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The Nonstationary Generalization of the Gödel Cosmological Model

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Abstract

The nonstationary generalization of the Gödel cosmological model, which represents the solution of Einstein's equations in the comoving system of reference, is derived, and some properties of the new model are investigated.

§(1): *Introduction*

Modern astronomical observational data do not exclude the possibility of a Metagalaxy rotation as a whole. In this connection it seems reasonable to find and investigate cosmological models describing the rotating universe. One such model is the Gödel cosmological model [1]. A great number of articles have been devoted to the investigation of the Gödel model (see e.g., [2-7] and references in them). Attempts to deduce the cosmological models, generalizing the Gödel model, were made (see e.g., [8-10, 11]). But such models for the perfect fluid without the energy flux were not obtained in those papers. This paper deals with the nonstationary generalization of the Gödel cosmological model which represents the solution of Einstein's equations with the perfect fluid energy-momentum tensor in the comoving system of reference.

In the general case, finding the solutions of Einstein's equations describing the rotating universe is connected with great mathematical difficulties; therefore, one must make certain simplifying suppositions (e.g., the supposition of space homogeneity). In [12], without using Einstein's equations, there were found all the nonstationary metrics with rotation satisfying the spatial homogeneity dif-

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ferential criterion, which is a direct generalization of the homogeneity criterion accepted in nonrelativistic physics, but taking into account the possible indefiniteness in the choice of the time coordinate ([13], see also [14]). The line element ds^2 of one of these metrics in the definite system of reference is reduced to the form

$$ds^2 = c^2\,dt^2 + \frac{2c^2}{ar}\frac{\partial \epsilon}{\partial x^1}\,dt\,dx^2 - a^2\,dx^{1^2} + \left[-\epsilon^2 + \frac{c^2}{a^2r^2}\left(\frac{\partial \epsilon}{\partial x^1}\right)^2\right]dx^{2^2}$$
$$+ 2a(\alpha\cos rt + \beta\sin rt)\,dx^1\,dx^3 + 2\epsilon(\alpha\sin rt - \beta\cos rt)\,dx^2\,dx^3$$
$$- b^2\,dx^{3^2} + A(a^2\cos 2rt\,dx^{1^2} + 2a\epsilon\sin 2rt\,dx^1\,dx^2 - \epsilon^2\cos 2rt\,dx^{2^2}) \tag{1}$$

Here

$$\frac{\partial^2\epsilon(x^1,x^2)}{\partial x^{1^2}} = k\epsilon(x^1,x^2),$$

$$k = -\frac{2a^2r\omega}{c^2}\,[b^2(1-A^2) - \beta^2(1-A) - \alpha^2(1+A)]^{1/2},$$

$$A, r, \alpha, \beta, a, b, \omega = \text{const}\ |A| < 1,$$
$$b^2(1-A^2) - \beta^2(1-A) - \alpha^2(1+A) > 0$$
$$2b^2 - \alpha^2 - \beta^2 > 0, \qquad a, b, \omega \neq 0$$

We shall show that this metric contains the nonstationary generalization of the Gödel cosmological model and we shall investigate some properties of this new model.

§(2): *The Nonstationary Generalization of the Gödel Cosmological Model*

When examining the problem of compatibility of the metric (1) with Einstein's equations, one can show that it is compatible with Einstein's equations with the energy-momentum tensor of the perfect or viscous fluid only when $\alpha = \beta = 0$. When $\alpha = \beta = 0$, Einstein's equations with the perfect fluid energy-momentum tensor

$$T^{\mu\nu} = \left(\rho_{00} + \frac{p_0}{c^2}\right)u^\mu u^\nu - \frac{p_0}{c^2}g^{\mu\nu}$$

limit the metric (1) in the following way:

$$r^2 = \omega^2 \beta^2 (1 - A^2) = \frac{\varkappa c^2}{2}\left(\rho_{00} + \frac{p_0}{c^2}\right)$$

$$\varkappa\left(\rho_{00} - \frac{p_0}{c^2}\right) = -2\Lambda, \qquad u^i = 0$$

$$\epsilon = N_1(x^2)\, e^{-kx^1} + N_2(x^2)\, e^{kx^1}, \qquad k = \frac{ar}{c}\sqrt{2}$$

Here ρ_{00} is the matter density in the comoving system of reference, p_0 is pressure, u^μ is the four-dimensional velocity vector, $\varkappa$ is Einstein's gravitational constant, Λ is the cosmological constant, c is the fundamental velocity, and N_1, N_2 are arbitrary functions of x^2; Greek indices run over the values 0, 1, 2, 3; Latin ones run only 1, 2, 3.

