General Relativity and Gravitation, Vol. 17, No. 8, 1985

On Bianchi Type-I Vacuum Solutions in $R + R^2$ Theories of Gravitation. I. The Isotropic Case

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Received November 14, 1983

From a Lagrangian $L = (1/\kappa)(-g)^{1/2}[R/2 + l^2(\alpha R_{ik}R^{ik} + \beta R^2)]$ one obtains fourth-order field equations for the metrical tensor g_{ik}. Inserting a 3-flat Robertson–Walker line element, the set of their vacuum solutions will be enumerated completely. The qualitative behavior, and especially the influence of the l^2-terms (which is possibly necessary for the renormalization of quantum gravity) in certain stages of evolution follow from a phase plane analysis. Depending on the sign of coupling, one obtains either exponentially increasing or oscillating solutions at late times as well as special solutions without an initial singularity.

1. INTRODUCTION

In the last decades field equations of higher than second order have raised some interest both in classical and quantum gravity. Here we want to discuss field equations stemming from a Lagrangian with linear and quadratic terms in the Ricci tensor R_{ik},

$$L = \frac{1}{\kappa}(-g)^{1/2}\left[\frac{R}{2} + l^2(\alpha R_{ik}R^{ik} + \beta R^2)\right] + L_{\mathrm{mat}} \tag{1}$$

where l is a coupling length, α, β are numerical constants of the order 1, and L_{mat} is the matter Lagrangian. Quadratic invariants have been introduced firstly by Weyl [1] and Bach [2] in the conformally invariant combination $\alpha + 3\beta = 0$ while attempting to unify classical gravitational

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0001-7701/85/0800-0769$04.50/0

and electromagnetic interactions. Later such a sum (1) has been taken as gravitational theory with phenomenological matter modifying Einstein's theory at small distances (Buchdahl [3], Pechlaner and Sexl [4], and Treder [5]). From this point of view especially the consequences for the gravitational collapse of massive bodies have been studied.

Recently, such a sum (1) appeared quite naturally as the renormalized action of quantized matter field coupled to classical general relativity (de Witt [6], Stelle [7], and Fischetti and others [8]). The quadratic terms describe phenomenologically the vacuum polarization at the one-loop level. Consequently, it is of interest to compare cosmological solutions of the generalized action (1) with solutions of the general relativity theory. From inspection of the Lagrangian one expects deviations from solutions of Einstein's equations at regions of a large space-time curvature, $|R_{ijkl}| \gtrsim l^{-2}$, i.e., near singularities. But the global behavior of solutions may be quite different, too.

Variation of (1) with respect to the metric g_{ik} yields the field equation

$$E_{ik} + l^2(\alpha H_{ik} + \beta G_{ik}) = \kappa T_{ik} \qquad \text{where} \tag{2}$$

$$H_{ik} = 2 \Box R_{ik} + g_{ik} \Box R - 2R_{;ik} + 4R_{ilmk}R^{lm} - g_{ik}R_{lm}R^{lm}$$

$$G_{ik} = 4g_{ik} \Box R - 4R_{;ik} + 4R\, R_{ik} - g_{ik}R^2$$

A semicolon means covariant differentiation, and $\Box(\cdot) = g^{ik}(\cdot)_{;ik}$ is the wave operator. In general, the field equations are of fourth order in the metric, which results in a wide enlargement of the class of solutions. Up to now cosmological solutions have been studied only in special cases by approximation methods (Gurovich and Starobinsky [9]), by numerical methods (Tomita and others [10]), and by power series expansion (Rusmaikin [11]), or one has selected special solutions as the de Sitter solution (Starobinsky [12, 13]) or Friedman's radiation cosmos (Macrae and Riegert [14]). Frenkel and Brecher [15] have discussed the horizon problem of Robertson–Walker metrics by means of an asymptotic expansion. Most papers restrict the discussion to the simpler case $\alpha = 0$.[2]

Because of qualitatively new properties of higher-order differential equations it seems fashionable to obtain a complete classification of the solutions and their asymptotic behavior. To elucidate the physical meaning of the quadratic modifications in the Lagrangian we discuss as a first step the matter-free case and study the corresponding field equations for isotropic (and in an accompanying paper II axially symmetric anisotropic) Bianchi type-I models.

