General Relativity and Gravitation, Vol. 16, No. 3, 1984

Spatially Homogeneous Cosmological Models

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Received December 6, 1982

Abstract

We study nonviscous and viscous fluids in Bianchi types II, VIII, and IX space-times under the restriction that the ratio of shear to expansion be constant.

§(1): *Introduction*

The roles played by the viscosity and the consequent dissipative mechanism in cosmology have been discussed by some authors [1–4]. The heat represented by the large entropy per baryon in the microwave background provides a useful clue to the early universe, and a possible explanation for this huge entropy per baryon is that it was generated by physical dissipative processes acting at the beginning of evolution. These dissipative processes may indeed be responsible for the smoothing out of initial anisotropies [5]. Misner [1] suggested that neutrino viscosity acting in the early era might have considerably reduced the present anisotropy of the blackbody radiation during the process of evolution. While Belinsky and Khalatnikov [4] presented some general characteristics of anisotropic cosmological models in the presence of viscosity and Murphy [3] attempted to construct a homogeneous isotropic model introducing the second viscosity coefficient in the energy-momentum tensor of the fluid content, we have investigated the cosmological solutions in Einstein's theory for Bianchi type II, VIII, and IX space-times, which are spatially homogeneous but anisotropic in motion. We consider the matter content to be either a perfect fluid without viscosity or an imperfect one associated with first and second viscosity coefficients. There are some other perfect fluid solutions in the literature [6-8], while no exact solutions for these space-times with viscous fluid source are known, nor have their properties being studied.

In this work there are attempts to find solutions under certain geometric restrictions such as $\sigma^2/\Theta^2 = D^2 = \text{const}$ [6] and also some simple relationships between the viscosity coefficients and the energy density. Small values of D will, however, make shear prominant only when the expansion or contraction is very large. Subject to the above restrictions we obtain a nonlinear differential equation in terms of one metric coefficient. Once it is solved the others can be determined without difficulty. The simplifying assumptions made in the following sections on the geometry of space-time and also in the viscosity coefficients are not always quite realistic, although they are useful in the construction of some approximate models and their study.

§(2): *The Field Equations*

The line elements for homogeneous anisotropic Bianchi types II ($\delta = 0$), VIII ($\delta = -1$), and IX ($\delta = +1$) in a locally rotating system (2) are given by

$$ds^2 = -dt^2 + S^2 dx^2 + R^2[dy^2 + f^2(y)\,dz^2] - S^2 h(y)[2dx - h(y)\,dz]\,dz \quad (1)$$

where $R = R(t)$, $S = S(t)$, and

$$f(y) = \begin{bmatrix} \sin y \\ y \\ \sinh y \end{bmatrix}, \quad h(y) = \begin{bmatrix} \cos y \\ -\frac{1}{2}\,y \\ -\cosh y \end{bmatrix} \quad \text{for } \delta = \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}$$

We build cosmological models in those space-times with a viscous fluid having the energy-momentum tensor [5] given by

$$\begin{aligned} T_{ij} &= (\rho + \bar{p})v_i v_j + p g_{ij} - \eta_s u_{ij} \\ \bar{p} &= p - (\eta_b - \tfrac{2}{3}\,\eta_s)v^a{}_{;a} \\ v_i v^i &= -1 \\ u_{ij} &= v_{i;j} + v_{j;i} + v_i v^a v_{j;a} + v_j v^a v_{i;a} \end{aligned} \quad (2)$$

where ρ is the matter density, p the pressure, v^i the four-velocity, and η_b and η_s are the bulk and shear viscosity coefficients. In homogeneous cosmological models these quantities are only time dependent. Choosing a comoving coordinate frame where $v^i = \delta^i_0$, the nonvanishing components of Einstein's field equations,

$$G_{ij} = R_{ij} - \tfrac{1}{2}\,g_{ij}R = -kT_{ij} \quad (3)$$

with (1) and (2) are

$$2\frac{\dot{R}}{R}\frac{\dot{S}}{S}+\frac{\dot{R}^2+\delta}{R^2}-\frac{1}{4}\frac{S^2}{R^4}=k\rho \tag{4}$$

