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A Class of Wormhole Solutions to Higher-Dimensional General Relativity

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Abstract

An ansatz is given which reduces the equations of sourceless $(n + p)$-dimensional general relativity to those of n-dimensional general relativity coupled to a repulsive $O(p)$ scalar field. Regular solutions are obtained for $(n = 2, p = 3)$, $(n = 3, p = 2)$, and $(n = 3, p = 4)$. All these solutions have the wormhole topology.

§(1): *Introduction*

Models of four-dimensional general relativity with classical fields as sources do not in general admit regular asymptotically flat solutions. Some exceptional models [1-3] do admit such solutions, which share a spatial geometry with two symmetrical, asymptotically flat regions. The possibility of a consistent particle interpretation of such solutions has been previously discussed [3, 4].

These models also share a common aesthetic defect: they admit regular solutions only because scalar fields (repulsively coupled to gravity) are put there to that effect, just as Higgs fields are added to gauge theories in order to break some symmetry. It would therefore be gratifying to find "natural" field-theoretical models which admit regular solutions.

Such a solution has indeed been found in sourceless five-dimensional general relativity [5]. It is our purpose in the present paper to search for regular asymptotically flat solutions to $(n + p)$-dimensional general relativity, the extra $(p - 1)$ dimensions being subsequently compactified so as to lead to a possible physical theory for the number of space dimensions $n = 3$.

In the second section of this paper we propose an ansatz which reduces the

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equations of sourceless $(n + p)$-dimensional general relativity to those of n-dimensional general relativity with a repulsive $O(p)$ scalar field as source. Regular solutions of these equations are constructed for $(n = 2, p = 3)$, $(n = 3, p = 2)$, and $(n = 3, p = 4)$ in the third section; they are all of the wormhole type. Our conclusions are stated in the last section.

§(2): *The Ansatz*

We look for solutions to the Einstein equations in n (space) + p (time and inner) dimensions, which are invariant under translations in the p dimensions, and we choose coordinates x^i $(i = 1, \ldots, n)$ and x^a $(a = n + 1, \ldots, n + p)$ such that

$$ds^2 = g_{ab}(x^k)\, dx^a\, dx^b + g_{ij}(x^k)\, dx^i\, dx^j \tag{1}$$

In this case the elements of the Ricci tensor are given by

$$\begin{aligned} R_{ab} &= -\tfrac{1}{2}\nabla_i\nabla^i g_{ab} + \tfrac{1}{2}g^{cd}(\nabla_i g_{ac}\nabla^i g_{bd} - \tfrac{1}{2}\nabla_i g_{ab}\nabla^i g_{cd}) \\ R_{ia} &= 0 \\ R_{ij} &= \bar{R}_{ij} - \tfrac{1}{2}\nabla_i(g^{ab}\nabla_j g_{ab}) + \tfrac{1}{4}\nabla_i g^{ab}\nabla_j g_{ab} \end{aligned} \tag{2}$$

where ∇_i is the spatial covariant derivative, and $\bar{R}_{ij}$ is the purely spatial Ricci tensor. Introducing for the cyclic coordinates x^a the matrix notation

$$G = \{g_{ab}\}, \qquad P = \{R_{ab}\} \tag{3}$$

the nonzero elements of the Ricci tensor may be rewritten as

$$\begin{aligned} P &= -\tfrac{1}{2}\nabla_i\nabla^i G + \tfrac{1}{2}\nabla_i G \cdot G^{-1} \cdot \nabla^i G - \tfrac{1}{4}\nabla_i G\, \mathrm{Tr}(G^{-1} \cdot \nabla^i G) \\ R_{ij} &= \bar{R}_{ij} - \tfrac{1}{2}\nabla_j\, \mathrm{Tr}(G^{-1} \cdot \nabla_i G) + \tfrac{1}{4}\, \mathrm{Tr}(\nabla_i G^{-1} \cdot \nabla_j G) \end{aligned} \tag{4}$$