[2] Power series expansions of singularity-free cosmological models are discussed by Kerner [16], but there instead of R^2 an arbitrary function of the Ricci-scalar appears in the action.

Taking an ideal fluid as source one finds a further enlargement of the space of solutions. But already the vacuum case leads to interesting non-trivial results. In Section 2, we give a qualitative argument for the behavior of general Bianchi type-I solutions. Next, in Section 3 we explain the method of the phase plane analysis in the isotropic case and present the results in Sections 4 and 5 for the two signs of coupling $\alpha + 3\beta \gtrless 0$ separately. A comparison with Einstein's theory ($\alpha = \beta = 0$) and a short discussion will be found in Section 6.

2. TRACE EQUATION

A first simple estimate concerning the behavior of solutions follows from the trace of equation (2). If one chooses a synchronized time coordinate t, the volume expansion h can be defined by

$$h = [\ln(-g)^{1/2}]^{\cdot}, \qquad g = \det g_{ik} \tag{3}$$

where the dot means differentiation with respect to t. Then, the trace of equation (2) gives for Bianchi type-I an oscillator equation for the Ricci scalar R:

$$R^{\cdot\cdot} + hR^{\cdot} - R/\mu = 0 \tag{4}$$

where $\mu = 4(\alpha + 3\beta)l^2$. In the vacuum case it is homogeneous but it contains a nonlinear damping. For large classes of functions $h(t)$ comparison theorems for linear differential equations of second order (Swanson [17]) determine the global and asymptotic behavior. For $\mu < 0$, $R(t)$ oscillates around the general relativistic solution $R = 0$ with a damping $R \to 0$ for $t \to \infty$ ($h > 0$). If $h(t)$ becomes too large then $R(t)$ decays exponentially.

$\mu > 0$ implies exponential behavior, i.e., unbounded growth of $R(t)$ in past or future. The following section demonstrates the essential correctness of this first estimate for 3-flat Robertson–Walker space-times.

3. PHASE PLANE ANALYSIS

In the homogeneous and isotropic case with flat space slices, the metric depends on the scale factor $a(t)$ only, and is given by

$$ds^2 = dt^2 - a^2(t)(dx_1{}^2 + dx_2{}^2 + dx_3{}^2) \tag{5}$$

Here, the cosmological expansion is characterized by the Hubble parameter $h = 3\dot{a}/a$, and the energy component of the field equation (2) simply reads

$$\mu(2h\ddot{h} - \dot{h}^2 + 2h^2\dot{h}) = h^2 \tag{6}$$

The remaining components of the field equation follow already from this one. In the following we consider expanding solutions ($h > 0$), i.e., we fix the time direction. As will be seen later, a transition from contraction to expansion is excluded by the field equations.

The deciding question for prefering a modified Lagrangian (1) is whether the spatial contraction in the negative time direction can be stopped before reaching the cosmological singularity $h \to \infty$. This would require $\dot{h} = 0$ at some finite $h > 0$, i.e., with equation (6) $\ddot{h} = h/2\mu$. So only for $\mu < 0$ could one find a maximum of the expansion velocity which seems necessary to avoid the cosmological singularity. But on the other hand, the above conclusion from the trace equation (4) indicates oscillating solutions, i.e., a non-Friedman-like character for this sign of the coupling of the quadratic terms. It was just this behavior that caused the pessimism expressed in References 10 and 11 with respect to the fourth-order equations.

Now equation (6) shall be investigated in detail. In all but the isolated points where $\dot{h} = 0$, we may take h as a new independent variable, and one obtains

$$\frac{d\dot{h}(h)}{dh} = \frac{\dot{h}}{2h} + \frac{h}{2\mu\dot{h}} - h \tag{7}$$

Let us discuss equation (7) in the h–$\dot{h}$ phase plane which is covered by the trajectories $\dot{h}(h)$. The only point where these trajectories may intersect is the singular point $h = \dot{h} = 0$.