The line element in the new model takes the form

$$\begin{aligned} ds^2 = {} & c^2\, dt^2 + 2\sqrt{2}\, c(-N_1 e^{-kx^1} + N_2 e^{kx^1})\, dt\, dx^2 - a^2\, dx^{1^2} \\ & + [(N_1 e^{-kx^1} + N_2 e^{kx^1})^2 - 8N_1 N_2]\, dx^{2^2} - b^2\, dx^{3^2} \\ & + 2A\,[a \cos rt\, dx^1 + (N_1 e^{-kx^1} + N_2 e^{kx^1}) \sin rt\, dx^2]^2 \\ & - A\,[a^2\, dx^{1^2} + (N_1 e^{-kx^1} + N_2 e^{kx^1})^2\, dx^{2^2}] \end{aligned} \tag{2}$$

We shall further assume $N_1 = 0, N_2 = N_0 =$ const. After the elementary transformations $ax^1 \to x^1, \sqrt{2}N_0 x^2 \to x^2, \sqrt{2}\, bx^3 \to x^3$ the line element ds^2 takes the form

$$\begin{aligned} ds^2 = {} & ds_0^2 + A\{\sqrt{2} \cos rt\, dx^1 + \sin rt \exp\, [\sqrt{2}\,(r/c)\, x^1]\, dx^2\}^2 \\ & - A\{dx^{1^2} + \tfrac{1}{2} \exp\, [2\sqrt{2}\,(r/c)\, x^1]\, dx^{2^2}\} \end{aligned} \tag{3}$$

where $r^2 = \Omega^2(1 - A^2)$, $|A| < 1$, Ω^2 is the square of the angular velocity of the comoving system of reference; Ω^2 and A are constants;

$$\begin{aligned} ds_0^2 = {} & c^2\, dt^2 - dx^{1^2} + \tfrac{1}{2} \exp\, [2\sqrt{2}\,(r/c)\, x^1]\, dx^{2^2} \\ & + 2c \exp\, [\sqrt{2}\,(r/c)\, x^1]\, dt\, dx^2 - \tfrac{1}{2}\, dx^{3^2} \end{aligned}$$

is the line element in the Gödel model [1].

Now one can see that the metric (3) as well as the Gödel metric describe space-time of the rotating system of reference which falls freely in each of its point and which comoves with the perfect fluid. Space deformation takes place in such a way that the comoving space element volume does not change in the course of time. The state equation $\varkappa(\rho_{00} - p_0/c^2) = -2\Lambda$ in our model takes the form of the state equation $\varkappa\rho_{00} = -2\Lambda$ in the Gödel model, when $p_0 = 0$.

When $A = 0, p_0 = 0$, our model transforms into the Gödel model. When the rotation vanishes ($\Omega^2 = 0$) in the new model, as well as in the Gödel model, we come to the stationary universe with $\rho_{00} = 0$, $\Lambda = 0$ (the empty universe with $\Lambda = 0$, i.e., Minkovski's universe).

§(3): *The Closed Timelike Curves in the New Cosmological Model*

As is known, one of the interesting properties of the Gödel cosmological model is the presence of the closed timelike curves [1-3]. This fact was taken as a base for the idea of "traveling into the past" in the Gödel model. It is interesting to find out whether the nonstationary generalization of the Gödel cosmological model also contains closed timelike curves.

After defining

$$\sigma ct = z^0, \quad \sqrt{2}\sigma x^1 = z^1, \quad \sigma x^2 = z^2, \quad \sigma x^3 = z^3, \quad \sigma = r/c$$

the metric (3) takes the form

$$ds^2 = \frac{1}{\sigma^2}\left[(dz^0 + e^{z^1}\, dz^2)^2 - \frac{1}{2}({dz^1}^2 + e^{2z^1}\, {dz^2}^2 + {dz^3}^2) + A(\cos z^0\, dz^1 + \sin z^0\, e^{z^1}\, dz^2)^2 - \frac{A}{2}({dz^1}^2 + e^{2z^1}\, {dz^2}^2)\right]$$

We shall show that our model also contains closed timelike curves. For this we introduce new coordinates τ, ρ, θ, η (cf. [1, 2]):

$$\tan\left(\theta + \frac{z^0 - \tau}{2}\right) = e^{-\rho}\tan\theta$$

$$e^{z^1} = e^{\rho}\cos^2\theta + e^{-\rho}\sin^2\theta$$

$$z^2 e^{z^1} = (e^{\rho} - e^{-\rho})\sin\theta\cos\theta$$

$$\frac{z^3}{\sigma} = \eta.$$

It is evident that a change of θ by π, with no change in τ, p, η, will leave all the conditions undisturbed; therefore, we may regard θ as ranging, not over the real numbers, but over the interval with the length of π. Thus, $-\infty < \tau < \infty$, $0 \leqslant \rho < \infty$, $0 \leqslant \theta \leqslant \pi$, while $\theta = 0$ is identified with $\theta = \pi$.

