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Axisymmetric Regular Multiwormhole Solutions in Five-Dimensional General Relativity

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Abstract

A class of regular, asymptotically flat solutions to the five-dimensional vacuum Einstein equations with a two-parameter Abelian isometry group is constructed, under the additional assumption of axial symmetry in three-dimensional space. The possibility of interpreting these multiwormhole solutions as multiparticle systems is discussed.

§(1): *Introduction*

Sourceless five-dimensional general relativity is known to admit a variety of spherically symmetric solutions with a two-parameter Abelian isometry group [1, 2]. Particularly interesting among these solutions are the regular wormhole solutions obtained by Chodos and Detweiler [2].

A simple ansatz which leads to such regular wormhole solutions to the equations of sourceless $(n + p)$-dimensional general relativity has been given in Reference 3. This ansatz, which reduces the vacuum Einstein equations in $(n + p)$ dimensions to the equations of n-dimensional general relativity with a repulsive $O(p)$ scalar field as source, has in particular been applied to construct multiwormhole solutions of five-dimensional general relativity in the case $n = 2, p = 3$, [3, 4].

In this paper we shall construct multiwormhole solutions in the physically more interesting case $n = 3, p = 2$, under the additional assumption of axial symmetry. The ansatz, stated in the second section, enables us to reduce the five-dimensional axisymmetric Einstein equations to the three-dimensional axisymmetric Laplace equation. The properties of the known spherically symmetric regular wormhole solution in Weyl coordinates are studied in Section 3.

From this solution, multiwormhole axisymmetric solutions are constructed in Section 4. The case of coincident wormholes is studied in Section 5. Finally, the possibility of interpreting our solutions as systems of elementary particles is discussed in the last section.

§(2): *Local Axially Symmetric Solutions*

We assume that $(n + p)$-dimensional space-time (n space dimensions, p time and inner dimensions) admits a p-parameter Abelian isometry group ($p \geqslant 2$), and that coordinates x^i ($i = 1, \cdots, n$) and x^a ($a = n + 1, \cdots, n + p$) can be found such that

$$ds^2 = g_{ab}(x^k)\, dx^a dx^b + g_{ij}(x^k)\, dx^i dx^j \tag{1}$$

with the signature $(+ - \cdots -)$.

Following Reference 3, we further assume that g_{ab} has the simple form

$$g_{ab} = 2\phi_a \phi_b - \delta_{ab} \tag{2}$$

where the field $\boldsymbol{\phi}$ is constrained to vary on the $(p - 1)$ sphere:

$$\boldsymbol{\phi}^2 = 1 \tag{3}$$

This ansatz (which for $n = 3$ is a special case of the more general construction given in Reference 1), reduces the $(n + p)$-dimensional vacuum Einstein equations to the system

$$\underset{3}{R}_{ij} = -2\nabla_i \boldsymbol{\phi} \cdot \nabla_j \boldsymbol{\phi} \tag{4}$$

and

$$\nabla_i \nabla^i \boldsymbol{\phi} + (\nabla_i \boldsymbol{\phi} \cdot \nabla^i \boldsymbol{\phi}) \boldsymbol{\phi} = 0 \tag{5}$$

(where ∇_i and $\underset{3}{R}_{ij}$ are the spatial covariant derivative and spatial Ricci tensor). Actually, this system may be redundant, as is often the case in general relativity, because of the Bianchi identities. Indeed, we prove in the Appendix the following

Theorem. If the number of linearly independent vectors $\nabla_i \boldsymbol{\phi}$ is $(p - 1)$ (which can only happen for $n \geqslant p - 1$), then equation (5) is a consequence of equations (4).

Specializing now to $n = 3$, we make the additional assumption of axial symmetry, in which case the spatial metric can be written in the form

$$d\sigma^2 \equiv -g_{ij} dx^i dx^j = e^{2k}(d\rho^2 + dz^2) + W^2\, d\varphi^2 \tag{6}$$

where $k(\rho, z)$ and $W(\rho, z)$ are the metric functions (we have, without loss of generality, chosen isotropic coordinates in the planes ρ, z). Assuming that the field $\boldsymbol{\phi}$ is also independent of the azimuthal angle φ (in the case $p \geqslant 3$, this assumption is probably too restrictive [3]), the Einstein equations (4) for the mixed components $\underset{3}{R}_{\rho\varphi}, \underset{3}{R}_{z\varphi}$ of the Ricci tensor are identically satisfied, while the remaining equations give