Now we make the ansatz

$$G^2 = 1 \tag{5}$$

which reduces equations (4) to

$$\begin{aligned} P &= -\tfrac{1}{2}G\nabla_i(G \cdot \nabla^i G) \\ R_{ij} &= \bar{R}_{ij} + \tfrac{1}{4}\, \mathrm{Tr}(\nabla_i G \cdot \nabla_j G) \end{aligned} \tag{6}$$

enabling us to write the action for $(n + p)$-dimensional general relativity as

$$\begin{aligned} S &= -\frac{1}{2\kappa}\int d^{n+p}x|g|^{1/2}g^{\mu\nu}R_{\mu\nu} \\ &= -\frac{1}{2\kappa}\int d^p x \int d^n x|\bar{g}|^{1/2}g^{ij}\left[\bar{R}_{ij} + \frac{1}{4}\, \mathrm{Tr}(\nabla_i G \cdot \nabla_j G)\right] \end{aligned} \tag{7}$$

[$\bar{g} = \det(g_{ij})$]. Thus, the solution of vacuum Einstein equations in $(n+p)$ dimensions has been reduced to that of the Einstein equations in n dimensions with a matrix source field G subject to the constraint (5). Note that this source field enters the action with the "wrong" sign (the corresponding energy density is negative); previous experience [1-3] therefore suggests that regular solutions to (7) will be of the wormhole type.

Let us now specialize to a $(n+p)$-dimensional manifold with the signature $(+ - \cdots -)$. It then follows from (5) that the diagonal form of G is

$$G_0 = \begin{pmatrix} +1 & & & & 0 \\ & -1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & -1 \end{pmatrix} \tag{8}$$

The general matrix $G(x^k)$ solution of (5) with the prescribed signature is obtained from (8) by an arbitrary rotation $R(x^k)$ with the result

$$g_{ab} = 2\phi_a\phi_b - \delta_{ab} \tag{9}$$

where $\boldsymbol{\phi}$ varies on the $(p-1)$ sphere:

$$\boldsymbol{\phi}^2 = 1 \tag{10}$$

[A metric similar to (9) has been investigated, in a different context, by Williams and Zia [7].] The action (7) then reduces to that of an $O(p)$ field $\boldsymbol{\phi}$ coupled repulsively to n-dimensional gravity:

$$S = -\frac{1}{2\kappa}\int d^px \int d^nx |\bar{g}|^{1/2} g^{ij}(\bar{R}_{ij} + 2\nabla_i\boldsymbol{\phi}\cdot\nabla_j\boldsymbol{\phi}) \tag{11}$$

The problem remains to solve the corresponding field equations with boundary conditions at spatial infinity which we now state: (B1) the spatial metric $g_{ij}(x^k)$ is asymptotically Euclidean with a negative signature; (B2) the matrix G is asymptotic to its diagonal form (8), which means that $\boldsymbol{\phi}(x^k)$ is asymptotic to the constant vector $\boldsymbol{\phi}_0 = (1, 0, \ldots, 0)$.

The ansatz (1) and (9), together with the boundary condition (B2), has broken down the original G_{n+p} invariance (invariance under general coordinate transformations in $(n+p)$ dimensions) of the theory to $G_n \times SL_1 \times SL_{p-1}$. To make contact with the physical world, one should further break down the SL_{p-1} symmetry of the solutions (if not already broken) by compactifying, *à la* Kaluza-Klein, each of the extra $(p-1)$ dimensions.

§(3): *Regular Solutions*

(a) $n = 2, p = 3$. For $n = 2$, the model defined by

$$\bar{R}_{ij} = -2\partial_i\boldsymbol{\phi}\cdot\partial_j\boldsymbol{\phi} \qquad (\boldsymbol{\phi}^2 = 1) \tag{12}$$

has no nontrivial solutions for $p = 2$. Therefore we take $p = 3$, which gives the two-dimensional $O(3)$ nonlinear σ model coupled repulsively to gravity. All the regular solutions have already been constructed in the case of an attractive coupling [8], and we repeat the construction in the present case.

