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GRAVITATION IN THE LIGHT CONE GAUGE†

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ABSTRACT

The formulation of gravitation theory in the light cone gauge is studied. After a brief discussion of Yang-Mills theory for purposes of illustration, tensor and scalar-tensor gravitation are investigated. We show that if the gauge conditions are properly chosen the constrained components of the metric tensor can be explicitly solved for by quadrature, so that the field theory can be reformulated entirely in terms of the physical transverse fields. It is also shown that the light cone gauge is useful for finding wave solutions of classical field equations. Occasional reference is made to dual models, primarily to explain our motivation, but familiarity with them is not required for an understanding of this paper.

§(1): *INTRODUCTION*

The study of gravitation in the light cone gauge is a useful exercise, in our opinion, for a number of reasons. First of all, the connections with the field theory of dual strings suggests that it might possess some surprising elegance and simplicity. Secondly, this simplicity could be useful for finding new exact solutions of the classical field equations. These possibilities are born out, at least to a certain extent, by the calculations presented in this paper. There are additional possible benefits of a light cone gauge formulation that are not pursued here. For one thing, since a light cone gauge formulation involves only physical transverse fields, there is no need to use a Fadeev-Popov (or other) formalism for the elimination of spurious states from the internal lines of higher order quantum corrections to the theory.

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Also a light cone gauge formalism might be well-suited for studying high energy properties of amplitudes.

We have been led to these investigations as an outgrowth of our work on dual models. Therefore we cannot resist saying a little bit about dual models and the motivations that got us to this point. Before doing so we wish to assure the reader that no knowledge of dual models is required to understand this paper.

Dual string models embody deep connections with practically all previously known theoretical constructions. Einstein's theory of gravitation is no exception. The first clue that this might be the case is provided by the occurrence of a massless spin 2 particle in the spectrum of states of closed strings. In a previous work [*1*] the on-mass-shell dual scattering amplitude for N of these 'gravitons' was investigated. It was proved that in the tree approximation these amplitudes precisely coincide on the mass shell with those given by the Einstein theory to leading order in the Regge slope parameter α'. A massless scalar particle is also present in the dual model spectrum. Furthermore, a massless antisymmetric tensor state also is present in the closed string spectrum in certain versions of the theory [*2*]. It was proved [*3*] that the rôle of this state could be interpreted classically as introducing torsion into the geometry of space-time. Torsion fields are not considered in this paper, even though it would be a relatively straightforward generalization to do so.

During the past year, dual string models have been reformulated as (multilocal) second quantized field theories of strings [*4*]. This work has led to a number of important new insights. In particular, it has put the rules for determining the relative weights of diagrams required for unitarization on a firmer foundation. One curious feature of the field theory of strings is that it appears to be essential to make a non-covariant choice of gauge. The choice that has been investigated so far is based on the use of a light cone gauge. It may also be possible to formulate the field theory of strings in a radiation gauge [*5*] at the price of some increase in complexity.

The analysis of reference [*1*] establishing the connection between closed strings and gravitation in the zero-slope limit was based on covariant rules for the dual amplitudes. If the small-slope expansion is applied to the string field theory, on the other hand, one should obtain the field theory of gravitons (and massless scalars) formulated in the light cone gauge. An intriguing aspect of this expansion is that the only interaction of closed strings with themselves is cubic. Therefore, if it is valid to apply this limit directly on the Lagrangian, one would obtain a formulation of scalar-tensor gravitation involving cubic couplings only. The conjecture that such an astounding simplification of the formulation of gravitation does occur in the light cone gauge has been put forward by Cremmer and Gervais [*6*].

The validity of applying the zero-slope limit directly to the Lagrangian of the dual string field theory is not obvious. The dual model contains massive states that disappear at infinity in the limit. One could imagine that contributions to the amplitude arising from exchange of these states are non-vanishing in the limit, in which case it would be necessary to add higher order contact terms to the limting field theory to represent these effects. There is another example that is easier to study. Namely, for the case of open strings the limiting field theory is Yang-Mills theory [7]. This theory is obtained by applying the zero-slope limit directly to the string Lagrangian. This may appear encouraging for the truth of the Cremmer-Gervais conjecture, but the case of closed strings is somewhat different and it is by no means clear that it should work in the same way.

Since it is unclear whether the zero-slope limit of the closed string Lagrangian should yield scalar-tensor gravitation theory, we have chosen not to study that limit directly. (Also, it is technically rather difficult to carry out). Instead, we start with tensor gravity or scalar-tensor gravity, formulated in the usual covariant and gauge symmetric manner, since we know (from the work of reference [1]) that this is the theory corresponding to the zero-slope limit of the dual model. Light cone gauge conditions are then imposed and properties of the resulting formulation of the theory are explored.

