*Letters in Mathematical Physics* **31:** 279–288, 1994.



# $SU_q(2)$ Covariant $\hat{R}$-Matrices for Reducible Representations

A. LOREK[1], W. B. SCHMIDKE[1], and J. WESS[1,2]
[1] *Max-Planck-Institut für Physik, Werner-Heisenberg-Institut für Physik, Föhringer Ring 6, D-80805 Munich, Germany*
[2] *Sektion Physik, Universität München, Theresienstr. 37, D-80333 Munich, Germany*

(Received: 17 December 1993)

**Abstract.** We consider $SU_q(2)$ covariant $\hat{R}$-matrices for the reducible $\mathbf{3} \oplus \mathbf{1}$ representation. There are three solutions to the Yang–Baxter equation. They coincide with the previously known $\hat{R}$-matrices for $SO_q(3)$ and $SO_q(3, 1)$. Also, they are the three $\hat{R}$-matrices which can be constructed by using four different $SU_q(2)$ doublets. Only two of the three $\hat{R}$-matrices allow a differential structure on the reducible four-dimensional quantum space.

**Mathematics Subject Classifications (1991).** 16W30, 81R05.

## 1. Introduction

From the universal $R$-matrix of the $SU_q(2)$ algebra, there follows an $\hat{R}$-matrix for the product of any two irreducible representations of $SU_q(2)$. However, these $\hat{R}$-matrices do not allow a differential structure on the representation spaces with angular momentum two or higher. The reason is that in the decomposition of the $\hat{R}$-matrix into projectors on irreducible subspaces of the product space, all projectors enter with different eigenvalues. It is then not possible to build a differential structure based on this $\hat{R}$-matrix that would satisfy the Poincaré–Birkhoff–Witt theorem. The situation is exactly the same if one tries to define creation and annihilation operators for particles with angular momentum two or larger. Thus, it is not possible to construct a Fock space based on the universal $\hat{R}$-matrix.

There are, however, $\hat{R}$-matrices defined on reducible representations of $SU_q(2)$ that are $SU_q(2)$ covariant but do not decompose the same way as the $SU_q(2)$ generators do. If in the reduction of the product the same irreducible representation occurs several times, the generators and the $\hat{R}$-matrix mix them differently. Among these $\hat{R}$-matrices, there are candidates for a differential structure. The problem is that these reducible $\hat{R}$-matrices are not known in general.

In this Letter, we give an example of such $\hat{R}$-matrices and construct them by a method that could be generalized to higher spin. The emphasis of this Letter, however, is a search for all possible $\hat{R}$-matrices for a given reducible representation – in our case a representation with one triplet and one singlet under $SU_q(2)$.

This Letter is organized as follows. In the next section, we discuss the notion of $SU_q(2)$ covariance and $\hat{R}$-matrices. After two simple examples of irreducible representations we solve the Yang–Baxter equation for the $\mathbf{3} \oplus \mathbf{1}$ representation and find exactly three classes of solutions. In the third section, comparison is made with the $q$-spinor approach and we find that the three classes of $\hat{R}$-matrices can be constructed from the $q$-spinors as well. In the final section, we discuss the suitability of these $\hat{R}$-matrices for a differential calculus.

## 2. $\mathcal{U}_q(SU_q(2))$ and Covariance

We begin by describing the algebra $\mathcal{U}_q(SU_q(2))$. It is generated by the generators $T^+$, $T^-$ and $T^3$ which obey the relations

$$\begin{aligned} &q^{-1}T^+T^- - qT^-T^+ = T^3, \\ &q^2T^3T^+ - q^{-2}T^+T^3 = (q + q^{-1})T^+, \\ &q^{-2}T^3T^- - q^2T^-T^3 = -(q + q^{-1})T^-, \end{aligned} \tag{2.1}$$

where $q$ is the deformation parameter. $\mathcal{U}_q(SU_q(2))$ is a Hopf algebra and as such has a coproduct $\Delta$. It essentially describes how the generators act on a tensor product of representation spaces. For $\mathcal{U}_q(SU_q(2))$, we have

$$\begin{aligned} &\Delta(T^+) = T^+ \otimes 1 + (1 - \lambda T^3)^{1/2} \otimes T^+, \\ &\Delta(T^-) = T^- \otimes 1 + (1 - \lambda T^3)^{1/2} \otimes T^-, \\ &\Delta(T^3) = T^3 \otimes 1 + (1 - \lambda T^3) \otimes T^3. \end{aligned} \tag{2.2}$$

Here we define $\lambda = q - q^{-1}$. This coproduct is a homomorphism of the algebra (2.1) and is coassociative.