The function $h(t)$ can be reobtained from $\dot{h}(h)$ via the inverted function

$$t(h) = \int \frac{1}{\dot{h}(h)}\, dh \tag{8}$$

The inflection points of the function $h(t)$ ($\ddot{h} = 0$) are just the extrema of the curve $\dot{h}(h)$. They are described by $d\dot{h}/dh = 0$, i.e., $h = |\dot{h}|\,(2\dot{h} - 1/\mu)^{-1/2}$, and exist for $\dot{h} > 1/2\mu$ only; cf. the dashed line in Figures 1 and 2. The asymptotic behavior is $\dot{h} = 1/2\mu + 1/8\mu^2h^2$ and $\dot{h} = 2h^2$ for $h \to \infty$. The dashed line together with the h axis separates phase space regions with increasing and decreasing phase trajectories $\dot{h}(h)$. The slope $d\dot{h}/dh$ of the curves $\dot{h}(h)$ is indicated by future-directed arrows.

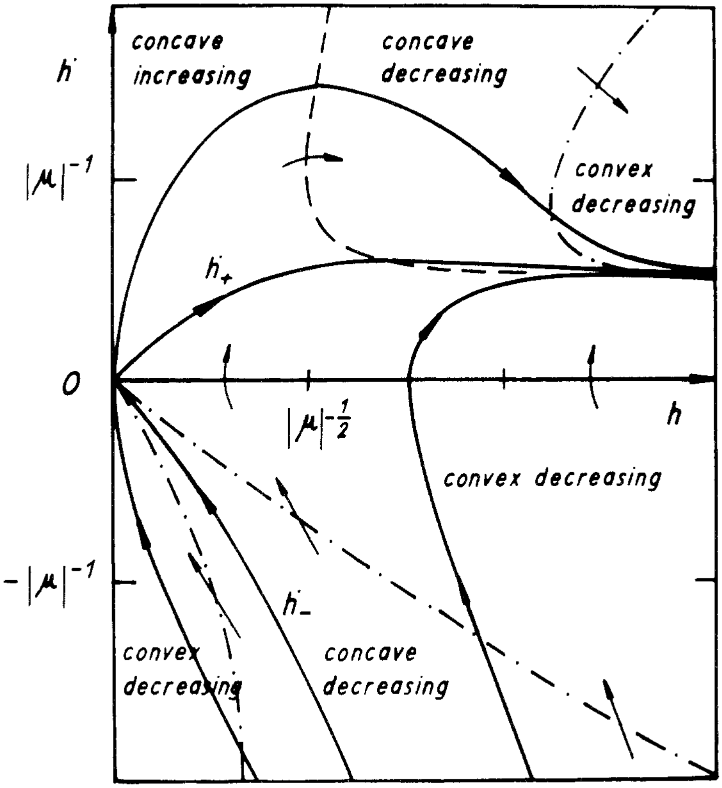

Fig. 1. Phase plane pattern for $\mu > 0$.

In a similar way we look for the regions where the trajectories $\dot{h}(h)$ are convex and concave, respectively. To this end we differentiate equation (7),

$$\frac{d^2\dot{h}}{dh^2} = \frac{-3}{2} + \frac{1}{2\mu\dot{h}} - \frac{\dot{h}}{4h^2} - \frac{h^2}{4\mu^2\dot{h}^3} + \frac{h^2}{2\mu\dot{h}^2} \tag{9}$$

and solve the equation $d^2\dot{h}/dh^2 = 0$, which describes the inflection points of the function $\dot{h}(h)$. This dash–dot line together with the h axis separates the regions with convex and concave functions $\dot{h}(h)$; cf. Figures 1 and 2. Equation (9) is biquadratic in h and can be solved explicitly. Now the different signs of μ lead to quite different results and we shall discuss them separately.