$$\frac{\ddot{S}}{S}+\frac{\ddot{R}}{R}+\frac{\dot{S}}{S}\frac{\dot{R}}{R}+\frac{1}{4}\frac{S^2}{R^4}=-k\left(\bar{p}-2\eta_s\frac{\dot{R}}{R}\right) \tag{5}$$

$$2\frac{\ddot{R}}{R}+\frac{\dot{R}^2+\delta}{R^2}-\frac{3}{4}\frac{S^2}{R^4}=-k\left(\bar{p}-2\eta_s\frac{\dot{S}}{S}\right) \tag{6}$$

where the dot stands for differentiation with respect to time.

We have a system of three independent equations (4)-(6) and six unknown functions, namely, R, S, ρ, p, η_b, and η_s. Hence, we can assume three appropriate relations between these variables in order to obtain solutions of the system of equations, which we do so by considering

$$\eta_b=\alpha_b\rho, \qquad \eta_s=\alpha_s\rho \tag{7}$$

$$\frac{\sigma^2}{\Theta^2}\doteq D^2 \tag{8}$$

where α_b, α_s, and D are constants. σ^2 and Θ^2 indicate shear and expansion, respectively. One of the above relations, (7), was used by Murphy [3] in some ideal situations of the fluid content. The asymptotic forms of the viscosity coefficients for small and large values of the energy density for some simple cases were discussed by Belinsky and Khalatnikov. These could be expressed approximately in terms of power functions of the energy density such as $\eta_b=\alpha_b\rho^m$ and $\eta_s=\alpha_s\rho^n$. For small values of ρ, m and η are large, and in the extreme case are equal to 1, whereas for large values of ρ the exponents are small. For simplification, following Murphy we consider $m = 1$ and also $n = 1$. The dependence of the viscosity coefficients on the energy density would be more complicated in reality and our task of obtaining solutions would be more difficult. The shear σ and the expansion Θ are given explicitly by

$$\sigma^2=\sigma_{ij}\sigma^{ij} \tag{9}$$

where, $\sigma_{ij}=v_{(i;j)}+\frac{1}{2}(v_{i;a}v^a v_j+v_{j;a}v^a v_i)-\frac{1}{3}\Theta(g_{ij}+v_i v_j)$

$$\Theta=v^i_{;i} \tag{10}$$

which become with metric (1)

$$\sigma^2=\left(\frac{\dot{S}}{S}-\frac{1}{3}\Theta\right)^2+2\left(\frac{\dot{R}}{R}-\frac{1}{3}\Theta\right)^2 \tag{11}$$

$$\Theta=2\frac{\dot{R}}{R}+\frac{\dot{S}}{S} \tag{12}$$

Substituting relations (11) and (12) into (8) we obtain

$$\frac{\dot{S}}{S} = \lambda \frac{\dot{R}}{R} \tag{13}$$

$$\lambda = \frac{1}{2/3 - D^2}\left[2\left(\frac{1}{3} + D^2\right) \pm \sqrt{6}\, D\right] \tag{14}$$

Integrating equation (13) we obtain $S = CR^{\lambda}$ and absorbing the integration constant C into S or R we can write

$$S = R^{\lambda} \tag{15}$$

Now subtracting equation (6) from (5) and considering equations (4), (7), and (15) we obtain

$$\frac{\ddot{R}}{R} + a_1\left(\frac{\dot{R}}{R}\right)^3 + a_2\left(\frac{\dot{R}}{R}\right)^2 + \left(a_3 + a_4\frac{\dot{R}}{R}\right)R^{2\lambda-4} + \left(a_5 + a_6\frac{\dot{R}}{R}\right)R^{-2} = 0 \tag{16}$$

where

$$\begin{aligned} a_1 &= 2\alpha_s(2\lambda + 1), & a_4 &= -\frac{\alpha_s}{2} \\ a_2 &= \lambda + 1, & a_5 &= -\frac{\delta}{\lambda - 1} \\ a_3 &= \frac{1}{\lambda - 1}, & a_6 &= 2\alpha_s\delta \end{aligned} \tag{17}$$