The representations of $\mathcal{U}_q(SU_q(2))$ are well known [1]. As in the classical case there is a Casimir operator with eigenvalues labeled by $j$, the total angular momentum. States within each representation have eigenvalues of $T^3$ labeled by $m$. The action of the generators on a state $|j, m\rangle$ is

$$\begin{aligned} &T^+|j, m\rangle = q^{-1}\sqrt{[j + m + 1]_{q^{-2}}[j - m]_{q^2}}\,|j, m + 1\rangle, \\ &T^-|j, m\rangle = q\sqrt{[j + m]_{q^{-2}}[j - m + 1]_{q^2}}\,|j, m - 1\rangle, \\ &T^3|j, m\rangle = q^{-1}[2m]_{q^{-2}}|j, m\rangle. \end{aligned} \tag{2.3}$$

$[n]_r$ is the $q$-number defined by $[n]_r = (r^n - 1)/(r - 1)$.

We will be interested in tensor products of these representation spaces. Consider two different sets of states $|j_1, m_1\rangle_1$ and $|j_2, m_2\rangle_2$. As in the classical case, their tensor product can be written as a sum of states with different total angular momentum:

$$|j_1, m_1\rangle_1 \otimes |j_2, m_2\rangle_2 = \sum_{J=|j_1-j_2|}^{j_1+j_2} C_q(j_1, m_1, j_2, m_2, J)\,|J, m_1 + m_2\rangle_{12}. \tag{2.4}$$

The subscripts on the product state indicate the ordering of the underlying spaces. The form of the Clebsch–Gordan coefficients $C_q$ is determined by requiring the coproduct (2.2) acting on the tensor product to be compatible with the representation (2.3). Several explicit examples will be given in the following.

The $\hat{R}$-matrix defines how the order of two different sets of states in a tensor product can be reversed. It is a set of numerical factors defining this action:

$$|j_1, m_1\rangle_1 \otimes |j_2, m_2\rangle_2 = \hat{R}^{(j_1,m_1)(j_2,m_2)}{}_{(j_2',m_2')(j_1',m_1')} |j_2', m_2'\rangle_2 \otimes |j_1', m_1'\rangle_1, \tag{2.5}$$

where repeated indices are summed. On such an $\hat{R}$-matrix several properties are imposed. First, it should satisfy the quantum Yang–Baxter equation:

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{2.6}$$

Here the subscripts indicate which of three spaces in a tensor product is acted on by the $\hat{R}$-matrix. A consequence of this equation is that reordering of a product of three or more spaces is independent of the order in which adjacent spaces are swapped.

Another feature of $\hat{R}$-matrices is that they can be decomposed into a sum of projectors: $\hat{R} = \Sigma_i \lambda_i P_i$. The projectors obey $P_i P_j = \delta_{ij} P_i$ and sum to the identity matrix: $\mathbb{1} = \Sigma_i P_i$. It is useful to characterize $\hat{R}$-matrices by their eigenvalues and projector decomposition. This will be especially important for constructing a differential calculus, as will be discussed in Section 4.

A final property of our $\hat{R}$-matrices is that they should be $\mathcal{U}_q(\mathrm{SU}_q(2))$ covariant. One way of stating this is as follows. Product states may be arranged into states of definite quantum numbers $J$ and $M$ by inverting (2.4). Covariance of $\hat{R}$ means that when the corresponding product states are reordered according to (2.5), their quantum numbers are conserved and the relative normalizations within multiplets of a given $J$ are preserved. When the underlying spaces are both irreducible, the product multiplets all have different total angular momentum $J$. However, when an underlying space is reducible, there may be different multiplets with the same $J$, and in reordering these multiplets may mix. Labeling different multiplets with the same $J$ as $|J, M\rangle^i$, (2.5) may be rewritten in the product state basis:

$$|J, M\rangle^i_{12} = \hat{R}^i_k |J, M\rangle^k_{21}. \tag{2.7}$$

This is the most general form of a covariant $\hat{R}$-matrix.