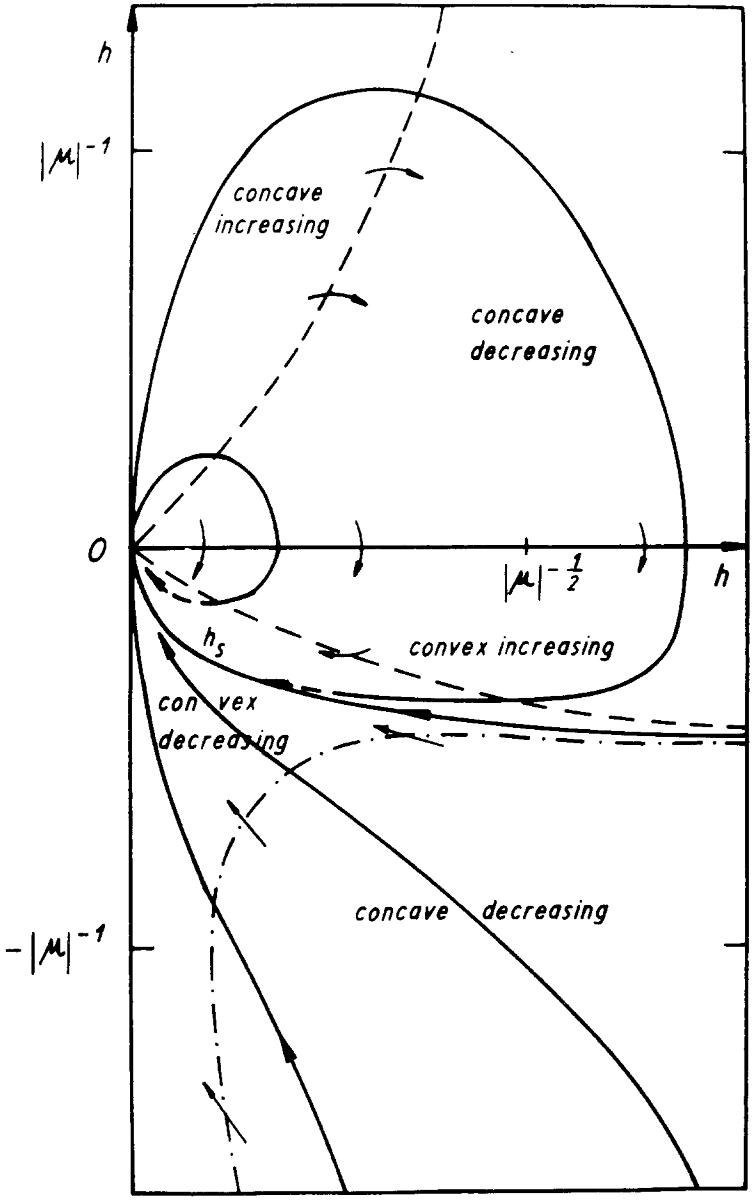

Fig. 2. Phase plane pattern for $\mu < 0$.

4. THE CASE $\mu > 0$

The zeros of equation (9) are the following: For $\dot{h} > 1/2\mu$ one has

$$h = \dot{h}(2\dot{h} - 1/\mu)^{-1/2}\{3\mu\dot{h} - 1 + [\mu\dot{h}(9\mu\dot{h} - 4)]^{1/2}\}^{1/2} \tag{10}$$

and for $\dot{h} < 0$ the two solutions

$$h = -\dot{h}(-2\dot{h} + 1/\mu)^{-1/2}\{-3\mu\dot{h} + 1 \pm [-\mu\dot{h}(-9\mu\dot{h} + 4)]^{1/2}\}^{1/2} \tag{11}$$

Other solutions do not exist. The asymptotic behavior is

$$\dot{h} = h/(3\mu)^{1/2}, \qquad \dot{h} = 1/2\mu + 1/8\mu^2 h^2, \qquad \dot{h} = -h/(3\mu)^{1/2}$$

and $\dot{h} = -6h^2$ for $h \to \infty$; and $\dot{h} = -h/\sqrt{\mu}$ for $h \to 0$; cf. the dash–dot line in Figure 1.

After this analysis the inspection of the h–$\dot{h}$ phase plane shows the existence of two open regions containing one-parameter families of phase trajectories. The boundary between them consists of two special solutions.