The expansion given by (12) can be written with (15) as

$$\Theta = (\lambda + 2)\frac{\dot{R}}{R} \tag{18}$$

which transforms (16) into

$$\dot{\Theta} + b_1\Theta^3 + \Theta^2 + (b_2 + b_3\Theta)R^{2\lambda-4} + (b_4 + b_5\Theta)R^{-2} = 0 \tag{19}$$

where

$$\begin{aligned} b_1 &= \frac{2\alpha_s(2\lambda + 1)}{(\lambda + 2)^2}, & b_4 &= -\frac{\lambda + 2}{\lambda - 1}\delta \\ b_2 &= \frac{\lambda + 2}{\lambda - 1}, & b_5 &= 2\alpha_s\delta \\ b_3 &= -\frac{1}{2}\alpha_s \end{aligned} \tag{20}$$

Equations (4) and (5) or (6) with (15) and (16) allow us to write

$$k\rho = (2\lambda + 1)\left(\frac{\dot{R}}{R}\right)^2 + \frac{\delta}{R^2} - \frac{1}{4}R^{2\lambda-4} \tag{21}$$

$$k\bar{p} = 2\alpha_s(2\lambda + 1)(\lambda + 2)\left(\frac{\dot{R}}{R}\right)^3 + (2\lambda + 1)\left(\frac{\dot{R}}{R}\right)^2 - \left(\frac{\lambda + 1}{\lambda - 1}\right)\frac{\delta}{R^2}$$

$$+ 2\alpha_s(\lambda + 2)\left(\frac{\dot{R}}{R}\right)\frac{\delta}{R^2} - \frac{\alpha_s}{2}(\lambda + 2)\left(\frac{\dot{R}}{R}\right)R^{2\lambda-4} + \left(\frac{\lambda + 1}{\lambda - 1} - \frac{1}{4}\right)R^{2\lambda-4} \tag{22}$$

The equation (16) above is a highly nonlinear differential equation involving only R as a function of time. Once it is solved for R one can easily determine S from (15) and the metric is known.

§(3): *Nonviscous and Viscous Cosmological Models*

Raychaudhuri's equation (9) is given by

$$\Theta_{;i}v^i = a^i_{\ ;i} + 2\omega^2 - 2\sigma^2 - \tfrac{1}{3}\Theta^2 + R_{ij}v^iv^j$$

where a^i is the acceleration vector, which vanishes for geodesic motion, ω^2 is the rotation, σ^2, the shear, and Θ, the expansion. So in a comoving coordinate system ($v^i = \delta^i_0$) this equation reduces to

$$\dot{\Theta} = -2\sigma^2 - \tfrac{1}{3}\Theta^2 + R_{ij}\,v^iv^j$$

The Hawking-Penrose energy condition (10) is that $R_{ij}v^iv^j \leqslant 0$. Hence in such a case $\dot{\Theta} < 0$ and there can be only a maximum and no minimum for the expansion and from (2) we obtain that

$$R_{ij}v^iv^j = -k\left(\frac{\rho + 3\bar{p}}{2} - \eta_s\Theta\right) = -k\left(\frac{\rho + 3p}{2} - \frac{3}{2}\eta_b\Theta\right)$$

Now we make some general observations about the properties of the expressions obtained so far.

(i) When $\Theta = 0$, we have from (21)

$$k\rho = \frac{\delta}{R^2} - \frac{1}{4}R^{2\lambda-4} \tag{23}$$

So for $\delta = 0$ and $\delta = -1$, i.e., for Bianchi types II and VIII models, $\rho < 0$ at turning points. Hence the models which have turning points matter density are not positive throughout their life cycle.