$$\underset{3}{R}_{\varphi\varphi} \equiv We^{-2k}\Delta_2 W = 0$$

$$\underset{3}{R}_{ij} \equiv \underset{2}{R}_{ij} - W^{-1}\nabla_i\nabla_j W = -2\nabla_i\boldsymbol{\phi}\cdot\nabla_j\boldsymbol{\phi} \tag{7}$$

where the indices i, j refer now to the two-space (ρ, z), $\underset{2}{R}_{ij}$ is the Ricci tensor of this two-space, and Δ_2 is the two-dimensional Euclidean Laplacian.

Defining the complex coordinates

$$\zeta = \rho + iz \tag{8}$$

we note that equation (6) and the first of equations (7) are form invariant under conformal transformations $\zeta \to f(\zeta)$.

We may thus choose our (ρ, z) coordinates such that

$$W = \rho \tag{9}$$

This choice reduces our metric (6) to the Weyl canonical form [5]. Then the Einstein equations (7) for the components $\underset{3}{R}_{ij}$ give

$$\partial_\rho k = -\rho[(\partial_\rho\boldsymbol{\phi})^2 - (\partial_z\boldsymbol{\phi})^2]$$

$$\partial_z k = -\rho 2\partial_\rho\boldsymbol{\phi}\cdot\partial_z\boldsymbol{\phi}$$

$$\Delta_2 k = (\partial_\rho\boldsymbol{\phi})^2 + (\partial_z\boldsymbol{\phi})^2 \tag{10}$$

while equation (5) reduces to the flat-space non-linear Laplace equation

$$\Delta_3\boldsymbol{\phi} + [(\partial_\rho\boldsymbol{\phi})^2 + (\partial_z\boldsymbol{\phi})^2]\boldsymbol{\phi} = 0 \tag{11}$$

where Δ_3 is the three-dimensional Euclidean Laplacian. It can be checked that the last of equations (10) is a consequence of the other two and of equation (11). Furthermore, the first two equations (10) may be combined into the complex equation

$$\partial_\zeta k = -2\rho(\partial_\zeta\boldsymbol{\phi})^2 \tag{12}$$

The similarity of equations (11) and (12) to the four-dimensional static axisymmetric vacuum Einstein equations [5] is obvious [notice, however, the minus sign in (12)]. The strategy to solve this system is simply, first to solve equation (11) for a real field $\boldsymbol{\phi}$ independent of the azimuthal coordinate φ, then to integrate equation (12) for k.

In the case of five-dimensional general relativity, $p = 2$, and we may parametrize the vector field $\boldsymbol{\phi}$ by an angle η (complementary to the angle η used in Reference 3) such that

$$\phi_4 = \cos\eta, \qquad \phi_5 = \sin\eta \tag{13}$$

Then equation (11) reduces to the flat-space Laplace equation

$$\Delta_3 \eta = 0 \tag{14}$$

while equation (12) gives

$$\partial_\xi k = -2\rho(\partial_\xi \eta)^2 \tag{15}$$

Every local axisymmetric solution of the linear equation (14) thus yields a local solution to the vacuum Einstein equations in five dimensions. The global solutions we are interested in must be (1) asymptotically flat and (2) regular. Asymptotic flatness means (if we take x^4 as the time coordinate)

$$\begin{aligned} \eta(\infty) &= n\pi \qquad (n \in \mathbb{N}) \\ k(\infty) &= 0 \end{aligned} \tag{16}$$

(If the three-dimensional space of metric g_{ij} has more than one end point, then we expect these boundary conditions to be satisfied, with possibly different values for n, at each end point.) The general axially symmetric solution of equations (14) and (15) with boundary conditions (16) is given in spherical coordinates (r, θ) by [5]

$$\begin{aligned} \eta &= n\pi + \sum_{l=0}^{\infty} a_l r^{-l-1} P_l(\cos\theta) \\ k &= \sum_{l,m=0}^{\infty} a_l a_m \frac{(l+1)(m+1)}{l+m+2} r^{-l-m-2} \left(P_l P_m - P_{l+1} P_{m+1}\right) \end{aligned} \tag{17}$$

[where the $P_l = P_l(\cos\theta)$ are the Legendre polynomials].