In this paper formulas appropriate to four-dimensional spacetime are used throughout, even though they could be easily generalized to the arbitrary case. It should perhaps be remarked that even though the dual string models select a preferred 'critical' dimension (26 and 10 are the best known cases), the limiting field theories do not 'remember' this number. To put it another way, when the point-particle field theory is formulated for arbitrary dimension, there are no special features associated with the critical dimension of the corresponding string field theory.

In section 2 a brief discussion of Yang-Mills theory in the light cone gauge is presented. The purpose in doing this is to illustrate techniques in a context that is less mathematically cumbersome than gravitation theory. In section 3 pure tensor gravity is formulated in the light cone gauge. It is shown that when the gauge conditions are chosen suitably it is possible to solve for the remaining constrained components of the metric tensor by quadrature. The Lagrangian is then rewritten entirely in terms of the physical transverse components. In section 4 the light cone gauge is shown to be appropriate for finding exact wave solutions of the classical Einstein equations in empty space. A particular solution is presented as an illustration. Section 5 deals with scalar-tensor gravitation, since this is the theory that actually emerges from the dual models.

§(2): *YANG-MILLS THEORY IN THE LIGHT CONE GAUGE*

The techniques required to express gravitation in the light cone gauge can be illustrated using the simple example of Yang-Mills theory [8]. This example is sufficiently non-trivial to be of some interest in its own right, but the mathematics is substantially less complicated than in the case of gravitation. The Lagrangian for Yang-Mills theory is

$$\mathcal{L} = \frac{1}{4} \sum_{a} G_{\mu\nu}{}^{a} G^{\mu\nu a} \tag{2.1}$$

where

$$G_{\mu\nu}{}^{a} = \partial_\mu W_\nu{}^{a} - \partial_\nu W_\mu{}^{a} + g f_{abc} W_\mu{}^{b} W_\nu{}^{c}, \tag{2.2}$$

g is the coupling constant, and f_{abc} are the structure constants of a simple Lie algebra. For simplicity we will only discuss the algebra $SU(2)$, which requires taking $f_{abc} = \varepsilon_{abc}$.

The Lagrange equations of motion are easily obtained in the usual way. One finds

$$\Box W_\mu{}^{a} - \partial_\mu \partial \cdot W^{a} = g\varepsilon_{abc}[W_\nu{}^{b}\partial_\mu W^{\nu c} - W_\mu{}^{c}\partial_\nu W^{\nu b} - 2W^{\nu b}\partial_\nu W_\mu{}^{c}]$$

$$+ g^2[W_\mu{}^{a}W_\nu{}^{b}W^{\nu b} - W_\mu{}^{b}W_\nu{}^{a}W^{\nu b}]. \tag{2.3}$$

The light cone gauge is specified by the choice

$$W_+{}^{a} = \frac{1}{\sqrt{2}}(W_0{}^{a} + W_3{}^{a}) = 0. \tag{2.4}$$

Such a choice is possible because of the gauge invariance of the Yang-Mills theory. Our metric conventions are such that a dot product of two vectors A_μ and B_μ is

$$A \cdot B = A_\mu B^\mu = A_+ B_- + A_- B_+ - A_i B_i. \tag{2.5}$$

An index i, j, k, or ℓ is understood to run over the two transverse directions, and the summation convention is used. Setting $\mu = +$ in equation (2.3) and using equation (2.4), one obtains

$$\partial_+{}^2 W_-{}^{a} = \partial_+\partial_i W_i{}^{a} + g\varepsilon_{abc} W_i{}^{b}\partial_+ W_i{}^{c}. \tag{2.6}$$

This may be regarded as a constraint equation determining $W_-{}^{a}$ in terms of the transverse components $W_i{}^{a}$. Equation (2.6) may be formally integrated to read

$$W_-{}^{a} = \frac{\partial_i}{\partial_+} W_i{}^{a} + g\varepsilon_{abc}\frac{1}{\partial_+{}^2}[W_i{}^{a}\partial_+ W_i{}^{c}]. \tag{2.7}$$

The use of the operator $1/\partial_+$ deserves a few words of explanation. It is an indefinite integral and as such has the usual integration constant ambiguity. A particular choice of boundary conditions that is frequently made is described by the formula

$$\frac{1}{\partial_+} f(x^+) = \tfrac{1}{2}\int_{-\infty}^{+\infty} \varepsilon(x^+ - y^+) f(y^+) dy^+, \tag{2.8}$$

at least if this integral exists. In the context of formulating Feynman rules for calculating amplitudes, it is clear that the integration constant issue only arises for lines with the momentum component $p_+ = 0$. When calculating tree diagrams one can always choose a frame for which no line, external or internal, has such a momentum. In the case of loops, however, one cannot circumvent the fact that the integrated internal lines can have $p_+ = 0$. Loop calculations are beyond the scope of this paper, so we will not consider the matter any further at this time.