(ii) In (19) and (22) it is clear that the terms associated with shear viscosity contain odd powers of $(\dot{R}/R)$, so that when the sign of time-rate changes the relevant terms changes their signatures. It means that expanding and contracting models do have different time behaviors due to the presence of shear viscosity. A bulk viscosity term does not have such influence and is contained only within $\bar{p}$.

(iii) The matter density (21) does not explicitly depend on the viscosity term except implicitly through the metric, whereas the pressure term (22) is modified explicitly due to viscosity.

Bianchi Type II Nonviscous Case. In Bianchi type II ($\delta = 0$) for nonviscous fluid ($\alpha_s = \alpha_b = 0$) equation (16) reduces to

$$\ddot{R}R + a_2\dot{R}^2 + a_3R^{2\lambda-2} = 0 \tag{24}$$

which after the first integration reduces to

$$\dot{R} = \pm R^{-(\lambda+1)}\left[\frac{1}{2\lambda(1-\lambda)}R^{4\lambda} + C_1\right]^{1/2} \qquad \text{for} \quad \lambda \neq 0 \tag{25}$$

where C_1 is an integration constant. In view of (25) we obtain the matter density (21) and the pressure (22) given by

$$k\rho = \left[\frac{2\lambda+1}{2\lambda(1-\lambda)} - \frac{1}{4}\right]R^{2\lambda-4} + C_1(2\lambda+1)R^{-2\lambda-4} \tag{26}$$

$$k\bar{p} = \left[\frac{2\lambda+1}{2\lambda(1-\lambda)} - \frac{1}{4} + \frac{\lambda+1}{\lambda-1}\right]R^{2\lambda-4} + C_1(2\lambda+1)R^{-2\lambda-4} \tag{27}$$

There is no turning point ($\dot{R} \neq 0$) only in the case $0 < \lambda < 1$ and $C_1 > 0$. In this case as $R \to 0, \dot{R} \to \pm R^{-(\lambda+1)}\, C_1$ so that $|\dot{R}| \to \infty$ and ρ and p both increase to infinitely large values.

Now from (26) and (27) we have

$$k(\rho + 3\bar{p}) = \frac{2(\lambda+1)}{\lambda}R^{2\lambda-4} + 4(2\lambda+1)C_1R^{-(2\lambda+4)} \tag{28}$$

Throughout the history from R very small to R very large $(\rho + 3\bar{p}) \geqslant 0$. Hawking-Penrose energy condition is not violated during the evolution of the model from zero proper volume to infinitely large proper volume. Since $(2\lambda+1)/[2\lambda(1-\lambda)] > 1/4$ for $0 < \lambda < 1$, we get ρ decreasing from infinitely large to vanishingly small values as the model expands.

Cases different from $0 < \lambda < 1$ and $C_1 > 0$ lead to unphysical situations.

Bianchi Type II Viscous Case. Owing to the difficulty in obtaining the general solution of the differential equation (16) for the Bianchi type II ($\delta = 0$) viscous model $\alpha_s \neq 0$, $\alpha_b \neq 0$, we restrict ourselves to make the analysis mainly of the particular case $\lambda = 2$. For this case (19) becomes

$$\dot{\Theta} = \frac{5\alpha_s}{8}\Theta^3 - \Theta^2 - \frac{\alpha_s}{2}\Theta - 4 \tag{29}$$

and the expansion rate is given by $\Theta = 4\dot{R}/R$. From (21) and (22) for $\delta = 0$ and $\lambda = 2$, the matter density and the pressure become

$$k\rho = 5\left(\frac{\dot{R}}{R}\right)^2 - \frac{1}{4}R^{2\lambda-4} \tag{30}$$

$$k\bar{p} = 40\alpha_s\left(\frac{\dot{R}}{R}\right)^3 + 5\left(\frac{\dot{R}}{R}\right)^2 - 2\alpha_s\left(\frac{\dot{R}}{R}\right) + \frac{11}{4} \tag{31}$$