We may now construct covariant $\hat{R}$-matrices. For any two underlying representations, the $\hat{R}$-matrix can be parameterized according to (2.7). The parameters are then determined by solving the Yang–Baxter equation. (Since the Yang–Baxter equation is purely cubic in $\hat{R}$, the overall normalization of $\hat{R}$ is not fixed.) The $\hat{R}$-matrix may then be analyzed to determine its projector decomposition.

The first and simplest example we will consider is the case of two $\mathcal{U}_q(\mathrm{SU}_q(2))$ doublets. According to the rule $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$, their product will have a singlet and a triplet. Using the coproduct and representation rules, the Clebsch–Gordan coefficients for the singlet are found to be

$$|0,0\rangle_{12} = q^{-1}|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2 - |\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2 \tag{2.8}$$

and for the triplet we have

$$\begin{aligned} |1,-1\rangle_{12} &= q^{-1}|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2, \\ |1,0\rangle_{12} &= (q^2+1)^{-1/2}(|\tfrac{1}{2},-\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2 + q^{-1}|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle_2), \\ |1,1\rangle_{12} &= q^{-1}|\tfrac{1}{2},\tfrac{1}{2}\rangle_1 \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle_2. \end{aligned} \tag{2.9}$$

(The overall normalizations have been chosen for convenience.) Following (2.7), we parameterize the $\hat{R}$-matrix:

$$|0,0\rangle_{12} = a|0,0\rangle_{21}, \qquad |1,m\rangle_{12} = b|1,m\rangle_{21}. \tag{2.10}$$

In the classical limit $q \to 1$, $a = -1$ and $b = 1$. Fixing the normalization by setting $b = 1$, we find two solutions to the Yang–Baxter equation:

$$a = -q^{-2} \quad \text{or} \quad a = -q^2. \tag{2.11}$$

(There is also a solution for $a$ which has the wrong classical limit.) When rewritten in the basis of (2.5), it is seen that these solutions correspond to the usual $\hat{R}$-matrix for $\mathcal{U}_q(\mathrm{SU}_q(2))$ and its inverse [2]. For the former, the eigenvalues are 1 with multiplicity 3 and $-q^{-2}$ with multiplicity 1 and the projector decomposition is

$$\hat{R} = P_S - q^{-2} P_A. \tag{2.12}$$

Here $P_S$ is the $q$-deformed symmetrizer and $P_A$ is the $q$-deformed antisymmetrizer.

For our next example, we will take two $\mathcal{U}_q(\mathrm{SU}_q(2))$ doublets. The product space will have three multiplets: $\mathbf{3} \otimes \mathbf{3} = \mathbf{5} \oplus \mathbf{3} \oplus \mathbf{1}$. The singlet is

$$\begin{aligned} &|0,0\rangle_{12} \\ &\quad = q^2|1,1\rangle_1 \otimes |1,-1\rangle_2 - q|1,0\rangle_1 \otimes |1,0\rangle_2 + |1,-1\rangle_1 \otimes |1,1\rangle_2 \end{aligned} \tag{2.13}$$

and the triplet states are

$$\begin{aligned} |1,-1\rangle_{12} &= q^{-1}|1,-1\rangle_1 \otimes |1,0\rangle_2 - q|1,0\rangle_1 \otimes |1,-1\rangle_2, \\ |1,0\rangle_{12} &= |1,-1\rangle_1 \otimes |1,1\rangle_2 - |1,1\rangle_1 \otimes |1,-1\rangle_2 - \lambda|1,0\rangle_1 \otimes |1,0\rangle_2, \\ |1,1\rangle_{12} &= q^{-1}|1,0\rangle_1 \otimes |1,1\rangle_2 - q|1,1\rangle_1 \otimes |1,0\rangle_2. \end{aligned} \tag{2.14}$$

