] = 0$ if $a_{ii} \leqslant 0$, $a_{jj} \leqslant 0$ and $a_{ij} = 0$.

The elements h_i form a basis for an Abelian subalgebra of $\hat{\mathfrak{g}}(\hat{A})$, called its *Cartan subalgebra* $\hat{\mathfrak{h}}(\hat{A})$. $\hat{\mathfrak{g}}(\hat{A})$ has a triangular decomposition

$$\hat{\mathfrak{g}}(\hat{A}) = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+,$$

where $\hat{\mathfrak{n}}_-$ (resp. $\hat{\mathfrak{n}}_+$) is the algebra obtained by dividing the free algebra $\tilde{\mathfrak{n}}_-$ ($\tilde{\mathfrak{n}}_+$) generated by the f_i (e_i) by the ideal $\mathfrak{r}_-$ ($\mathfrak{r}_+$) generated by the f_{ij} (e_{ij}).

Note that if $a_{ii} = 2$ for all i, then $\hat{\mathfrak{g}}(\hat{A})$ is the same as the ordinary Kac–Moody algebra with symmetrized Cartan matrix $\hat{A}$. In general, $\hat{\mathfrak{g}}(\hat{A})$ has almost all the properties that ordinary Kac–Moody algebras have, and the only major difference is that generalized Kac–Moody algebras are allowed to have imaginary simple roots. In what follows, we will exclusively deal with the case of a Borcherds algebra with one imaginary simple root.

It is clear that if we delete in $\hat{A}$ the row and the column corresponding to the imaginary root then the resulting submatrix A is a generalized Cartan matrix in the sense of Kac [13] with associated Kac–Moody algebra $\mathfrak{g}(A)$. Recall the triangular decomposition

$$\mathfrak{g}(A) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

and the induced decomposition of the universal enveloping algebra:

$$\mathfrak{U}(\mathfrak{g}(A)) = \mathfrak{U}(\mathfrak{n}_-) \otimes \mathfrak{U}(\mathfrak{h}) \otimes \mathfrak{U}(\mathfrak{n}_+).$$

An irreducible $\mathfrak{g}(A)$-module $\mathscr{F}_\lambda$ is called *integrable highest-weight module* if there exists a dominant integral weight $\lambda \in \mathfrak{h}^*$ and a nonzero vector $\omega \in \mathscr{F}_\lambda$ such that

$$h(\omega) = \lambda(h)\omega \quad \text{for } h \in \mathfrak{h},$$

$$\mathfrak{n}_+(\omega) = 0,$$

$$\mathfrak{U}(\mathfrak{n}_-)(\omega) = \mathscr{F}_\lambda.$$

We denote by $\mathfrak{T}(\mathscr{F}_\lambda)$ the tensor algebra over $\mathscr{F}_\lambda$,

$$\mathfrak{T}(\mathscr{F}_\lambda) := \bigoplus_{n=0}^{\infty} \mathscr{F}_\lambda^n \equiv \mathbb{C}\cdot\mathbf{1} \oplus \mathscr{F}_\lambda \oplus (\mathscr{F}_\lambda \otimes \mathscr{F}_\lambda) \oplus (\mathscr{F}_\lambda \otimes \mathscr{F}_\lambda \otimes \mathscr{F}_\lambda) \oplus \cdots$$

Now we are ready to state our result.

3. The Theorem

THEOREM 1. *Let $\hat{A} = (a_{ij})$, $0 \leqslant i, j \leqslant n$, be a symmetric integer matrix satisfying the following properties*:

(i) $a_{00} \leqslant 0, \qquad a_{ii} = 2$ *for* $1 \leqslant i \leqslant n$,
(ii) $a_{ij} \leqslant 0$ *if* $i \neq j$.

Let $\mathscr{F}_\lambda$ be the integrable highest-weight module over the Kac–Moody algebra $\mathfrak{g}(A)$ associated to the Cartan matrix $A = (a_{ij})$, $1 \leqslant i, j \leqslant n$, with highest weight λ defined by $\lambda(h_i) := -a_{0i}$, $1 \leqslant i \leqslant n$, and highest-weight vector ω. Then the tensor algebra $\mathfrak{T}(\mathscr{F}_\lambda)$ over $\mathscr{F}_\lambda$ is $\hat{\mathfrak{g}}(\hat{A})$-module isomorphic to the highest-weight module $L(\Lambda), \Lambda(h_i) = \delta_{i0}, 0 \leqslant i \leqslant n$ of $\hat{\mathfrak{g}}(\hat{A})$.