Writing (30) and (31) in terms of expansion scalars we obtain

$$k\rho = \tfrac{1}{4}\left(\tfrac{5}{4}\Theta^2 - 1\right) \tag{32}$$

and

$$k\bar{p} = \tfrac{5}{8}\alpha_s\Theta^3 + \tfrac{5}{16}\Theta^2 - (\alpha_s/2)\Theta + \tfrac{11}{4} \tag{33}$$

From (29) we see that $\Theta = 0$ corresponds to $\dot{\Theta} = -4$, which indicates that there is a maximum for R, there being no minimum when $\Theta < 0$ and $|\Theta| = 2/\sqrt{5}$, $\rho = 0$ and $\dot{\Theta} = -(\Theta^2 + 4)$, which is less than zero. Θ decreases subsequently and becomes more and more negative until $|\Theta| \to \infty$ so that the density increases monotonically to an infinitely large value. In the reverse picture we do not get the model exploding from the infinitely large value for ρ. This is mainly because $\rho \to \infty$ corresponds to $\Theta \to \infty$, which in turn yields $\dot{\Theta} \sim \Theta^3$, so that $\dot{\Theta} > 0$ and Θ increases further.

The Hawking-Penrose energy conditions (9) demand $R_{ij}v^iv^j \leqslant 0$, which again as has been shown earlier, yields the condition

$$-R_{ij}v^iv^j = k\left(\frac{\rho + 3\bar{p}}{2} - \eta_s\Theta\right) \geqslant 0$$

This again putting expressions for ρ and $\bar{p}$ reduces to the condition

$$\tfrac{1}{2}\alpha_s\Theta\left(\tfrac{5}{4}\Theta^2 - 1\right) + \left(\tfrac{5}{8}\Theta^2 + 4\right) = \dot{\Theta} + \tfrac{13}{8}\Theta^2 + 8 \geqslant 0$$

Therefore at $\Theta = 0$ or at $\Theta^2 = 4/5$ the energy condition is satisfied and so long as $\rho > 0$ that is $\Theta^2 > 4/5$ it is satisfied at every stage of expansion $\Theta > 0$. On the other hand for a contracting model $\Theta < 0$ the energy condition is satisfied only when $-(\alpha_s/2)\,|\Theta|\left(\tfrac{5}{4}\Theta^2 - 1\right) + \left(\tfrac{5}{8}\Theta^2 + 4\right) \geqslant 0$ and it is violated when $|\Theta|$ becomes large.

We consider now the situation where the model expands in course of time that is $\Theta > 0$: When $\Theta = 2/\sqrt{5}$ the energy density vanishes and $\dot{\Theta} < 0$. But for $\Theta \to \infty$ we have $\dot{\Theta} > 0$, since $\dot{\Theta} \sim \Theta^3$ as discussed previously. In between these two instants there must be an instant when $\dot{\Theta} = 0$. From (29) this corresponds to the vanishing of higher time derivatives also, so that one has a steady state situation (cf. Murphy [3]) with $\tfrac{5}{4}\Theta^2 > 1$ that is, with energy density positive. Later if Θ decreases the expansion rate monotonically approaches zero with the density vanishing at $\Theta = 2/\sqrt{5}$.

§(4): *Conclusions*

We can now summarize the previous results obtained for Bianchi type II universes with perfect or viscous fluids.

For perfect fluid distributions there are situations where the model explodes from the initial singularity of infinitely large energy density and zero proper volume and monotonically expands towards vanishingly small density. For an expanding model the Hawking-Penrose energy condition is not violated at any stage.

The behavior is in general difficult to study for a viscous fluid. In a simple case $\lambda = 2$ the model either collapses or expands monotonically, there being no oscillation possible. For a contracting model with positive energy density ($\rho > 0$) the rate of contraction increases in the course of time and the energy density increases indefinitely. The energy condition of Hawking-Penrose is not satisfied at all stages of collapse. The behavior is different for an expanding model due to the presence of viscosity. There is a steady state for a finite magnitude of energy density depending on the magnitude of the shear viscosity coefficient, which when perturbed may expand monotonically as a consequence towards lower energy density states.

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