Since the normalizations for the **5** are complicated and not important for our purposes, we give only the appropriate linear combinations:

$$\begin{aligned} |2,-2\rangle_{12} &\propto q^2|1,-1\rangle_1 \otimes |1,-1\rangle_2, \\ |2,-1\rangle_{12} &\propto q^3|1,-1\rangle_1 \otimes |1,0\rangle_2 + q|1,0\rangle_1 \otimes |1,-1\rangle_2, \\ |2,0\rangle_{12} &\propto q^4|1,-1\rangle_1 \otimes |1,1\rangle_2 + q(q^2+1)|1,0\rangle_1 \otimes |1,0\rangle_2 + |1,1\rangle_1 \otimes |1,-1\rangle_2, \\ |2,1\rangle_{12} &\propto q^{-1}|1,1\rangle_1 \otimes |1,0\rangle_2 + q|1,0\rangle_1 \otimes |1,1\rangle_2, \\ |2,2\rangle_{12} &\propto q^{-2}|1,1\rangle_1 \otimes |1,1\rangle_2. \end{aligned} \tag{2.15}$$

As in the previous case, the multiplets all have different $J$ and the $\hat{R}$-matrix has the simple form

$$|0,0\rangle_{12} = a|0,0\rangle_{21}, \qquad |1,m\rangle_{12} = b|1,m\rangle_{21}, \qquad |2,m\rangle_{12} = c|2,m\rangle_{21}. \quad (2.16)$$

For $q \to 1$, $a = 1$, $b = -1$, and $c = 1$. Normalizing so that $c = 1$ there are two solutions to the Yang–Baxter equation with parameter values

$$\begin{aligned} a &= q^{-6} \\ b &= -q^{-4} \end{aligned} \quad \text{or} \quad \begin{aligned} a &= q^{6}, \\ b &= -q^{4}. \end{aligned} \quad (2.17)$$

These solutions correspond to the usual $\hat{R}$-matrix for $\mathscr{U}_q(SO_q(3))$ and its inverse [2]. For the former the eigenvalues are 1 with multiplicity 5, $-q^{-4}$ with multiplicity 3 and $q^{-6}$ with multiplicity 1. The projector decomposition is

$$\hat{R} = P_S + q^{-6}P_T - q^{-4}P_A, \quad (2.18)$$

where the $q$-deformed projectors are the trace projector $P_T$, the traceless symmetrizer $P_S$ and the antisymmetrizer $P_A$.

We now consider the interesting case of two $\mathbf{3} \oplus \mathbf{1}$ reducible representations of $\mathscr{U}_q(SU_q(2))$. Their product has six multiplets according to the rule

$$(\mathbf{3} \oplus \mathbf{1}) \otimes (\mathbf{3} \oplus \mathbf{1}) = \mathbf{5}_{TT} \oplus \mathbf{3}_{TT} \oplus \mathbf{1}_{TT} \oplus \mathbf{3}_{ST} \oplus \mathbf{3}_{TS} \oplus \mathbf{1}_{SS}.$$

The subscripts indicate whether the two spaces in the product are triplets or singlets. In the following we will append a corresponding superscript to the state kets. The $TT$ multiplets have the same form as (2.13)–(2.15) and we write them as $|J,M\rangle_{12}^{TT}$. There are three additional multiplets, a singlet:

$$|0,0\rangle_{12}^{SS} = |0,0\rangle_1 \otimes |0,0\rangle_2 \quad (2.19)$$

and two triplets:

$$\begin{aligned} |1,-1\rangle_{12}^{ST} &= |0,0\rangle_1 \otimes |1,-1\rangle_2, \qquad & |1,-1\rangle_{12}^{TS} &= |1,-1\rangle_1 \otimes |0,0\rangle_2, \\ |1,0\rangle_{12}^{ST} &= |0,0\rangle_1 \otimes |1,0\rangle_2, & |1,0\rangle_{12}^{TS} &= |1,0\rangle_1 \otimes |0,0\rangle_2, \\ |1,1\rangle_{12}^{ST} &= |0,0\rangle_1 \otimes |1,1\rangle_2, & |1,1\rangle_{12}^{TS} &= |1,1\rangle_1 \otimes |0,0\rangle_2. \end{aligned} \quad (2.20)$$