Proof. We define an action of the generators of $\hat{\mathfrak{g}}(\hat{A})$ on the tensor algebra $\mathfrak{T}(\mathscr{F}_\lambda)$ as follows. Our convention for indices will be that i, j, k run from 1 to n unless otherwise stated!

The Kac–Moody generators e_i, h_i, f_i act trivially on the 'vacuum' vector $\mathbf{1}$ and as highest weight representation on $\mathscr{F}_\lambda$. We extend this action to the tensor algebra $\mathfrak{T}(\mathscr{F}_\lambda)$ by Leibnitz' rule.

The generator h_0 acts diagonal:

$$h_0(\mathbf{1}) := \mathbf{1}, \tag{1}$$

$$h_0(\omega) := (1 - a_{00})\omega, \tag{2}$$

$$h_0(f_k\varphi) := -a_{0k}f_k\varphi + f_k h_0(\varphi) \quad \text{for } \varphi \in \mathscr{F}_\lambda, \tag{3}$$

$$h_0(\Phi \otimes \Psi) := h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi) - \Phi \otimes \Psi \quad \text{for } \Phi, \Psi \in \mathfrak{T}(\mathscr{F}_\lambda). \tag{4}$$

The 'imaginary' generator f_0 adjoins one tensor factor of the highest weight vector ω, i.e.,

$$f_0(\Psi) := \omega \otimes \Psi \quad \text{for } \Psi \in \mathfrak{T}(\mathscr{F}_\lambda). \tag{5}$$

For e_0, we put

$$e_0(\mathbf{1}) := 0, \tag{6}$$

while for the definition on $\mathscr{F}_\lambda^n, n \geqslant 1$, we observe that $\mathscr{F}_\lambda^n = \mathfrak{U}(\mathfrak{n}_-)(\omega \otimes \mathscr{F}_\lambda^{n-1})$, so that it is sufficient to require, inductively,

$$e_0(f_i(\Psi)) := f_i(e_0(\Psi)), \tag{7}$$

$$e_0(\omega \otimes \Psi) := h_0(\Psi) + \omega \otimes e_0(\Psi), \tag{8}$$

for $\Psi \in \mathscr{F}_\lambda^n, n \geqslant 0$.

Having defined the action of the generators on the tensor algebra we will now check that $\mathfrak{T}(\mathscr{F}_\lambda)$ carries the claimed $\hat{\mathfrak{g}}(\hat{A})$-module structure. First we note that h_0 and the h_i's are defined to act diagonal on the tensor algebra. Hence, the h's commute with each other. Secondly, all commutation relations involving only Kac–Moody generators e_i, h_i, f_i are valid by assumption. Next, we have a look at those commutation relations which are more or less trivial since they can be checked immediately on the whole tensor algebra,

$$\begin{aligned}
(e_0 f_0 - f_0 e_0)(\Psi) &= e_0(\omega \otimes \Psi) - \omega \otimes e_0(\Psi) = h_0(\Psi),\\
(e_0 f_i - f_i e_0)(\Psi) &= 0,\\
(e_i f_0 - f_0 e_i)(\Psi) &= e_i(\omega \otimes \Psi) - \omega \otimes e_i(\Psi) = 0,\\
(h_0 f_0 - f_0 h_0)(\Psi) &= h_0(\omega \otimes \Psi) - \omega \otimes h_0(\Psi)\\
&= (h_0 - 1)(\omega) \otimes \Psi\\
&= -a_{00} f_0(\Psi),\\
(h_i f_0 - f_0 h_i)(\Psi) &= h_i(\omega \otimes \Psi) - \omega \otimes h_i(\Psi)\\
&= h_i(\omega) \otimes \Psi\\
&= -a_{0i} f_0(\Psi).
\end{aligned}$$