The condition of regularity is much more stringent. Indeed, the associated problem in four dimensions is known [6] to admit no regular solution other than flat space. However, in our case the minus sign in equation (15) is responsible for the existence of regular wormhole solutions.

Instead of attacking frontally the problem of finding regular solutions, our strategy shall be to start from the known spherically symmetric wormhole solution of our problem [3], and exploit the linearity of equation (14) to construct multiwormhole solutions as linear superpositions of one-wormhole solutions.

§(3): *The One-Wormhole Solution*

The spherically symmetric solution of the three-dimensional equations

$$\underset{3}{R}_{ij} = -2\partial_i \eta \partial_j \eta \tag{18}$$

with the boundary conditions (16) is given in isotropic coordinates X^i ($i = 1, 2, 3$) by [3]

$$
\begin{aligned}
d\sigma_1^2 &= \left(1 + \frac{\rho_0^2}{4R^2}\right)^2 dX^2 \\
\eta_1 &= 2 \arctan\left(\frac{2R}{\rho_0}\right)
\end{aligned}
\tag{19}
$$

($R = |X|$, ρ_0 is an arbitrary constant, and we have chosen $n = +1$ for the end point $R = \infty$).

This solution, a special case ($a_1 = a_2 = 1/2, \lambda = 0, B^2 = -\rho_0^2/4$) of the class III regular solutions of Chodos and Detweiler [2], is such that the combined inversion

$$
\begin{aligned}
X^i &\to -\frac{\rho_0^2}{4R^2} X^i \\
x^4 &\to x^4 \\
x^5 &\to -x^5
\end{aligned}
\tag{20}
$$

is an isometry of our five-dimensional space-time. This shows, in particular, that the three-dimensional space of metric $d\sigma_1^2$ consists of two asymptotically flat sheets connected by the spherical neck $R = \rho_0/2$.

To go over to the Weyl form of the metric (19), we first introduce cylindrical coordinates P, Z, φ such that

$$
dX^2 = dP^2 + dZ^2 + P^2\, d\varphi^2 \tag{21}
$$

and the complex coordinates

$$
\chi = P + iZ \tag{22}
$$

which are such that $|\chi| = R$. The Weyl coordinates and metric of our manifold are then given by

$$
\begin{aligned}
\operatorname{Re} \zeta(\chi) &= \left(1 + \frac{\rho_0^2}{4R^2}\right) \operatorname{Re} \chi \\
k_1 &= \log\left(1 + \frac{\rho_0^2}{4R^2}\right) - \log\left|\frac{d\zeta}{d\chi}\right|
\end{aligned}
\tag{23}
$$

The solution of the first of these equations is given by the Joukovski function [7]

$$
\zeta = \frac{1}{2}\left(2\chi + \frac{\rho_0^2}{2\chi}\right) \tag{24}
$$

For φ = const, this function maps both the exterior and the interior of the circle $R = \rho_0/2$ into the exterior of the segment

$$C = \{z = 0, \ -\rho_0 \leqslant \rho \leqslant \rho_0, \ \varphi = \text{const}\} \tag{25}$$

The inverse function

$$\chi = \tfrac{1}{2}\ [\zeta + (\zeta^2 - \rho_0^2)^{1/2}] \tag{26}$$

is therefore bivalued, and the ζ plane must be considered as a two-sheeted Riemann surface with the cut C. Therefore, our three-dimensional wormhole is represented in Weyl coordinates as a two-sheeted three-dimensional Riemann manifold, the two sheets being connected along the disk

$$D = \{z = 0, \ \rho \leqslant \rho_0\} \tag{27}$$

The function η_1, given in terms of χ by (19), is also bivalued. To continue it analytically through the cut C, we note that under the three-dimensional isometry [corresponding to (20)]

$$\chi \to \frac{\rho_0^2}{4\chi}, \qquad \varphi \to \varphi + \pi \tag{28}$$

which exchanges the two ζ-Riemann sheets, the function η_1 is changed to $(\pi - \eta_1)$. Therefore the two determinations $\eta_{1\pm}$ of the function η_1 are related by

$$\eta_{1+} + \eta_{1-} = \pi \tag{29}$$

The second equation (23) gives the metric function k_1 in terms of χ as

$$k_1 = \log\left(R + \frac{\rho_0^2}{4R}\right) - \log\left|\chi - \frac{\rho_0^2}{4\chi}\right| \tag{30}$$

which is a uniform function of ζ because of the isometry (28). This function is singular on the "branch circle" ($z = 0, \rho = \rho_0$), in the vicinity of which it behaves as

$$k_1 \underset{\zeta \to \rho_0}{\sim} -\frac{1}{2}\log\left|\frac{\zeta^2}{\rho_0^2} - 1\right| \tag{31}$$

This apparent singularity of the Weyl metric is characteristic of the wormhole structure.