First we note that for $\dot{h} = 0$, $h \neq 0$, equation (7) is singular. One finds from (6) $\ddot{h} = h/2\mu > 0$, and $h(t)$ has simply a regular local minimum. Local maxima do not exist for $h > 0$, therefore this minimum is a global one.

If this minimum h_0 runs from $h_0 > 0$ up to ∞ we obtain the first one-parameter family of solutions; they fill some open region G of the h–$\dot{h}$ plane. G lies fully beyond the line $\dot{h} = 1/\mu$ and above the curve $\dot{h} = -h^2$. Its boundary ∂G consists of the point $h = \dot{h} = 0$ and two smooth curves starting from the origin with slopes $\pm 1/\sqrt{\mu}$. They represent two special solution $\dot{h}_+(h)$ and $\dot{h}_-(h)$ (cf. Figure 1) for they are envelopes of a family of solutions of the first-order differential equation (7).

Near $h = 0$ they have the expansion

$$\dot{h}_\pm = \pm h/\sqrt{\mu} - h^2/2 \pm \sqrt{\mu}\, h^3/24 + O(h^4) \tag{12}$$

If one takes initial conditions outside $G \cup \partial G$ one obtains the second class of solutions. They go through the singular point at the origin and can be parametrized by the second time derivative $\ddot{h} = \lim_{\dot{h} \to 0}(\dot{h} \cdot d\dot{h}/dh)$ at this point. A continuous $\ddot{h} > 0$ forbids a change of the sign of h. The asymptotic behavior of the phase trajectories follows from a comparison of the slopes of the curves which separate increasing and decreasing, concave and convex regions as well as slopes of curves $\dot{h} = -a^2h^2$ ($a = \text{const}$) with $d\dot{h}/dh$ as given by equation (7). For accelerated expansion $\dot{h} > 0$, it follows $h \to \infty$ and $\dot{h} \to 1/2\mu$. For retarded expansion $\dot{h} < 0$, one finds $\dot{h}/h^2 \to -2/3$ as $h \to \infty$. Now we turn to the $h(t)$ picture. All but the special solution $\dot{h}_+$ start with retarded expansion. With the exception of the special solution $\dot{h}_-$

all trajectories reach the line $h=0$ for some time t_0 and undergo accelerated expansion thereafter. Besides $\dot{h}_+$, the asymptotic behavior of all solutions for decreasing times is given by

$$\dot{h}=(-2/3)h^2, \qquad h=3/2t, \qquad a=a_0\sqrt{t} \tag{13}$$

i.e., they start from a big bang singularity with the expansion law of Friedman's radiation cosmos. For the special solution $\dot{h}_+$ one finds

$$\dot{h}=h/\sqrt{\mu}, \qquad h=\exp(t/\sqrt{\mu}), \qquad a=a_0\exp[\exp(t/\sqrt{\mu})] \tag{14}$$

It starts from a finite $a=a_0$ and $R=0$ at $t=-\infty$.[3] For increasing time $t\to\infty$ the asymptotic expansion law of all solutions except $\dot{h}_-$ again contains the coupling constant μ. It holds that

$$\dot{h}=1/2\mu, \qquad h=t/2\mu, \qquad a=a_0\exp(t^2/12\mu) \tag{15}$$

Hence $a(t)$ expands faster than exponentially, and $R\to-\infty$. The special solution $\dot{h}_-$ expands to a maximal value $a=a_0$ and $R(t)\to 0$:

$$\dot{h}=-h/\sqrt{\mu}, \quad h=\exp(-t/\sqrt{\mu}), \quad \text{and } a=a_0\exp[-\exp(-t/\sqrt{\mu})] \tag{16}$$

as $t\to\infty$.

It is an unstable solution which describes the typical behavior of solutions in the past. For small deviations there result solutions which go over into the region of accelerated expansion and behave like the singularity-free stable solution $\dot{h}_+$ for large times for $t\to\infty$.