Now there can be mixing among the two singlets and among the three triplets. We parameterize the $\hat{R}$-matrix as

$$\begin{aligned} |0,0\rangle_{12}^{SS} &= a_1|0,0\rangle_{21}^{SS} + a_2|0,0\rangle_{21}^{TT}, \\ |0,0\rangle_{12}^{TT} &= b_1|0,0\rangle_{21}^{SS} + b_2|0,0\rangle_{21}^{TT}, \\ |1,m\rangle_{12}^{TT} &= c_1|1,m\rangle_{21}^{TT} + c_2|1,m\rangle_{21}^{ST} + c_3|1,m\rangle_{21}^{TS}, \\ |1,m\rangle_{12}^{ST} &= d_1|1,m\rangle_{21}^{TT} + d_2|1,m\rangle_{21}^{ST} + d_3|1,m\rangle_{21}^{TS}, \\ |1,m\rangle_{12}^{TS} &= e_1|1,m\rangle_{21}^{TT} + e_2|1,m\rangle_{21}^{ST} + e_3|1,m\rangle_{21}^{TS}, \\ |2,m\rangle_{12}^{TT} &= f|2,m\rangle_{21}^{TT}. \end{aligned} \quad (2.21)$$

In the limit

$$q \to 1, \qquad a_1 = b_2 = d_3 = e_2 = f = 1, \qquad c_1 = -1$$

and the rest of the parameters vanish.

Solving the Yang–Baxter equation, we find exactly three classes of solutions, each class having an $\hat{R}$-matrix and its inverse. Each of the three solutions corresponds to a known $\hat{R}$-matrix. So we learn that the known $\hat{R}$-matrices are the only possible $\mathcal{U}_q(\mathrm{SU}_q(2))$ covariant $\hat{R}$-matrices. The parameter values for these solutions are listed in the Appendix. Here we will discuss the general features of the solutions.

The first solution is quite simple. Only the parameters which do not vanish classically are nonzero, and there is no mixing of multiplets. When written in the basis of (2.5) the $16 \times 16$ matrix is found to be block diagonal. There is a $9 \times 9$ block which is precisely the $\hat{R}$-matrix for $\mathrm{SO}_q(3)$ from the previous example. The rest of the $\hat{R}$-matrix involves product states with an underlying singlet $|0,0\rangle_1$ or $|0,0\rangle_2$. The parameters for these terms ($a_1, d_3$ and $e_2$) are undetermined and may be any functions of $q$ with the correct limit for $q \to 1$.

The second and third solutions should be discussed together since they are different linear combinations of the same projectors. The second solution has some mixing of multiplets, between the $|1,m\rangle^{TT}$ multiplet and either the $|1,m\rangle^{ST}$ multiplet or the $|1,m\rangle^{TS}$ multiplet. The parameters $a_2$ and $b_1$ are always zero. For the third solution all parameters are nonvanishing and there is maximal mixing between all multiplets with the same $J$. For both solutions there is a free parameter, but it may be absorbed in the relative normalization between the underlying singlet and triplet. When written in the basis of (2.5), it is found that these are the $\hat{R}$-matrices as constructed for the $q$-Lorentz group, where a triplet and a singlet were combined into a four-vector★ [3, 4]. The second solution has eigenvalues 1, $-q^2$ and $-q^{-2}$ with multiplicities 10, 3 and 3, respectively. The third solution has eigenvalues 1, $q^{-4}$ and $-q^{-2}$ with multiplicities 9, 1 and 6. Since each of these matrices has three distinct eigenvalues three projectors may be extracted from each of them. However, taken together we have four projectors. Again there is a trace projector $P_T$ and a traceless symmetrizer $P_S$. Here, however, the antisymmetrizer splits into selfdual and anti-selfdual projectors $P_\pm$. The projector decomposition for the second solution is

$$\hat{R} = P_T + P_S - q^2 P_- - q^{-2} P_+ \tag{2.22}$$

and for the third solution is

$$\hat{R} = P_S + q^{-4} P_T - q^{-2} P_+ - q^{-2} P_- . \tag{2.23}$$

As noted in [3, 4], these two $\hat{R}$-matrices and their inverses are the only linear combinations of these projectors consistent with the algebra of $q$-Minkowski coordinates.

★The second solution corresponds to $\hat{R}_{\mathrm{I}}$ and the third solution corresponds to $\hat{R}_{\mathrm{II}}$ in [3, 4].