Thus we have a regular, asymptotically flat, spherically symmetric wormhole solution (η_1, k_1) to the system (14), (15). As pointed out in Reference 3, this solution is also a five-dimensional metrical kink [8, 9]. Following five-dimensional light cones from one end point to the other, we see that a future oriented light cone at the end point $\eta = \pi$ tumbles gradually over, becomes oriented along the x^5 axis on the neck $\eta = \pi/2$, and continues tumbling over until it is past oriented at the other end point $\eta = 0$. More generally, the kink number may be defined, for any $(n + p)$-dimensional metric for which g_{ab} has the form (2), as the number of zeros of ϕ_4 [9]. Thus, the solution (η_1, k_1) is a one-kink.

§(4): *Multiwormhole Solutions*

The system (14), (15) is invariant under the two-parameter group of continuous point transformations preserving axial symmetry:

$$\begin{aligned} \rho &\to \lambda^{-1}\rho \\ z &\to \lambda^{-1}(z-a) \end{aligned} \tag{32}$$

$(\lambda \in \mathbb{R}_+, a \in \mathbb{R})$. These transformations, applied to the solution (η_1, k_1), yield a continuous family of solutions $(\eta_1(a, \lambda), k_1(a, \lambda))$ which we may interpret as wormholes of radius $\lambda\rho_0$ centered at the point $z = a$ of the z axis.

The linear superposition of n such functions η_1 is obviously again a solution of (14). We first restrict ourselves to addition, obtaining a function

$$\eta_n = \sum_{i=1}^{n} \eta_1(a_i, \lambda_i) \tag{33}$$

which is an axially symmetric solution of equation (14) together with the first boundary condition (16). The function η_n is a uniform function in the ζ plane with the n cuts

$$C_i = \{z = a_i, \;\; -\lambda_i\rho_0 \leqslant \rho \leqslant \lambda_i\rho_0\} \tag{34}$$

From this function, the function k_n solution of equation (15) with the second boundary condition (16) is obtained by a line integral.

Is this solution regular? A necessary condition for this is the regularity condition of the Weyl metric:

$$\lim_{\rho\to 0} k_n = 0 \tag{35}$$

To check that this condition is satisfied, we compute the behavior of the function η_n in the vicinity of the z axis (for $z \neq 0$):

$$\eta_n \underset{\rho\to 0}{\sim} \sum_{i=1}^{n} \left(\pi - \arctan\frac{\lambda_i\rho_0}{z - a_i}\right) + O(\rho^2) \tag{36}$$

and deduce from equations (10) the behavior of the function k_n:

$$k_n \underset{\rho\to 0}{\sim} \text{const} - \left[\sum_{i=1}^{n} \frac{\lambda_i\rho_0}{\lambda_i^2\rho_0^2 + (z - a_i)^2}\right]^2 \frac{\rho^2}{2} + O(\rho^4) \tag{37}$$

and the constant is equal to zero because of the boundary condition (16).

The function k_n itself is obviously regular everywhere except on the branch circles ($z = a_i$, $\rho = \lambda_i \rho_0$). In the vicinity of such a circle,

$$\partial_\zeta \eta_n \sim \tfrac{1}{2}\,[(\zeta - ia_i)^2 - \lambda_i^2 \rho_0^2]^{-1/2} \tag{38}$$

and therefore, from (15),

$$k_n \sim -\frac{1}{2} \log \left| \frac{(\zeta - ia_i)^2}{\lambda_i^2 \rho_0^2} - 1 \right| + O(1) \sim k_1(a_i, \lambda_i) + O(1) \tag{39}$$

which is the same behavior as that (31) of the one-wormhole solution. We are thus led to consider the disks

$$D_i = \{z = a_i, \quad \rho \leqslant \lambda_i \rho_0\} \tag{40}$$

as wormholes connecting one Riemann sheet with one or more other Riemann sheets.

