General Relativity and Gravitation, Vol. 16, No. 5, 1984

A Class of Static Charged Dust Spheres in General Relativity

RAMESH TIKEKAR

Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar–388 120, India

Received May 23, 1983

Abstract

Using static spherically symmetric space-times with associated 3-spaces obtained as hypersurfaces t = const as 3-spheroidal, a class of physically viable relativistic models for spherical distributions of uniformly charged dust in equilibrium is obtained. The charged analog of Schwarzschild interior solution given by Cooperstock and de la Cruz follows as a particular case of this class.

§(1): *Introduction*

The problem of determination of exact solutions of coupled Einstein-Maxwell equations for static spherical distributions of charged matter has attracted wide attention. These distributions constitute possible sources for a Reissner-Nordström metric which uniquely describes the exterior field of a spherically symmetric charged distribution of matter. Bonnor [1, 2] has shown that for spherical distributions of uniformly charged dust in equilibrium, the absolute value of charge density must be equal to the matter density and given an explicit solution of Einstein-Maxwell equations. De and Raychaudhari [3] have shown that the equality of the magnitudes of charge density and matter density is a direct consequence of Einstein-Maxwell equations. Cooperstock and de la Cruz [4] have studied relativistic spherical charged distributions of perfect fluids in equilibrium and obtained an explicit solution of coupled Einstein-Maxwell equations in the interior of a sphere containing uniformly charged dust in equilibrium, assumming a constant nongravitational energy density. Cooperstock and de la Cruz's solution is a generalization of Schwarzschild interior solution, with matter

0001-7701/84/0500-0445$03.50/0

density decreasing outward. Bonnor and Wickramsuriya [5] have obtained a static interior dust metric with matter density increasing outward.

In this paper, a class of exact solutions of the coupled Einstein-Maxwell equations for static spherical distributions of uniformly charged dust is obtained assuming a peculiar geometry for the associated 3-space of the distributions. It is found that Cooperstock and de la Cruz's solution also belongs to the class as a particular case. The relativistic space-time associated with Schwarzschild's interior metric representing the gravitational field within a sphere of homogeneous perfect fluid at rest has its physical 3-space, i.e., hypersurface t = const, in the form of a 3-sphere, the radius R of which is related with the matter density. Vaidya and Tikekar [6] have shown that the space-times which have the associated 3-spaces obtained as hypersurfaces t = const, 3-spheroids can be used to develop exact relativistic models of superdense stars in which the collapse under gravitational attraction is countered by repulsive fluid pressure. Following the same approach here, it is shown that these space-times can also be used to develop models describing the field in the interior of spherical distributions of matter in the form of dust, whose collapse under gravitational attraction is countered by the repulsive Coulombian electric field due to the presence of charge.

In the following section, the metric for the space-time with hypersurfaces t = const, as 3-spheroids is obtained. The Einstein-Maxwell equations for static charged fluid distributions are developed in Section 3. The field equations are integrated for charged dust distributions and a new class of exact solutions is obtained in Section 4. The matching of the solutions on the boundary with a Reissner-Nordström metric and other relevant features of the class of solutions are discussed in the concluding section.

§(2): *Geometrical Considerations*

A 3-spheroid immersed in the four-dimensional Euclidean space with metric,

$$d\Sigma^2 = dx^2 + dy^2 + dz^2 + dw^2$$

will have the "Cartesian" equation

$$\frac{x^2 + y^2 + z^2}{R^2} + \frac{w^2}{b^2} = 1$$

The sections w = const of the 3-spheroid are concentric spheres, while sections x = const, y = const, or z = const, represent systems of confocal ellipsoids. The parametrization

$$x = R \sin\lambda \sin\alpha \cos\beta$$

$$y = R \sin\lambda \sin\alpha \sin\beta$$

$$z = R \sin\lambda \cos\alpha$$

$$w = b \cos\lambda$$

of the 3-spheroid gives

$$d\Sigma^2 = (R^2 \cos^2 \lambda + b^2 \sin^2 \lambda)\, d\lambda^2 + R^2 \sin^2 \lambda\, (d\alpha^2 + \sin^2 \alpha\, d\beta^2) \tag{1}$$