Finally we check the remaining four types of commutators:

$$\begin{aligned}
(h_0 f_i - f_i h_0)(\mathbf{1}) &= -f_i(\mathbf{1}) = 0 = -a_{0i} f_i(\mathbf{1}),\\
(h_0 f_i - f_i h_0)(\varphi) &= -a_{0i} f_i \varphi,\\
(h_0 f_i - f_i h_0)(\Phi \otimes \Psi) &= h_0(f_i(\Phi) \otimes \Psi + \Phi \otimes f_i(\Psi)) -\\
&\quad - f_i(h_0(\Phi) \otimes \Psi + \Phi \otimes h_0(\Psi) - \Phi \otimes \Psi)\\
&= (h_0 f_i - f_i h_0)(\Phi) \otimes \Psi + \Phi \otimes (h_0 f_i - f_i h_0)(\Psi)\\
&= -a_{0i} f_i(\Phi \otimes \Psi) \quad \text{by induction,}
\end{aligned}$$

$$(h_0e_i - e_ih_0)(\mathbf{1}) = 0 = a_{0i}e_i(\mathbf{1}),$$

$$(h_0e_i - e_ih_0)(\omega) = 0 = a_{0i}e_i(\omega),$$

$$\begin{aligned}(h_0e_i - e_ih_0)(f_k\varphi) &= h_0(\delta_{ik}h_i(\varphi) + f_ke_i(\varphi)) - e_i(-a_{0k}f_k\varphi + f_kh_0(\varphi))\\ &= a_{0k}(e_if_k - f_ke_i)(\varphi) + f_k(h_0e_i - e_ih_0)(\varphi)\\ &= a_{0k}\delta_{ik}h_i(\varphi) + a_{0i}f_ke_i(\varphi) \quad \text{by induction}\\ &= a_{0i}e_i(f_k\varphi),\end{aligned}$$

$$\begin{aligned}(h_0e_i - e_ih_0)(\Phi\otimes\Psi) &= h_0(e_i(\Phi)\otimes\Psi + \Phi\otimes e_i(\Psi)) -\\ &\quad - e_i(h_0(\Phi)\otimes\Psi + \Phi\otimes h_0(\Psi) - \Phi\otimes\Psi)\\ &= (h_0e_i - e_ih_0)(\Phi)\otimes\Psi + \Phi\otimes(h_0e_i - e_ih_0)(\Psi)\\ &= a_{0i}e_i(\Phi\otimes\Psi) \quad \text{by induction,}\end{aligned}$$

$$(h_ie_0 - e_0h_i)(\mathbf{1}) = 0 = a_{i0}e_0(\mathbf{1}),$$

$$(h_ie_0 - e_0h_i)(\omega) = h_i(\mathbf{1}) + a_{0i}e_0(\omega) = a_{0i}\mathbf{1} = a_{i0}e_0(\omega),$$

$$(h_ie_0 - e_0h_i)(f_k\varphi) = 0 = a_{i0}e_0(f_k\varphi),$$

$$\begin{aligned}(h_ie_0 - e_0h_i)(\omega\otimes\Psi) &= h_i(h_0(\Psi) + \omega\otimes e_0(\Psi)) - e_0(h_i(\omega)\otimes\Psi + \omega\otimes h_i(\Psi))\\ &= a_{0i}h_0(\Psi) + \omega\otimes(h_ie_0 - e_0h_i)(\Psi) + (h_ih_0 - h_0h_i)(\Psi)\\ &= a_{i0}e_0(\omega\otimes\Psi),\end{aligned}$$

$$\begin{aligned}(h_ie_0 - e_0h_i)(f_k\varphi\otimes\Psi) &= h_i(-e_0(\varphi\otimes f_k(\Psi)) + f_k(e_0(\varphi\otimes\Psi))) -\\ &\quad - e_0(h_i(f_k\varphi)\otimes\Psi + f_k\varphi\otimes h_i(\Psi))\\ &= -h_i(e_0(\varphi\otimes f_k(\Psi))) + h_i(f_k(e_0(\varphi\otimes\Psi))) -\\ &\quad - a_{ik}e_0(\varphi\otimes f_k(\Psi)) + a_{ik}f_k(e_0(\varphi\otimes\Psi)) +\\ &\quad + e_0(h_i(\varphi)\otimes f_k(\Psi)) - f_k(e_0(h_i(\varphi)\otimes\Psi)) +\\ &\quad + e_0(\varphi\otimes f_k(h_i(\Psi))) - f_k(e_0(\varphi\otimes h_i(\Psi)))\\ &= (e_0h_i - h_ie_0)(\varphi\otimes f_k(\Psi)) +\\ &\quad + f_k((h_ie_0 - e_0h_i)(\varphi\otimes\Psi))\\ &= a_{i0}(-e_0(\varphi\otimes f_k(\Psi)) + f_k(e_0(\varphi\otimes\Psi))) \quad \text{by induction}\\ &= a_{i0}e_0(f_k\varphi\otimes\Psi),\end{aligned}$$