How many sheets does this three-dimensional manifold have? This question is answered by the study of the analytical continuation of the function η_n. First let us show that two sheets cannot be connected directly by more than one wormhole.

Assume that two sheets A and B are connected by at least two wormholes D_i and D_j, and let us follow the function η_n, defined in A as

$$\eta_n = \eta_{1+}(1) + \cdots + \eta_{1+}(i) + \cdots + \eta_{1+}(j) + \cdots + \eta_{1+}(n) \tag{41}$$

[the symbol (i) stands here for the couple (a_i, λ_i)] through wormhole D_i into B [where the determination of $\eta_1(i)$ is $\eta_{1-}(i) = \pi - \eta_{1+}(i)$], then back through wormhole D_j [changing the determination of $\eta_1(j)$ to $\eta_{1-}(j) = \pi - \eta_{1+}(j)$] to A. The sum η_n has now the determination

$$\eta_n = 2\pi + \eta_{1+}(1) + \cdots - \eta_{1+}(i) + \cdots - \eta_{1+}(j) + \cdots + \eta_{1+}(n) \tag{42}$$

which contradicts (41).

Now, let us try to build a model of our three-dimensional manifold as a system of boxes, standing for different sheets, each of which is connected by n lines (wormholes) to n different other boxes. Each box is labeled by the corresponding signature of the function η_n [each + or − denotes the determination of one of the functions $\eta_1(i)$]. Our model space is $\mathbb{R}^n$, and the lines corresponding to a wormhole D_i are always parallel to the basis vector $\boldsymbol{e}_i$ (Figure 1). It is obvious from the figure that the solution of our problem of analytical continuation is an n-dimensional hypercube.

In any given sheet, the metric function k_n is related to the function η_n by equation (15), the integration constant being fixed by analytical continuation through any one of the disks. The behavior of k_n in the vicinity of the z axis is given by equation (37), where the sum is replaced by an algebraic sum (with signs depending on the signature of η_n), and the constant is found, by sheet-to-sheet analytical continuation through the centers of the disks, to be equal to

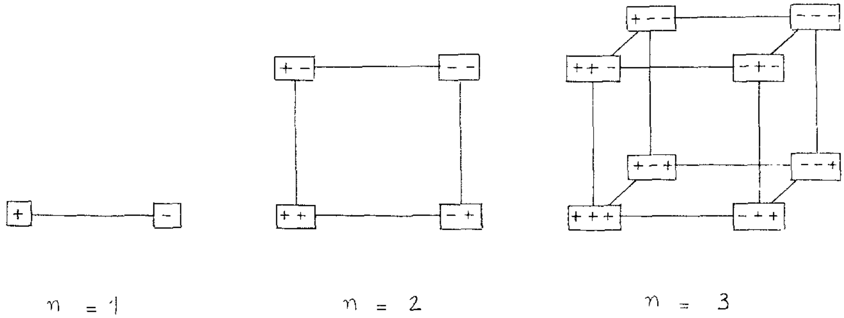

Fig. 1. Models of multisheeted spaces.

zero. Thus the regularity condition (35) is satisfied in all the sheets. The function k_n itself is obviously regular in any given sheet, except on the branch circles. This ensures that the given sheet has $\zeta = \infty$ as its sole end point. It follows from the asymptotic behavior

$$\partial_\zeta k_n \underset{\zeta \to \infty}{\sim} -(p-q)^2 \frac{\rho_0^2}{2} \frac{\rho}{r^2 \zeta^2} \tag{43}$$

[where $r = |\zeta|$, and p and q are the numbers of + and − determinations of $\eta_1(i)$] that $\zeta = \infty$ is a regular point of k_n. The value $k_n(\infty)$ is then found, from the regularity condition (35) applied for $z = \infty$, to be zero, in accordance with the boundary condition (16).

Thus, a global, regular solution of equations (14)-(16) is the analytical continuation of the solution (η_n, k_n) to a three-dimensional manifold consisting of 2^n asymptotically flat sheets interconnected by $2^{n-1}n$ wormholes. It is now obvious that, had we replaced at the beginning of our construction the sum (33) by an algebraic sum, with plus and minus signs, we would have obtained the same global solution after analytical continuation.