as the metric on the 3-spheroid. The introduction of the space variable

$$r = R \sin \lambda$$

transforms the metric (1) to

$$d\Sigma^2 = \frac{1 - K(r^2/R^2)}{1 - (r^2/R^2)}\, dr^2 + r^2 (d\alpha^2 + \sin^2 \alpha\, d\beta^2) \tag{2a}$$

where

$$K = 1 - \frac{b^2}{R^2} \tag{2b}$$

For $K < 1$, the metric (2) is regular and positive definite at all points $r < R$. In the case $K = 1$ the spheroidal 3-space degenerates into flat 3-space and in the case $K = 0$, it becomes spherical. In the Schwarzschild coordinates, the space-time metric

$$\begin{aligned} ds^2 &= -d\Sigma^2 + e^{\nu(r)}\, dt^2 \\ &= -\frac{1 - K(r^2/R^2)}{1 - (r^2/R^2)}\, dr^2 - r^2 (d\alpha^2 + \sin^2 \alpha\, d\beta^2) + e^{\nu(r)}\, dt^2 \end{aligned} \tag{3}$$

has its associated 3-space obtained as hypersurface t = const a spheroidal 3-space.

The metric (3) with $K = 0$ and

$$e^{\nu(r)} = [A + B(1 - r^2/R^2)^{1/2}]^2$$

is the metric of the Schwarzschild interior solution.

§(3): *Einstein-Maxwell Equations*

We will develop Einstein-Maxwell field equations for static, spherically symmetric distributions of matter in the form of a charged perfect fluid with the metric (3) as the space-time metric associated with the distribution. For a perfect fluid with charge, the energy-momentum tensor is

$$T_i^k = (\rho + p) u_i u^k - p \delta_i^k + \frac{1}{4\pi} \left(-F_{ij} F^{kj} + \frac{1}{4} F_{mn} F^{mn} \delta_i^k \right) \tag{4}$$

Here ρ, p, and u^i denote, respectively, the matter density, fluid pressure, and the unit timelike four-velocity field of the fluid. F_{ik} are the components of electromagnetic field tensor satisfying Maxwell's equations,

$$F_{ik,j} + F_{kj,i} + F_{ji,k} = 0 \tag{4a}$$

$$\frac{\partial}{\partial x^j}\left[F^{ij}(-g)^{1/2}\right] = 4\pi(-g)^{1/2}\, J^i \tag{4b}$$

The four-current vector (for fluids with null conductivity) is

$$J^i = \sigma u^i \tag{5}$$

where σ denotes the charge density. Since the field is static

$$u^i = (0, 0, 0, e^{-\nu/2}) \tag{6}$$

For spherical charged perfect fluid distributions under consideration Maxwell's equation (4b) gives

$$F_{41} = -\frac{e^{\nu/2}}{r^2}\left[\frac{1 - K(r^2/R^2)}{1 - (r^2/R^2)}\right]^{1/2} \int_0^r 4\pi\sigma r^2 \left[\frac{1 - K(r^2/R^2)}{1 - (r^2/R^2)}\right]^{1/2} dr \tag{7}$$

as the only surviving independent component of the electromagnetic field tensor F_{ij}.

We write

$$-F_{41}F^{41} = E^2 \tag{8}$$

where $E(r)$ can be interpreted as the electric field intensity. Equations (7) and (8) provide a relation between the charge density σ and $E(r)$ as

$$4\pi\sigma = \frac{1}{r^2}\left\{\frac{d}{dr}(r^2 E)\right\}\left(1 - \frac{r^2}{R^2}\right)^{1/2}\left(1 - K\frac{r^2}{R^2}\right)^{-1/2} \tag{9}$$

Subsequently,

$$q(r) = 4\pi \int_0^r \left(1 - K\frac{r^2}{R^2}\right)^{1/2}\left(1 - \frac{r^2}{R^2}\right)^{-1/2} \sigma r^2\, dr \tag{10}$$

represents the total charge contained within the sphere of coordinate radius r.