## 3. $q$-Deformed Coordinates

In this section, we will consider $q$-deformed coordinates as examples of the representation spaces discussed in the previous section. We will show how the $\hat{R}$-matrices for the $\mathbf{3} \oplus \mathbf{1}$ representation can be constructed from spinor coordinates corresponding to the doublet of $\mathcal{U}_q(SU_q(2))$.

Consider coordinates $X^i$ corresponding to some representation $|j, m\rangle_1$ and a second copy $\tilde{X}^i$ corresponding to a second space $|j, m\rangle_2$. Dropping the tensor product notation, (2.5) may be written as

$$X^i \tilde{X}^j = k \hat{R}^{ij}{}_{kl} \tilde{X}^k X^l, \tag{3.1}$$

where $k$ is an arbitrary normalization. Since $\hat{R}$ satisfies the Yang–Baxter equation any product of the coordinates may be reordered and the result is independent of the order in which adjacent coordinates are swapped.

The simplest example of such coordinates is the quantum plane, or quantum spinors. The spinors $x^\alpha, \alpha = 1, 2$ are a $\mathcal{U}_q(SU_q(2))$ doublet with the assignment

$$x^1 = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \quad \text{and} \quad x^2 = q^{-1}|\tfrac{1}{2}, \tfrac{1}{2}\rangle.$$

With a second copy $y^\alpha$ the appropriate $\hat{R}$-matrix for (3.1) is the $\hat{R}$-matrix constructed using (2.10), (2.11) and using this coordinate identification:

$$\hat{R}^{\alpha\beta}{}_{\gamma\delta} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{3.2}$$

As in the previous section, these two copies of the quantum plane may be combined into a four-dimensional $\mathbf{3} \oplus \mathbf{1}$ reducible representation of $\mathcal{U}_q(SU_q(2))$. We will denote these bispinors by $X^i, 1 \leqslant i \leqslant 4$.

For two copies of the $\mathbf{3} \oplus \mathbf{1}$ $X^i$ and $\tilde{X}^i$ we will need four copies of the quantum spinor plane: $x^\alpha$, $y^\alpha$, $u^\alpha$ and $v^\alpha$. Following (3.1), we choose the following normalization and ordering:

$$\begin{aligned} xy &= q^{-1}\hat{R}yx, & yu &= q^{-1}\hat{R}uy, \\ xu &= q^{-1}\hat{R}ux, & yv &= q^{-1}\hat{R}vy, \\ xv &= q^{-1}\hat{R}vx, & uv &= q^{-1}\hat{R}vu, \end{aligned} \tag{3.3}$$

where we have suppressed indices. Then there are three distinct ways to group the spinors into two $\mathbf{3} \oplus \mathbf{1}$'s:

$$\begin{aligned} &\textit{Case I:} \quad X = xy,\ \tilde{X} = uv, \\ &\textit{Case II:} \quad X = xv,\ \tilde{X} = yu, \\ &\textit{Case III:} \quad X = xu,\ \tilde{X} = yv. \end{aligned} \tag{3.4}$$

A direct calculation shows that these case I, II, and III lead to the first, second, and third solutions, respectively, from the previous section. Replacing $q^{-1}\hat{R}$ by $q\hat{R}^{-1}$ in (3.3) gives the inverse solutions. From the previous section, we know that these three constructed $\hat{R}$-matrices are unique assuming $\mathscr{U}_q(\mathrm{SU}_q(2))$ covariance. In this section, we showed that these $\hat{R}$-matrices may be constructed from the basic $\hat{R}$-matrix (3.2). Note that we have constructed $\hat{R}$-matrices which cannot be expressed in terms of the generators of $\mathscr{U}_q(\mathrm{SU}_q(2))$, as is the case for the universal $\hat{R}$-matrix. The method of construction followed here could be generalized to higher-dimensional representations. Similar results have been obtained in [5] from a different approach.

## 4. Differential Calculus

In this section we will consider a differential calculus on the quantum spaces discussed in the previous section. It will be seen that there are constraints on the projector decomposition of the $\hat{R}$-matrix used to formulate the calculus.