5. THE CASE $\mu<0$

The zeros of equation (9) are of a similar type like those of equations (10) and (11). The asymptotic behavior is

$$\dot{h}=-6h^2 \quad \text{and} \quad \dot{h}=1/2\mu+1/8\mu^2h^2 \qquad \text{for} \quad h\to\infty$$

Cf. Figure 2 for details. All phase trajectories from the region $\dot{h}<1/2\mu$ go with positive curvature $\ddot{h}$ through the singular point $h=\dot{h}=0$ and form thereafter distorted circles in the region $h>0$, $\dot{h}>1/2\mu$. In the case of negative coupling, $\mu<0$, we have at the intersections with the axis $h=0$, local maxima $(\ddot{h}=h/2\mu<0)$.

Now similar to the case of positive μ we construct a special solution $\dot{h}_s$ by a limiting process. Let us consider all phase trajectories which start with

[3] The Ricci scalar is given by $R=-2\dot{h}-4h^2/3$.

different curvature $\ddot{h}>0$ from the singular point $h=\dot{h}=0$ into the future direction. They cover some open region G of phase space, and its lower boundary (observe that G does not intersect the line $\dot{h}=1/2\mu$), ∂G, gives the special solution $\dot{h}_s(h)$. For $h\to\infty$ we have $\dot{h}_s\to 1/2\mu$ (cf. the heavy line in Figure 2). Let us note that $\dot{h}_s$ is a stable solution. The asymptotic behavior of phase trajectories outside $G\cup\partial G$ is given by $h\to\infty$ and $\dot{h}/h^2\to -2/3$ as in Section 2.1.

For the discussion of the phase trajectories inside G it is useful to define an auxiliary function $H=\dot{h}^2/h-h/\mu$. Then equation (5) can be transformed to

$$\dot{H}=-2\dot{h}^2 \tag{17}$$

i.e., $H(t)$ is monotonically decreasing. For $h>0$ we have $H\geqslant 2|\dot{h}|(-\mu)^{1/2}$. With equation (17) we find $H(t)\to 0$ and therefore h, $\dot{h}\to 0$ as $t\to\infty$. Each phase trajectory spirals inward to the origin and cuts the h axis an infinite number of times. It follows that the space of solutions is the one-dimensional sphere $S^1=R/Z$ (here R denotes the real numbers and Z the integers). The monotonic function $H(t)$ gives the decreasing of amplitudes of these spirals.

Now let us turn to the $h(t)$ picture. The asymptotic behavior for decreasing times of all solutions but $\dot{h}_s$ is given by Friedman's radiation cosmos. The special curve $\dot{h}_s$ has

$$\dot{h}=1/2\mu, \qquad h=t/2\mu, \qquad a=a_0\exp(t^2/12\mu) \tag{18}$$

i.e., the scale factor goes to zero for $t\to-\infty$, and only in this limit $R(t)\to-\infty$. Such a faster-than-exponential increase is denoted by Starobinsky [13] as a quasi-de Sitter stage which resembles the properties of an inflationary universe. $\dot{h}_s$ is the only singularity-free solution for $\mu<0$. For $t\to\infty$, $h(t)$ oscillates, but $a(t)$ increases monotonously because $h=0$ holds only at singular points.

Now we are interested in the averaged behavior of $a(t)$ as $t\to\infty$. To this end we choose the time scale such that $\mu=-1$. For some solution $h(t)$ we denote by $h_n=h(t_n)$ its nth local maximum. In Figure 3 the behavior of this solution is plotted for $t_n\leqslant t\leqslant t_{n+1}$ (bold line). At $t=t_n$, $H=h_n$ holds, and therefore $h_{n+1}\leqslant H\leqslant h_n$ for this time interval. The curve $\dot{h}(h)$ is situated between the circles $H=h_n$ and $H=h_{n+1}$ (dashed line in Figure 3). Now we define ϑ_n by $t_n<\vartheta_n<t_{n+1}$ and $\dot{h}(\vartheta_n)=0$. With equation (8) we obtain $t_{n+1}-\vartheta_n\lesssim\pi$, $\vartheta_n-t_n\gtrsim\pi$ and after a short calculation,