We may always choose a system of isotropic coordinates x^i such that

$$g_{ij} = -e^{2u}\delta_{ij} \tag{13}$$

With those coordinates, equations (12) give

$$\bar{R}_{ij} = -\Delta u \delta_{ij} = -2\partial_i \boldsymbol{\phi} \cdot \partial_j \boldsymbol{\phi} \tag{14}$$

(Δ being the ordinary Laplacian). Going over to complex fields ψ and complex coordinates z defined by

$$\phi_4 + i\phi_5 = \frac{2\psi}{1 + |\psi|^2}, \qquad x^1 + ix^2 = z \tag{15}$$

equations (14) may be rewritten as

$$\begin{gathered} 0 = \frac{\partial \psi^*}{\partial z} \cdot \frac{\partial \psi}{\partial z} \\ \frac{\partial^2 u}{\partial z \partial z^*} = 2F\left[\left|\frac{\partial \psi}{\partial z}\right|^2 + \left|\frac{\partial \psi}{\partial z^*}\right|^2\right] \end{gathered} \tag{16}$$

with

$$F(\psi) = \frac{1}{(1 + |\psi|^2)^2} \tag{17}$$

The first of equations (16) means that ψ is either an analytic function of z or of z^*:

$$\psi = \psi(z) \qquad \text{or} \qquad \psi = \psi(z^*) \tag{18}$$

In either case the second of equations (16) reduces to

$$\frac{\partial^2 u}{\partial \psi \partial \psi^*} = 2F(\psi) \tag{19}$$

which has the general solution

$$e^{2u} = \frac{(1 + |\psi|^2)^4}{|f(\psi)|^2} \tag{20}$$

where f is an analytic function of ψ.

We now enforce the boundary conditions. The boundary condition (B2) is satisfied by $\psi(\infty) = \infty$ [the other possibility $\psi(\infty) = 0$ is equivalent to this, modulo the transformation $\psi \to 1/\psi^*$, which leaves (15) invariant], which means that ψ is a polynomial

$$\psi = P_k(z) \qquad \text{or} \qquad \psi = P_k(z^*) \tag{21}$$

The boundary condition (B1) then implies that f cannot be constant, and so has at least one zero ψ_0. The antecedents z_0 of $\psi_0 = \psi(z_0)$ are necessarily end points (points at infinity) of the spatial sections x^a = const., because the differential proper distance to z_0 varies as

$$dl = e^u d|z - z_0| \sim \text{const.}\, \frac{d|z - z_0|}{|z - z_0|^q} \tag{22}$$

where q is the order of the zero z_0.

So the spatial sections of our solution have at least two end points ($z = \infty$, and $z = z_0$), and thus have the wormhole topology. The boundary conditions (B1) and (B2) at $z = \infty$ should then be supplemented by boundary conditions at the other end points. An interesting condition is that our solution remain invariant under rotations in the (x^4, x^5) subspace, i.e., phase transformations $\psi \to \psi e^{i\alpha}$, which is equivalent to the boundary condition (B3);

$$\psi_0 = 0 \tag{23}$$

Conditions (B1) and (B3) give $f(\psi) = \psi^4$, so that

$$e^{2u} = \left(1 + \frac{1}{|\psi|^2}\right)^4 \tag{24}$$

(One could, instead of (B1), enforce symmetrical spatial boundary conditions for the two end-points $\psi = \infty$ and $\psi = 0$; this possibility is explored elsewhere [9].)

The solution given by (21) and (24) has k spatial end-points other than $z = \infty$, and so is a k wormhole. The simplest solution is the 1 wormhole

$$\psi = \lambda z \qquad \text{or} \qquad \psi = \lambda z^* \tag{25}$$

which is spherically symmetric (other spherically symmetric solutions are given by $\psi = \lambda z^k$ or λz^{*k}).