The Einstein-Maxwell equations give

$$8\pi T_1^1 = -8\pi p + E^2 = \left\{\frac{1-K}{R^2} - \frac{\nu'}{r}\left(1 - \frac{r^2}{R^2}\right)\right\}\left(1 - K\frac{r^2}{R^2}\right)^{-1} \tag{11a}$$

$$8\pi T_2^2 = -8\pi p - E^2 = -\left\{\frac{\nu''}{2} + \frac{\nu'^2}{4} + \frac{\nu'}{2r}\right\}\left(1 - \frac{r^2}{R^2}\right)\left(1 - K\frac{r^2}{R^2}\right)^{-1} + \frac{(1-K)r}{R^2}\left\{\frac{\nu'}{2} + \frac{1}{r}\right\}\left(1 - K\frac{r^2}{R^2}\right)^{-2} \tag{11b}$$

$$8\pi T_3^3 = 8\pi T_2^2$$

$$8\pi T_4^4 = 8\pi\rho + E^2 = \frac{3(1-K)}{R^2}\left(1 - \frac{K}{3}\frac{r^2}{R^2}\right)\left(1 - K\frac{r^2}{R^2}\right)^{-2} \tag{11c}$$

The equation (11c) determines the nongravitational energy density at every point of the space for specified choice of K and R.

§(4): *Solution of Field Equations*

Cooperstock and de la Cruz obtained the analog of the Schwarzschild interior solution for the particular case of charged dust, $p = 0$, by assuming for the dust sphere a constant nongravitational energy density at every point of space. The expression (11c) assigns a constant value $3/R^2$ to $8\pi T_4^4$ at every point of the space with $K = 0$.

Charged dust spheres in equilibrium belong to the interior Papapetrou-Majumdar [7, 8] class and their metrics can be expressed in the form

$$ds^2 = -U^2(dx^2 + dy^2 + dz^2) + U^{-2}\, dt^2 \tag{12}$$

where $U = U(x, y, z)$. The metric (3) can be reduced to the form (12) if

$$1 + \frac{1}{2} r v' = \left(1 - K\frac{r^2}{R^2}\right)^{1/2} \left(1 - \frac{r^2}{R^2}\right)^{-1/2} \tag{13}$$

The equation (13) is found to have solutions given by

$$e^v = B^2 \left(\frac{\{[1 - K(r^2/R^2)]^{1/2} + \sqrt{K}\,[1 - (r^2/R)]^{1/2}\}^{\sqrt{K}}}{\{[1 - K(r^2/R^2)]^{1/2} + [1 - (r^2/R^2)]^{1/2}\}}\right)^2, \qquad 0 \leqslant K < 1 \tag{14}$$

$$e^v = B^2\, \frac{\exp\left(-2(-K)^{1/2} \tan^{-1} \{K(r^2/R^2 - 1)/[1 - K(r^2/R^2)]\}^{1/2}\right)}{\{[1 - K(r^2/R^2)]^{1/2} + (1 - r^2/R^2)^{1/2}\}^2}, \qquad K \leqslant 0 \tag{15}$$

where B is the arbitrary constant of integration. The expressions for e^v are regular and positive at all points $r < R$, and therefore metric (3) with e^v as given by (14) or (15) is well behaved metric. The matter density and the square of the electric field intensity are found to be

$$8\pi\rho = \frac{2}{r^2} \left\{\frac{(1 - r^2/R^2)^{1/2}}{[1 - K(r^2/R^2)]^{1/2}} - \frac{(1 - r^2/R^2)}{[1 - k(r^2/R^2)]}\right\} + \frac{2}{R^2}\, \frac{1 - K}{[1 - K(r^2/R^2)]^2} \tag{16}$$

and

$$E^2 = \frac{1}{r^2} \left\{1 - \frac{(1 - r^2/R^2)^{1/2}}{[1 - K(r^2/R^2)]^{1/2}}\right\}^2 \tag{17}$$

respectively. The electric charge density determined from (9) and (17) yields

$$\sigma = \pm\rho \tag{18}$$

in accordance with the De-Raychaudari requirement.

§(5): *Discussion*

The expressions (16) and (17) for matter density and electric field intensity E, show that in the limit $r \to 0$, one obtains at the center of the distribution

$$8\pi\rho_0 = \frac{3(1-K)}{R^2} \tag{19a}$$

and

$$E_0 = 0 \tag{19b}$$

implying regularity of the distribution at the center. Further, (19a) imposes the condition that $K < 1$ for the matter density at the center $\rho_0 > 0$. Matter density and the square of the electric field intensity are positive for $K < 1$ at all points $r^2 < R^2$ of the space, throughout the configuration.