$$t_{n+1}-t_n\to 2\pi \qquad \text{as} \quad n\to\infty \tag{19}$$

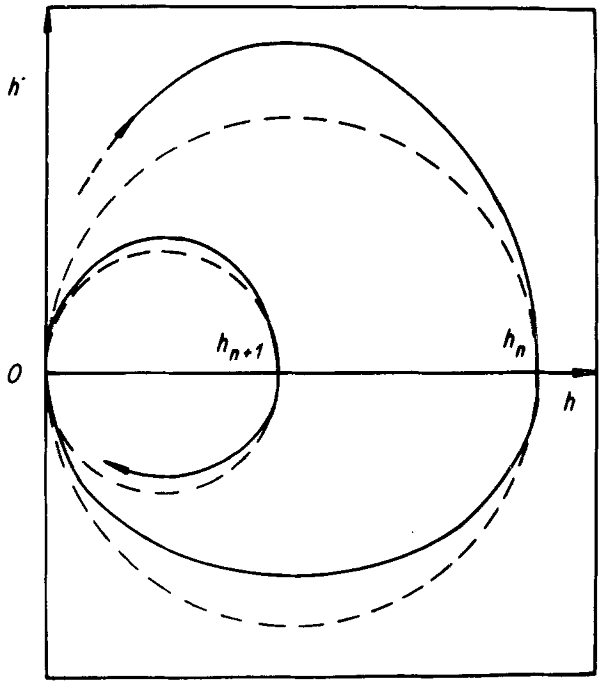

Fig. 3. Asymptotic oscillations for $\mu<0$, undetermined units.

Further, from equation (17) one has $dH/dh = -2\dot{h}$. Integrating this over one circle we find

$$h_n - h_{n+1} = -\int_{t_n}^{t_{n+1}} \dot{H}(t)\,dt = 2\left[\int_{h_n}^{0} \dot{h}(h)\,dh + \int_{0}^{h_{n+1}} \dot{h}(h)\,dh\right] \tag{20}$$

Hence, the quantity $(h_n - h_{n+1})/2$ lies between the surface areas of the circles $H = h_n$ and $H = h_{n+1}$. Therefore

$$\frac{\pi}{2}(h_{n+1})^2 < h_n - h_{n+1} < \frac{\pi}{2}(h_n)^2 \tag{21}$$

From this relation it follows $h_n \to 0$, and $h_{n+1}/h_n \to 1$ as $n \to \infty$. All sequences $(h_n)_n$ fulfilling relation (21) and $h_1 < 2/\pi$ have the same behavior for $n \to \infty$. Therefore we choose as representative series

$$h_n = 2/\pi n \tag{22}$$

which gives $h_n - h_{n+1} = \pi h_n h_{n+1}/2$.

From series (22) we obtain

$$a(t_{n+1}) = a(t_n) \exp\left[\int_{t_n}^{t_{n+1}} h(t)\, dt/3\right] \tag{23}$$

$\approx a(t_n) \exp(2/3n)$.

This relation describes the averaged behavior of the scale factor at late times, and it is that of Friedman's dust cosmos

$$a(t) = a_0 t^{2/3} \tag{24}$$

It should be noted that only the sign and not the absolute value of μ is important for this behavior.

6. DISCUSSION

For isotropic metrics all modifications depend on the combination of parameters $\alpha + 3\beta \neq 0$. The discussion of the trace equation shows that the evolution equation for the volume expansion is changed strongly. A phase plane analysis expressing acceleration as a function of expansion velocity yields a complete classification of solutions and their qualitative behavior. Afterward the dynamical behavior of the metric can be reobtained by series expansion to any desired precision.

As a first important result we have obtained that the sign of the expansion velocity cannot change, i.e., expanding solutions remain expanding. The vacuum field equations forbid an avoidance of the cosmological singularity by a bouncing at a certain minimal value of the scale factor. Really, all but one special solution starts with an initial singularity of the same type as Friedman's radiation cosmos in general relativity.