(b) $n = 3, p = 2$. For $n = 3$, our ansatz (7) is just a special case of the more general Dobiasch-Maison construction [6]. In the case $p = 2$, $\boldsymbol{\phi}$ varies on the circle, so that we may choose the parametrization

$$\phi_4 = \sin\eta, \qquad \phi_5 = \cos\eta \tag{26}$$

which leads to the equations of three-dimensional general relativity coupled repulsively to a massless scalar field:

$$\bar{R}_{ij} = -2\partial_i\eta\partial_j\eta \tag{27}$$

The spherically symmetric asymptotically flat solution with the boundary condition $\eta(\infty) = \pi/2$ [10, 11], which may be parametrized as

$$-g_{ij}\,dx^i\,dx^j = \frac{\rho_0^2}{\cos^4\eta}\,[d\eta^2 + \cos^2\eta\,(d\theta^2 + \sin^2\theta\,d\varphi^2)]$$
$$G = \begin{pmatrix} -\cos 2\eta & \sin 2\eta \\ \sin 2\eta & \cos 2\eta \end{pmatrix} \tag{28}$$

is found to correspond to a special case of the previously known wormhole solution of five-dimensional general relativity [5]. This special case (in the notation of [5], $a_1 = a_2 = \frac{1}{2}$, $\lambda = 0$, $B^2 = -\rho_0^2/4$) is singled out by the fact that it is invariant under the symmetry $\eta \to -\eta$, $x^5 \to -x^5$, which exchanges the two end points $\eta = \pm\pi/2$.

(c) $n = 3, p = 4$. The reduced model is that of an $O(4)$ ("chiral") field coupled repulsively to three-dimensional general relativity. We search for a spherically symmetric (both in physical space and inner space) solution, and so choose the parametrization:

$$g_{ij}(\mathbf{x}) = -e^{2u(r)}\delta_{ij}$$
$$\phi_4(\mathbf{x}) = \sin\eta(r), \qquad \phi_{4+i}(\mathbf{x}) = \cos\eta(r)\frac{x^i}{r} \tag{29}$$

(the possibility of such a spherically symmetric "hedgehog" solution has been overlooked in [6]). The reduced Einstein equations are then

$$\bar{R}_i^j = 2e^{-2u}\left[r^{-2}\cos^2\eta(\delta_{ij} - Q_{ij}) + \left(\frac{d\eta}{dr}\right)^2 Q_{ij}\right] \tag{30}$$

where

$$Q_{ij} \equiv \frac{x^i x^j}{r^2} \tag{31}$$

Going over to standard spherical coordinates, such that

$$-g_{ij}\,dx^i\,dx^j = \frac{d\rho^2}{y^2(\rho)} + \rho^2(d\theta^2 + \sin^2\theta\,d\varphi^2) \tag{32}$$

and using the relation between Cartesian and spherical components of the Ricci tensor

$$\bar{R}_i^j = \bar{R}_\rho^\rho Q_{ij} + \bar{R}_\theta^\theta(\delta_{ij} - Q_{ij}) \tag{33}$$

we rewrite equations (30) as

$$\bar{R}_\rho^\rho \equiv \frac{2y}{\rho}\frac{dy}{d\rho} = 2y^2\left(\frac{d\eta}{d\rho}\right)^2$$
$$\bar{R}_\theta^\theta \equiv \frac{y}{\rho}\frac{dy}{d\rho} - \frac{1}{\rho^2}(1 - y^2) = \frac{2}{\rho^2}\cos^2\eta \tag{34}$$

with the boundary conditions

$$y(\infty) = 1, \qquad \eta(\infty) = \frac{\pi}{2} \tag{35}$$

The system (34), (35) is equivalent to the following system:

$$\begin{aligned} \left(\frac{dy}{d\eta}\right)^2 + y^2 &= 1 + 2\cos^2\eta \qquad \left(y\left(\frac{\pi}{2}\right) = 1\right) \\ \rho\,\frac{d\eta}{d\rho} &= \frac{1}{y}\,\frac{dy}{d\eta} \qquad \left(\rho\left(\frac{\pi}{2}\right) = \infty\right) \end{aligned} \tag{36}$$