Inserting the function η_n in the metrical ansatz (2), we obtain from (η_n, k_n) a global, regular solution to the five-dimensional Einstein equations. This solution is asymptotically flat at each of the 2^n spatial end points. The highest value of η_n (at the end point $+ + \cdots +$) is $n\pi$, and its lowest value (at the end point $- - \cdots -$) is $n\pi - n\pi = 0$. Thus, our solution is a n-kink. Indeed, the method we have followed to construct a multikink from a one-kink is precisely that suggested in Section 2 of Reference 9. Our equation (2) means that the $g_{ab}(x^k)$ metric is obtained from the Minkowski η_{ab} by an Euclidean p-dimensional rotation which depends on x^k [3]. Thus $\eta(x^k)$ is a rotation angle, and equation (33) simply means that the multikink metric is obtained by a product of such rotations.

Finally, the five-dimensional space-time that we have constructed is globally invariant under the isometry [which generalizes (20)]

$$(\zeta, \varphi, x^4, x^5) \to (\zeta, \varphi + \pi, x^4, -x^5) \tag{44}$$

between opposite sheets (sheets represented, in model space, by opposite vertices of the n-cube). This follows from the fact that equations (14) and (15) are invariant under the transformation (44), which replaces η by $-\eta$, and thus reverses the boundary conditions for η (Fig. 1).

§(5): *Coincident Wormholes*

In the preceding section, it was assumed that the disks which represent our wormholes in $\boldsymbol{x}$ space do not coincide. What happens if they do?

The solution for n coincident disks of radii ρ_0 is $(\eta_n, k_n) = (n\eta_1, n^2 k_1)$. This solution is best written in χ coordinates, related to ζ coordinates by (24), as

$$\begin{aligned} \eta_n &= 2n \text{ arc tan} \left(\frac{2R}{\rho_0}\right) \\ d\sigma_n^2 &= \left(1 + \frac{\rho_0^2}{4R^2}\right)^2 \left[\left(\frac{R^2 + \rho_0^2/4}{|\chi^2 - \rho_0^2/4|}\right)^{2(n^2-1)} (dP^2 + dZ^2) + P^2\, d\varphi^2\right] \end{aligned} \tag{45}$$

$(\chi = P + iZ, R = |\chi|)$.

For $n \geqslant 2$, the two-dimensional χ manifolds have four end points, $\chi = \infty$ $(\eta = n\pi)$, $\chi = 0$ $(\eta = 0)$, $\chi = \pm\rho_0/2$ $(\eta = n\pi/2)$. In the case $n = 2$, this result admits a simple interpretation: the four-wormhole structure of Figure 1 (a torus with four points removed) has coalesced into a sphere with four points removed.

The three-dimensional metric (45) is regular, and characterizes a manifold with the two end points $R = \infty$ and $R = 0$, and a circle at infinity $P = \rho_0/2, Z = 0$. Such a non-simply-connected topology may support a nonvanishing magnetic dipole field (the circle at infinity acting as a current loop). This magnetic field might appear upon quantization of the Einstein equations against the background of our classical solution, the metric coefficients $g_{\mu 5}$ $(\mu = 1, \cdots, 4)$ being related, in the four-dimensional Jordan–Thiry interpretation of five-dimensional general relativity [6], to the components of the electromagnetic vector potential A_μ by

$$\beta A_\mu = \frac{g_{\mu 5}}{g_{55}} \tag{46}$$

The quantum excitations of our solution would then be characterized by a classical electric charge, because of the asymptotic behavior

$$\beta A_0 = -\tan(2\eta_n) \underset{R\to\infty}{\sim} -\frac{2n\rho_0}{R} \tag{47}$$

and a quantum magnetic dipole moment.

Going back to Weyl coordinates, we may again linearly superpose such functions $n\eta_i$ to construct new solutions to equations (14), (15). However, the interpretation of such solutions will be complicated by the fact that the branch circles ($z = a_i$, $\rho = \lambda_i \rho_0$) are also circles at infinity.

§(6): *Discussion*

Starting from a special case of the spherically symmetric wormhole solution to the Einstein equations in five dimensions, we have constructed axially symmetric solutions to these equations corresponding to systems of n wormholes centered on different points of the z axis. We have also studied other axially symmetric solutions corresponding to n coincident wormholes.

These solutions are characterized by a kink number n. If we consider the 1-kink solution as a possible classical model of a charged elementary particle [2], then our multikink solutions might be interpreted as multiparticle systems.