We consider a situation wherein the spherical charge dust distribution extends to a finite radius $a < R$. The interior metric (3) with e^v as given by (14) and (15) should then match with the exterior Reissner-Nordström metric

$$ds^2 = -\left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)^{-1} dr^2 - r^2(d\alpha^2 + \sin^2\alpha\, d\beta^2) + \left(1 - \frac{2m}{r} + \frac{q^2}{r^2}\right)dt^2 \tag{20}$$

across the radius $r = a$ of the distribution. Here m and q, respectively, denote the total mass and the total charge of the dust sphere. The appropriate boundary conditions

$$e^{v(a)} = \left(1 - \frac{a^2}{R^2}\right)\left(1 - K\frac{a^2}{R^2}\right)^{-1} = 1 - \frac{2m}{a} + \frac{q^2}{a^2} \tag{21}$$

determine the constants B and m as

$$B^2 = \frac{1 - a^2/R^2}{1 - K(a^2/R^2)}\left(\frac{\{[1 - K(a^2/R^2)]^{1/2} + (1 - a^2/R^2)^{1/2}\}}{\{[1 - K(a^2/R^2)]^{1/2} + \sqrt{K}(1 - a^2/R^2)^{1/2}\}^{\sqrt{K}}}\right)^2 \qquad 0 \leqslant K < 1 \tag{22a}$$

$$B^2 = \frac{1 - a^2/R^2}{1 - K(a^2/R^2)}\{[1 - K(a^2/R^2)]^{1/2} + (1 - a^2/R^2)^{1/2}\}^2 \cdot \exp\left\{2(-K)^{1/2}\tan^{-1}\left[\frac{K(a^2/R^2 - 1)}{1 - K(a^2/R^2)}\right]^{1/2}\right\}, \qquad 0 \leqslant K \tag{22b}$$

and

$$m = \frac{1}{2}\,\frac{1 - K}{1 - K(a^2/R^2)}\,\frac{a^3}{R^2} + \frac{q^2}{2a} \tag{23}$$

where the total charge $q(a)$ is obtained from (10) by setting $r = a$.

For specified values of geometric parameter K, the expression (19) determines the constant R^2 in terms of ρ_0, the matter density at the center of the configuration. Subsequently the total mass of the dust sphere at radius a is given by (23). The solution contains K, a geometric parameter which can be used to generate different solutions by assigning values to it such that $K < 1$. The matter density of charged dust spheres with $K \leqslant 0$ decreases radially outward. Numerical methods indicate that for charged dust spheres with $0 \leqslant K \leqslant 0.05$ the matter density decreases radially outward, whereas for spheres with $0.05 < K < 1$, it increases outward. The charged analog of Schwarzschild interior solution given by Cooperstock and de la Cruz is obtained by assigning $K = 0$.

Hence the space-time metric (3) with its hypersurfaces t = const as 3-spheroids gives physically viable models of charged dust spheres in equilibrium, acting as the interior sources for Reissner-Nordström metric.

Acknowledgment

The author is thankful to the U.G.C. of India for an award of an assistance grant supporting the work.

References

1. Bonnor, W. B. (1960). *Z. Phys.*, **160**, 59.
2. Bonnor, W. B. (1965). *Mon. Not. R. Astron. Soc.*, **129**, 443.
3. De, U. K., and Raychaudhari, A. K. (1968). *Proc. R. Soc. London Ser. A*, **303**, 87.
4. Cooperstock, F. I., and de la Cruz, V. (1978). *Gen. Rel. Grav.*, **9**, 835.
5. Bonnor, W. B., and Wickramsuriya (1975). *Mon. Not. R. Astron. Soc.*, **170**, 643.
6. Vaidya, P. C., and Tikekar, R. (1982). *J. Astrophys. Astron.*, **3**, 325.
7. Papapetrou, A. (1947). *Proc. R. Irish Acad.*, **A51**, 191.
8. Majumdar, S. D. (1947). *Phys. Rev.*, **72**, 390.