These equations are invariant under the symmetry $\eta \to -\eta$, so that we may choose

$$\frac{dy}{d\eta} > 0 \tag{37}$$

for $\eta \to \pi/2$. It follows from the second of equations (36) that η is an increasing function of ρ. Then, if η_1 is the highest zero of $y'(\eta)$,

$$y(\eta) < 1 \tag{38}$$

for $\eta_1 < \eta < \pi/2$. The first of equations (36) then shows that, as long as $y(\eta) > 0$, $y'(\eta) > 0$, so that $\eta_1 < \eta_0$, where η_0 is the highest zero of $y(\eta)$.

Consequently $y^2 < 1$ for $\eta > \eta_0$, so that we may replace the first of equations (36) by the inequality

$$\frac{dy}{d\eta} > \sqrt{2}\cos\eta \qquad \left(y\left(\frac{\pi}{2}\right) = 1\right) \tag{39}$$

(assuming $\eta_0 > -\pi/2$), which may be integrated to give

$$y < \sqrt{2}\left[\sin\eta - \left(1 - \frac{1}{\sqrt{2}}\right)\right] \tag{40}$$

It follows that $y(\eta)$ has a zero for

$$\eta_0 > \arcsin\left(1 - \frac{1}{\sqrt{2}}\right) \simeq 0.297 \tag{41}$$

The behavior of the metric functions in the vicinity of η_0 is found from equations (36) to be

$$\begin{aligned} y &\sim \beta(\eta - \eta_0) \qquad (\beta = (1 + 2\cos^2\eta_0)^{1/2}) \\ \rho &\sim \rho_0\left[1 + \frac{(\eta - \eta_0)^2}{2}\right] \end{aligned} \tag{42}$$

so that our solution has a minimum radius ρ_0 (which is of course an arbitrary constant). When η continues to decrease past η_0, ρ increases to infinity again (a maximum of ρ would necessitate, from the second of equations (36), another zero of y; but this can only happen if y first goes through a minimum $y(\eta_1)$ which, if ρ is an analytic function of η, can only occur for $\rho \to \infty$).

Thus, our solution to seven-dimensional general relativity again has the wormhole topology. As in case (a), the boundary conditions (B1) and (B2) are not fulfilled at the "other" end point η_1, because $\eta_1 \neq -\pi/2$ [if $\eta_1 = -\pi/2$, a reasoning similar to the above would lead to $\eta_0 < -0.297$, which contradicts (41)], and so

$$y(\eta_1) = (1 + 2\cos^2 \eta_1)^{1/2} > 1 \tag{43}$$

Again as in case (a), one could replace these boundary conditions by the symmetrical (but unphysical) condition $\eta_0 = 0$.

§(4): *Conclusion*

Our ansatz has enabled us to obtain in several instances regular solutions of $(n + p)$-dimensional general relativity which have a rich structure, in terms of symmetry as well as of topology. Besides being spatial wormholes, our solutions are also metrical kinks [7, 12, 13]. The $p \times p$ matrix G is everywhere regular and has signature $(+ - \cdots -)$. However, the ansatz (9) shows that g_{00} ($\equiv g_{n+1,n+1}$) is not positive definite, but goes from +1 at the North pole of the p sphere $\boldsymbol{\phi}^2 = 1$ to −1 at the equator and back to +1 at the South pole. The 1-wormhole solution of (b) is thus also a 1-kink, while the k-wormholes of (a) are also k-kinks [the wormhole solution of (c) is not strictly speaking a kink, because the end point η_1 corresponds not to the South pole, but to a parallel].

A drawback of our method is that the extra dimensions must be compactified toroidally so that the inner symmetry of the solutions is (in the case $p > 2$) lost. It would be interesting to generalize our ansatz to take into account the possibility of spontaneous (spherical) compactification.

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