Equation (3.1) gives the algebra between coordinates from two different copies of the quantum space. However, we have not specified the algebra among coordinates from the same copy. Suppose there is a set of $q$-deformed projectors on the space, symmetrizers $S_i$ and antisymmetrizers $A_i$, which sum to the identity matrix $\mathbb{1} = \Sigma_i S_i + \Sigma_i A_i$. Since the coordinates are classically commuting objects, we require the product of two coordinates to be annihilated by the antisymmetrizers. Suppressing indices, we write

$$A_i XX = 0. \tag{4.1}$$

This must be true for all of the antisymmetrizers.

We now establish a differential calculus on this space by introducing derivatives $\partial_i$ acting on the coordinates. This action is

$$\partial_i X^j = \delta_i^j + C^{jk}{}_{il} X^l \partial_k. \tag{4.2}$$

Here $C$ is some linear combination of the projectors $C = \Sigma_i \sigma_i S_i + \Sigma_i \alpha_i A_i$. If we apply a derivative to (4.1) and use (4.2) to move the derivative to the right, we get constraints on the eigenvalues for the antisymmetrizers:

$$\begin{aligned} \partial A_i XX &= A_i(\mathbb{1} + C)X + \text{cubic terms} \\ &= (1 + \alpha_i) A_i X + \cdots \end{aligned} \tag{4.3}$$

where in the last step we used $A_i A_j = \delta_{ij} A_i$ and $A_i S_j = 0$. Since the left-hand side vanishes for consistency we see that $\alpha_i = -1$. This must be true for all of the projectors, so we find

$$C = \sum_i \sigma_i S_i - \sum_i A_i. \tag{4.4}$$

In order to have a consistent differential calculus, the matrix $C$ in (4.2) must have all the antisymmetrizers with eigenvalue $-1$.

In [6, 7] the differential calculus has been developed with an exterior derivative $d = \xi^i \partial_i$, where $\xi^i = dX^i$ are the differentials of the coordinates. When all three objects $X^i$, $\partial_i$ and $\xi^i$ are considered together consistency requires that $C$ satisfy the Yang–Baxter equation. Thus, for a differential calculus $C$ must be an $\hat{R}$-matrix with all antisymmetrizers having eigenvalue $-1$.

We may now apply these considerations to the $\hat{R}$-matrices for the $\mathbf{3} \oplus \mathbf{1}$ representation. For the first solution the undetermined parameters may be chosen so that the antisymmetrizers all have eigenvalue $-q^{-4}$, the value for the $SO_q(3)$ part of the matrix. Then $C = q^4 \hat{R}$ is suitable for a differential calculus. Looking at (2.23), we see that the third solution can also be used to construct a differential calculus with $C = q^2 \hat{R}$. This was the matrix used for the $q$-Poincaré algebra in [3]. Finally, we see in (2.22) that the antisymmetrizers $P_\pm$ enter the $\hat{R}$-matrix with different eigenvalues. This $\hat{R}$-matrix cannot be used for a differential calculus.

## Appendix

In this appendix, we list the parameter values for solutions of the Yang–Baxter equation for the $\hat{R}$-matrix parameterized in (2.21). Recall that the Yang–Baxter equation does not fix the normalization of $\hat{R}$. We fix the normalization by choosing $f = 1$, the value obtained for all of the constructed $\hat{R}$-matrices.

The first solution is quite simple, with no mixing of different multiplets. We have

$$b_2 = q^{-6}, \qquad c_1 = -q^{-4}. \tag{A.1}$$

The parameters $a_1$, $d_3$ and $e_2$ are undetermined. They may be any functions of $q$ taking the value 1 in the limit $q \to 1$. (For the $\hat{R}$-matrix constructed in Section 3 we had $a_1 = d_3 = e_2 = q^{-2}$.) The rest of the parameters vanish. For the inverse solution we find

$$b_2 = q^6, \qquad c_1 = -q^4. \tag{A.2}$$

Again, the parameters $a_1$, $d_3$ and $e_2$ are undetermined and may be any appropriate functions of $q$. (For the constructed $\hat{R}$-matrix we had $a_1 = d_3 = e_2 = q^2$.) Again the rest of the parameters vanish.