$$(h_0e_0 - e_0h_0)(\mathbf{1}) = 0 = a_{00}e_0(\mathbf{1}),$$

$$(h_0e_0 - e_0h_0)(\omega) = h_0(\mathbf{1}) - (1 - a_{00})e_0(\omega) = a_{00}\mathbf{1} = a_{00}e_0(\omega),$$

$$(h_0e_0 - e_0h_0)(f_k\varphi) = 0 = a_{00}e_0(f_k\varphi),$$

$$\begin{aligned}(h_0e_0 - e_0h_0)(\omega\otimes\Psi) &= h_0(h_0(\Psi) + \omega\otimes e_0(\Psi)) - e_0(-a_{00}\omega\otimes\Psi + \omega\otimes h_0(\Psi))\\ &= a_{00}h_0(\Psi) + \omega\otimes(h_0e_0 - e_0h_0)(\Psi)\\ &= a_{00}e_0(\omega\otimes\Psi) \quad \text{by induction,}\end{aligned}$$

$$
\begin{aligned}
(h_0e_0 - e_0h_0)(f_k\varphi \otimes \Psi) &= h_0(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi))) - \\
&\quad - e_0(h_0(f_k\varphi) \otimes \Psi + f_k\varphi \otimes h_0(\Psi) - f_k\varphi \otimes \Psi) \\
&= - h_0(e_0(\varphi \otimes f_k(\Psi))) + h_0(f_k(e_0(\varphi \otimes \Psi))) - \\
&\quad - a_{0k}e_0(\varphi \otimes f_k(\Psi)) + a_{0k}f_k(e_0(\varphi \otimes \Psi)) + \\
&\quad + e_0(h_0(\varphi) \otimes f_k(\Psi)) - f_k(e_0(h_0(\varphi) \otimes \Psi)) + \\
&\quad + e_0(\varphi \otimes f_k(h_0(\Psi))) - f_k(e_0(\varphi \otimes h_0(\Psi))) + \\
&\quad + e_0(\varphi \otimes f_k(\Psi)) - f_k(e_0(\varphi \otimes \Psi)) \\
&= (e_0h_0 - h_0e_0)(\varphi \otimes f_k(\Psi)) + \\
&\quad + f_k((h_0e_0 - e_0h_0)(\varphi \otimes \Psi)) \\
&= a_{00}(-e_0(\varphi \otimes f_k(\Psi)) + f_k(e_0(\varphi \otimes \Psi))) \quad \text{by induction} \\
&= a_{00}e_0(f_k\varphi \otimes \Psi),
\end{aligned}
$$

for all $\varphi \in \mathscr{F}_\lambda$ and $\Phi, \Psi \in \mathfrak{T}(\mathscr{F}_\lambda)$.

Now we shall prove that $\mathfrak{T}(\mathscr{F}_\lambda)$ is indeed isomorphic to $L(\Lambda)$ as a $\hat{\mathfrak{g}}(\hat{A})$-module. Denote the highest-weight vector of $L(\Lambda)$ by v_Λ. Define a map $\nu\colon \mathfrak{U}(\tilde{\mathfrak{n}}_-)v_\Lambda \to \mathfrak{T}(\mathscr{F}_\lambda)$ by

$$\nu(f_{i_1}\dots f_{i_n}v_\Lambda) := f_{i_1}\dots f_{i_n}(\mathbf{1}),$$

where $i_1 \dots i_n \in \{0, \dots, n\}$, and linearity. To prove that ν reduces to a well-defined $\hat{\mathfrak{g}}(\hat{A})$-module homomorphism $\nu'\colon \mathfrak{U}(\hat{\mathfrak{n}}_-)v_\Lambda \to \mathfrak{T}(\mathscr{F}_\lambda)$, one has to check that the action of elements of $\mathfrak{r}_-$ on $\mathfrak{T}(\mathscr{F}_\lambda)$ vanishes, i.e. that the Serre relations are valid. For $f_{ij}, i, j = 1 \dots n$, this is part of the definition. To check the remaining ones, observe that