Such an interpretation is hampered by the fact that, according to our construction of Section 4, the three-space corresponding to a n-kink solution has a number of end points which grows exponentially with n. How can we define electric charge in such a space? According to equation (47), applied to a system of n distinct wormholes of identical radius, the total charge of the system, measured in units of the elementary charge, may take any of the values $n, n - 2, \cdots, -n$, depending on the end point where we measure it![1]

A possible way out of this difficulty is to modify the global three-space topology by identifying end points for which the values of η are equal. For instance, in the case $n = 2$, we identify the end points of sheets $(+\,-)$ and $(-\,+)$, thereby replacing in effect those end points by a wormhole which connects the two sheets (Figure 2). Such a procedure, which may be generalized to arbitrary n, leaves a three-space with only the two end points $(+ + \cdots +)$ and $(- - \cdots -)$ (such a topology is consistent with that obtained in Section 5 in the case of coincident wormholes). In this three-space, the total electric charge is defined up to a sign, which leads us to interpret the isometry (44) which exchanges the two end points as charge conjugation. It has been previously emphasized [11, 12] that such a globally two-sided geometry may be perfectly consistent with macroscopic observation by observers which are themselves multiparticle systems of the same kind; this interpretation has been shown to be viable in the case of quantum-mechanical scattering in a model two-sided geometry [12] provided that the wave functions associated with neutral test particles are odd under charge conjugation.

Another problem with the particle interpretation of our solution is their

[1] The difficulty in defining charge in non-simply-connected space-time manifolds is discussed in detail in Reference 10.

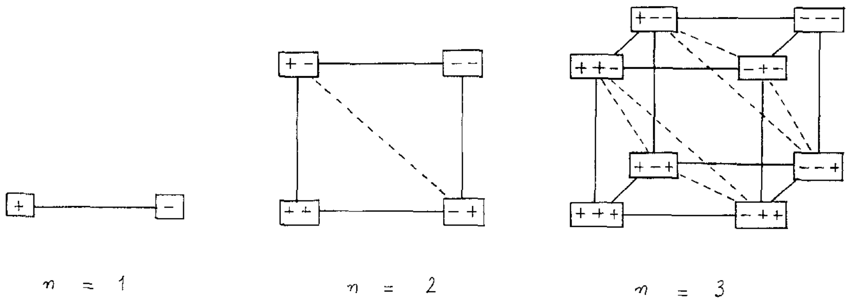

Fig. 2. Removing extra end points.

masslessness. The mass of any static solution of the five-dimensional Einstein equations with the ansatz (2), (13) may be obtained from the asymptotic behavior of the 44 component of the projected metric tensor [6]

$$\bar{g}_{44} = g_{44} - \frac{g_{45}^2}{g_{55}} = \frac{1}{\cos(2\eta)} \tag{48}$$

The asymptotic behavior of η which follows from (19) gives

$$M = 0 \tag{49}$$

for all our solutions. This defect shall be remedied in the following paper [13].

As far as multikink systems are concerned, there should be nothing special about axial symmetry, and so it would be desirable to find a construction of nonaxisymmetric multikink solutions to our ansatz. The situation recalls that of four-dimensional Euclidean Yang-Mills theory, in which multi-instanton solutions were first found by Witten [14], under the restriction of axial symmetry; this restriction was almost immediately removed by the more general 't Hooft ansatz (for a review, see reference [15]). Perhaps our multi-wormhole solutions can be similarly generalized.

Acknowledgment

The referee's helpful comments are duly acknowledged.

Appendix

The Bianchi identities derived from the Einstein equations (4) give

$$\nabla_i \Phi \cdot \nabla_j \nabla^j \Phi = 0 \tag{A1}$$

On the other hand, the gradient of equation (3) gives

$$\nabla_i \boldsymbol{\phi} \cdot \boldsymbol{\phi} = 0 \tag{A2}$$

Let $q \leqslant n$ be the number of linearly independent vectors $\nabla_i \boldsymbol{\phi} (i = 1, \cdots, n)$. From (A2) these vectors are orthogonal to the p-component vector field $\boldsymbol{\phi}$, so that $q \leqslant p - 1$. If $q = p - 1 \leqslant n$, then it follows from (A1) that

$$\nabla_j \nabla^j \boldsymbol{\phi} = \lambda \boldsymbol{\phi} \tag{A3}$$

from which, taking into account the divergence of (A2), follows equation (5).

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