For $\mu = 4(\alpha + 3\beta) > 0$ all solutions expand with constant acceleration for $t \to \infty$ which leads to a faster-than-exponential increase of the scale factor according to equation (15) and an asymptotically divergent Ricci scalar. On the other hand if $\mu < 0$, all solutions oscillate around an average behavior $a \sim t^{2/3}$ for $t \to \infty$. That means the quadratic modifications in the Lagrangian (or in other words the vacuum polarization of quantum fields) mimic solutions of general relativity with ordinary matter. The behavior of the asymptote (15) was already mentioned in [11], and approximate expressions for the oscillating behavior of the scale factor for $\mu < 0$ are given in [9]. For the interpretation as nonlinear oscillations around a power law according to Friedman's dust cosmos compare also [13]. These oscillations are damped and the scale factor grows monotonically.

Besides the one-parameter classes of solutions special solutions exist which start in the infinite past with a finite ($\mu > 0$) or zero ($\mu < 0$) initial value of the scale factor. The asymptotic expression for the metric of these solutions contains the coupling length, i.e., only by the combined action of the Einstein–Hilbert and quadratic Lagrangian can we obtain such singularity-free cosmological models. We would like to point out here that some authors (see, e.g., [11]) prefer the choice $\alpha, \beta > 0$ for having a minimal and not only stationary action. To be more general we did not follow this restriction in our discussion.

The chosen presentation of the material shows the power and the importance of the analytical phase-space analysis for such highly nonlinear differential equations. Numerical computations have been performed only to obtain some specific integral curves for illustrations. The complete enumeration of types of solutions gives the global behavior of solutions known previously only in asymptotic regions as well as an embedding of special solutions in the general class and a discussion of their stability. From linearizations around solutions of general relativity the claim results that the fourth-order terms become important only near the initial singularity. On the contrary, the Bianchi type-I solutions considered here show that the global behavior of the cosmological expansion is strongly modified, too.

ACKNOWLEDGMENTS

The authors are grateful to Dr. U. Kasper for reading the manuscript and for his interesting comments. It is a pleasure to thank Dr. H.-H. v. Borzeszkowski and Dr. A. Starobinsky for useful discussions in October 1983 during the 6th Potsdam Seminar on Gravitation Theory and Relativistic Astrophysics where these results were presented.

REFERENCES

1. Weyl, H. (1919). *Ann. Phys. Leipzig*, **59**, 101.
2. Bach, R. (1921). *Math. Zeitschr.*, **9**, 110.
3. Buchdahl, H. (1962). *Nuovo Cimento*, **123**, 141.
4. Pechlaner, E., and Sexl, R. (1966). *Commun. Math. Phys.*, **2**, 165.
5. Treder, H.-J. (1975). *Ann. Phys. Leipzig*, **32**, 383.
6. De Witt, B. S. (1975). *Phys. Rep.*, **19C**, 295.
7. Stelle, K. S. (1977). *Phys. Rev. D*, **16**, 953.
8. Fischetti, M. V., Hartle, J., and Hu, B. (1979). *Phys. Rev. D*, **20**, 1757.

9. Gurovich, V. T., and Starobinsky, A. A. (1979). *Zh. Eksp. Teor. Fiz.*, **77**, 1683 [(1979): *Sov. Phys.-JETP*, **50**, 844].
10. Tomita, K. Azuma, T., and Nariai, H. (1978). *Progr. Theor. Phys.*, **60**, 403.
11. Rusmaikin, T. V., and Rusmaikin, A. A. (1969). *Zh. Eksp. Teor. Fiz.* **57**, 680 [(1970): *Sov. Phys.-JETP*, **30**, 372]; Rusmaikin, A. A. (1977). *Astrofizika*, **13**, 345.
12. Starobinsky, A. A. (1980). *Phys. Lett. B*, **91**, 99.
13. Starobinsky, A. A. (1982). In *Quantum Gravity (Proceedings)* M. A. Markov, ed. (Inst. of Nucl. Res., Moscow), p. 58.
14. Macrae, K. I., and Riegert, R. J. (1981). *Phys. Rev. D*, **24**, 2555.
15. Frenkel, A., and Brecher, K. (1982). *Phys. Rev. D*, **26**, 368.
16. Kerner, R. (1982). *Gen. Rel. Grav.*, **14**, 453.
17. Swanson, C. A. (1968). *Comparison and Oscillation Theory of Linear Differential Equations* (Academic Press, New York).