For the second solution we find

$$\begin{aligned} &a_1 = 1, \qquad a_2 = 0, \\ &b_1 = 0, \qquad b_2 = 1, \\ &c_1 = -1 - \lambda^2, \qquad c_2 = 0, \qquad c_3 = -(q^2 + q^{-2})\lambda r, \\ &d_1 = \lambda r^{-1}, \qquad d_2 = 0, \qquad d_3 = 1 + \lambda^2, \\ &e_1 = 0, \qquad e_2 = 1, \qquad e_3 = 0 \end{aligned} \tag{A.3}$$

and for the inverse solution

$$\begin{aligned}
&a_1 = 1, \quad a_2 = 0,\\
&b_1 = 0, \quad b_2 = 1,\\
&c_1 = -1 - \lambda^2, \quad c_2 = -(q^2 + q^{-2})\lambda r, \quad c_3 = 0,\\
&d_1 = 0, \quad d_2 = 0, \quad d_3 = 1,\\
&e_1 = \lambda r^{-1}, \quad e_2 = 1 + \lambda^2, \quad e_3 = 0.
\end{aligned} \tag{A.4}$$

Here $r$ is a free parameter which may be absorbed in the relative normalization of the underlying singlet and triplet. The constructed $\hat{R}$-matrices are recovered for $r = 1$.

The third solution is more complicated, with no parameters vanishing. For convenience, we define the common factor $\eta = (q^2 + 1)^{-1}$. Then we have

$$\begin{aligned}
&a_1 = \eta(q^2 + q^{-2}), \quad a_2 = -\eta q^{-2}\lambda r^{-2},\\
&b_1 = \eta(q^{-3} - q^3)r^2, \quad b_2 = \eta(q^{-4} + 1),\\
&c_1 = \eta(-q\lambda - 2q^{-2}), \quad c_2 = -\eta(q^2 + q^{-2})\lambda r, \quad c_3 = -\eta(q^2 + q^{-2})\lambda r,\\
&d_1 = \eta\lambda r^{-1}, \quad d_2 = \eta q\lambda, \quad d_3 = \eta(q^2 + q^{-2}),\\
&e_1 = \eta\lambda r^{-1}, \quad e_2 = \eta(q^2 + q^{-2}), \quad e_3 = \eta q\lambda.
\end{aligned} \tag{A.5}$$

The inverse solution has parameters

$$\begin{aligned}
&a_1 = \eta(q^4 + 1), \quad a_2 = \eta q^2\lambda r^{-2},\\
&b_1 = \eta(q^7 - q)r^2, \quad b_2 = \eta(q^6 + q^2),\\
&c_1 = \eta(q\lambda - 2q^4), \quad c_2 = -\eta(q^4 + 1)\lambda r, \quad c_3 = -\eta(q^4 + 1)\lambda r,\\
&d_1 = \eta q^2\lambda r^{-1}, \quad d_2 = -\eta q\lambda, \quad d_3 = \eta(q^4 + 1),\\
&e_1 = \eta q^2\lambda r^{-1}, \quad e_2 = \eta(q^4 + 1), \quad e_3 = -\eta q\lambda.
\end{aligned} \tag{A.6}$$

Again, $r$ is a free parameter and the constructed $\hat{R}$-matrices are recovered for $r = 1$.

## References


1. Biedenharn, L. C., p. 61 in C. Bartocci, U. Bruzzo, and R. Cianci (eds), *Proc. Differential Geometric Methods in Theoretical Physics*, Rapallo, Italy, 1990, Springer-Verlag, New York, Berlin, 1991, and references therein.
2. Faddeev, L. D., Reshetikhin, N. Yu, and Takhtajan, L. A., *Leningrad Math. J.* **1**, 193 (1990).
3. Ogievetsky, O., Schmidke, W. B., Wess, J., and Zumino, B., *Comm. Math. Phys.* **150**, 495 (1992).
4. Ogievetsky, O., Schmidke, W. B., Wess, J., and Zumino, B., to appear in *Proc. First German–Polish Symposium on 'Particles and Fields'*, Rydzyna, Poland, 1992.
5. Jain, V. and Ogievetsky, O., *Modern Phys. Lett. A* **7**, 2199 (1992).
6. Wess, J. and Zumino, B., Covariant differential calculus, *Nuclear Phys. B* (*Proc. Suppl.*) **18**, 320 (1990).
7. Wess, J., Differential calculus on quantum planes and applications, Talk given at Third Centenary Celebrations of the Mathematische Gesellschaft, March 1990, based on work with B. Zumino, preprint KA-THEP-1990-22 (1990).