$$((\text{ad}\, f_i)^m f_0)(\Psi) = f_i^m\omega \otimes \Psi,$$

so that for $i = 1 \dots n$

$$f_{i0}(\Psi) = f_i^{1+\lambda(h_i)}\omega \otimes \Psi = 0$$

because of lemma 10.1 of [13]. According to [9] (see also [10]), the irreducible module $L(\Lambda)$ is obtained from the Verma module $M(\Lambda)$ by dividing out the subspace generated by the primitive vectors $f_i^{1+\Lambda(h_i)}v_\Lambda, i = 1 \dots n$. Because of $f_i^{1+\Lambda(h_i)}(\mathbf{1}) = f_i(\mathbf{1}) = 0$, ν' reduces further to a map $\nu''\colon L(\lambda) \to \mathfrak{T}(\mathscr{F}_\lambda)$. ν'' is injective because the kernel of ν'' would be a proper submodule of $L(\lambda)$, and surjective because $\mathfrak{T}(\mathscr{F}_\lambda)$ is spanned by vectors of the form

$$u_1\omega \otimes \dots \otimes u_n\omega = \nu(u_1(f_0)\dots u_n(f_0)v_\Lambda),$$

where

$$
\begin{aligned}
&u_i = F_{n_1(i)}\dots F_{n_{k(i)}(i)}, \quad F_{n_j(i)} \in \mathfrak{g}(A), \\
&u_i(f_0) = [F_{n_1(i)}, [\dots [F_{n_{k(i)}(i)}, f_0] \dots]].
\end{aligned}
$$

We observe that the theorem is not altered if we replace a_{00} by any nonpositive real number or $\Lambda(h_0)$ by any positive real number.

4. Outlook

According to a conjecture of Ginsparg [12], the special class of Borcherds algebras considered in the theorem might play a role in second quantization of a single particle theory. In this interpretation, we regard the module $\mathscr{F}_\lambda$ from above as one-particle Fock space so that $\mathfrak{T}(\mathscr{F}_\lambda)$ comprises all multiparticle states. In other words, within a single irreducible representation of the Borcherds algebra we encounter all possible multiparticle excitations. Thus, the 'imaginary' generators f_0 and e_0 act as particle creation and particle annihilation operators, respectively, whereas the vector $\mathbf{1}$ indeed deserves the name 'true vacuum' in contrast to the 'ground state' $\omega \in \mathscr{F}_\lambda$.

Applying this idea to string theory one should think about the underlying Kac–Moody algebra $\mathfrak{g}(A)$ as spectrum generating algebra for the physical states of the bosonic string. Consequently, the tensor algebra $\mathfrak{T}(\mathscr{F}_\lambda)$ would be intimately related to a string field theory. Note that in the special case of an underlying affine Lie algebra $\mathfrak{g}(A)$ we would end up with a string field theory on the group manifold associated to $\mathfrak{g}(A)$ (cf. [14]).

It is clear that the emergence of Borcherds algebras in quantum field theory is just a naive speculation since up to now at least one important point in dealing with particles is missing. The tensor algebra $\mathfrak{T}(\mathscr{F}_\lambda)$ carries no symmetry or antisymmetry constraints at all, which means that the concept of statistics is absent. At present, work is in progress to clarify how symmetrization of $\mathfrak{T}(\mathscr{F}_\lambda)$ can be implemented into $\hat{\mathfrak{g}}(\hat{A})$ algebraically via additional relations.

In view of these possible realizations of Borcherds algebras in physics, we shall finish with the useful construction of a 'number operator' which counts the number of f_0's (number of particles/strings) occurring in the expression for a homogeneous state vector $\Psi \in \mathfrak{T}(\mathscr{F}_\lambda)$. We are looking for an element N in the Cartan subalgebra $\hat{\mathfrak{h}}(\hat{A})$ satisfying

$$N(\Psi) \stackrel{!}{=} n\Psi \quad \forall \Psi \in \mathscr{F}_\lambda^n, n \geqslant 1,$$

or, equivalently,

$$[N, f_j] \stackrel{!}{=} \delta_{j0} f_j \quad \text{for } 0 \leqslant j \leqslant n.$$

The ansatz $N = \Sigma_{i=0}^n N_i h_i$ yields the following system of linear equations for the rational coefficients N_i:

$$\sum_{i=0}^{n} a_{ij} N_i \stackrel{!}{=} -\delta_{j0} \quad \text{for } 0 \leqslant j \leqslant n.$$

If $\hat{A}$ is invertible we obtain a unique solution for the number operator N. Note, however, that the eigenvalues of N give us the number of f_0's shifted by N_0 since we have $N(\mathbf{1}) = N_0 \mathbf{1}$ instead of $N(\mathbf{1}) = 0$. This annoying constant may be removed by defining the 'renormalized' number operator $\hat{N} := N - N_0$.

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Note added in proof. The referee has informed us that our theorem can also be proved by using the character formula for highest-weight modules and the results of the thesis by E. Jurisich.