After the cumbersome computations, the line element ds^2 in terms of the new coordinates takes the form

$$ds^2 = \frac{1}{\sigma^2}\left\{d\tau^2 - \left(\frac{1+A}{2} - A\mu^2\right) d\rho^2 + 2A\mu\delta\, d\rho\, d\theta\right.$$

$$+ \left[A\delta^2 + (e^{\rho/2} - e^{-\rho/2})^4 - \frac{1+A}{2}(e^{\rho} - e^{-\rho})^2\right] d\theta^2$$

$$\left. + 2(e^{\rho/2} - e^{-\rho/2})^2\, d\theta\, d\tau\right\} - \frac{1}{2} d\eta^2$$

where

$$\mu = \frac{(e^{\rho}\cos^2\theta - e^{-\rho}\sin^2\theta)\cos z^0 + \sin 2\theta \sin z^0}{e^{\rho}\cos^2\theta + e^{-\rho}\sin^2\theta}$$

$$\sigma = \frac{(e^{\rho} - e^{-\rho})\,[(e^{\rho}\cos^2\theta - e^{-\rho}\sin^2\theta)\sin z^0 - \sin 2\theta\cos z^0]}{e^{\rho}\cos^2\theta + e^{-\rho}\sin^2\theta}$$

We can show that μ, δ satisfy the inequalities $|\mu| \leqslant 1$, $|\delta| \leqslant e^{\rho} - e^{-\rho}$. For every $\rho = r_0 > \bar{r}_0$, where

$$\bar{r}_0 = \ln \frac{3 - |A| + 2[2(1 - |A|)]^{1/2}}{1 + |A|}$$

the inequality

$$A\delta^2 + (e^{r_0/2} - e^{-r_0/2})^4 - \frac{1+A}{2}(e^{r_0} - e^{-r_0})^2 > 0$$

takes place, consequently, the closed curves, defined by the conditions $\rho = r_0 =$ const, $\tau = \tau_0 =$ const, $\eta = \eta_0 =$ const, are timelike everywhere. Thus, the new model also contains the closed timelike curves. But in our model, in constrast with the Gödel model, these closed curves are not circles, as the distance of R of the points of the closed timelike curves from the point of $O(\rho = 0, \tau = \tau_0, \eta = \eta_0)$ depends on the angular coordinate θ:

$$R = R(\theta) = \frac{1}{\sigma}\int_0^{r_0}\left(\frac{1+A}{2} - A\mu^2\right)^{1/2} d\rho$$

Since

$$\frac{1 - |A|}{2} \leqslant \frac{1+A}{2} - A\mu^2 \leqslant \frac{1 + |A|}{2}$$

then R satisfies the inequality

$$\frac{r_0}{\sigma}\left(\frac{1-|A|}{2}\right)^{1/2} \leqslant R \leqslant \frac{r_0}{\sigma}\left(\frac{1+|A|}{2}\right)^{1/2}$$

In order to estimate the value of R, we assume $A = 0, p_0 = 0$ (the Gödel model). Then $R = r_0/\sigma\sqrt{2}$ and the closed timelike curves are the circles. The radius of the first circle insignificantly exceeds the value of $R_0 = (1/\sigma\sqrt{2})\ln(3 + 2\sqrt{2})$. (The circumference with the radius R_0 is a closed isotropic curve). As in the Gödel model $\sigma = r/c = (\varkappa\rho_{00}/2)^{1/2} = (R_E)^{-1}$, then $R_0 = (R_E/\sqrt{2})\ln(3 + 2\sqrt{2})$, where R_E is the Einstein stationary universe radius. When $\rho_{00} = 10^{-31}$ g/cm^3, $R_0 \sim 10^{28}$ cm. Thus, the minimum radius of the closed timelike curves in the Gödel model is excessively great, i.e., of the order of the Einstein stationary universe radius. Consequently, for "travel into the past" the astronaut should have traveled around the whole universe.

Finally, we state that the new model is the most complete generalization of the Gödel cosmological model in the space class satisfying the spatial homogeneity differential criterion, when Einstein's equations with the perfect or viscous fluid energy-momentum tensor are taken. In fact, the analysis of the problem of compatibility of the rotating metrics, which satisfy the spatial homogeneity differential criterion (shown in [12]), with Einstein's equations shows that only the metric (2) contains the Gödel cosmological model as some particular case. Therefore, the solutions of Einstein's equations with a perfect or viscous fluid energy-momentum tensor, which generalize the Gödel cosmological model more completely than our model, are to be searched for in the class of spaces not satisfying the spatial homogeneity differential criterion.

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