If we consider the transverse components of equation (2.3) and use equation (2.7) to eliminate $W_-{}^a$, we find the equation of motion

$$\begin{aligned}\Box W_i{}^a = {} & 2g\varepsilon_{abc}\left\{W_j{}^b\partial_j W_i{}^c - \frac{\partial_j}{\partial_+} W_j{}^b\partial_+ W_i{}^c + \frac{1}{\partial_+}\left[\partial_i W_j{}^b \partial_+ W_j{}^c\right]\right\} \\ & + g^2\left\{2\partial_+ W_i{}^b \frac{1}{\partial_+{}^2}\left[W_j{}^a\partial_+ W_j{}^b - W_j{}^b\partial_+ W_j{}^a\right]\right. \\ & \left. + 2W_i{}^b \frac{1}{\partial_+}\left[W_j{}^a\partial_+ W_j{}^b\right] - W_i{}^a W_j{}^b W_j{}^b\right\}. \end{aligned} \tag{2.9}$$

The Lagrangian that gives this equation of motion may be obtained by substituting equations (2.4,6) into equation (2.1). When this is done, and a certain number of total derivatives are dropped, one finds

$$\begin{aligned}\mathcal{L} = {} & -\tfrac{1}{2}\partial_\mu W_i{}^a\partial^\mu W_i{}^a + 2g\varepsilon_{abc}\frac{1}{\partial_+} W_i{}^a\partial_i W_j{}^b\partial_+ W_j{}^c \\ & + g^2\left\{\tfrac{1}{4}W_i{}^a W_i{}^a W_j{}^b W_j{}^b - \frac{1}{\partial_+}\left[W_i{}^a\partial_+ W_i{}^b\right]\frac{1}{\partial_+}\left[W_j{}^b\partial_+ W_j{}^a\right]\right\}. \end{aligned} \tag{2.10}$$

One can verify directly that equation (2.10) does in fact lead to equation (2.9). The reason that substitution of the constraint and gauge condition equations directly into the Lagrangian is valid can be understood by noting that equation (2.10) and equation (2.1) both give rise to the same Hamiltonian. This is easily proved by calculating the canonical momenta in each case.

The Lagrangian in equation (2.10) is the light cone gauge expression based entirely on the physical transverse fields. The price one pays for achieving this is a lack of manifest covariance and locality.

§(3): *SPIN 2 GRAVITY IN THE LIGHT CONE GAUGE*

In this section an analysis similar to that of the last section is applied to Einstein's theory of the self-interacting spin 2 field [9]. The Lagrangian for this theory is

$$\mathcal{L} = \frac{1}{16\pi G}\sqrt{g}R, \tag{3.1}$$

and the covariant equations of motion are

$$R_{\mu\nu} = \partial_\mu\Gamma_{\nu\rho}{}^\rho - \partial_\rho\Gamma_{\mu\nu}{}^\rho + \Gamma_{\mu\rho}{}^\tau\Gamma_{\nu\tau}{}^\rho - \Gamma_{\mu\nu}{}^\tau\Gamma_{\tau\rho}{}^\rho = 0, \tag{3.2}$$

where we follow the standard notation in which

$$\Gamma_{\mu\nu}{}^\rho = \tfrac{1}{2}g^{\rho\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\lambda\mu} - \partial_\lambda g_{\mu\nu}), \tag{3.3}$$

$$R = g^{\mu\nu}R_{\mu\nu}, \tag{3.4}$$

and $g^{\mu\nu}$ is the inverse of the metric tensor $g_{\mu\nu}$. This theory has coordinate transformations as its gauge group. The number of independent gauge functions is the same as the number of dimensions of space-time. Thus four relations may be imposed on $g_{\mu\nu}$ to constitute a choice of gauge.

The light cone gauge is specified by three conditions analagous to equation (2.4)

$$g_{++} = g_{+i} = 0. \tag{3.5}$$

This leaves one gauge condition still to be chosen. A simple choice for the remaining condition would be to set $g_{+-} = 1$. This choice does prove useful in the next section. However, for present purposes it is not the most convenient one because it results in unpleasant integral equations for the remaining constrained components of $g_{\mu\nu}$. To see this we set

$$g_{ij} = e^\psi\gamma_{ij}, \tag{3.6}$$

$$\det\gamma_{ij} = 1, \tag{3.7}$$

$$g_{+-} = e^\phi. \tag{3.8}$$

These equations simply serve to define the variables ψ, ϕ, and γ_{ij}. Using equations (3.5-8), the relation $R_{++} = 0$ now takes the form

$$2\partial_+\phi\partial_+\psi - 2\partial_+{}^2\psi - (\partial_+\psi)^2 + \tfrac{1}{2}\partial_+\gamma^{ij}\partial_+\gamma_{ij} = 0, \qquad (3.9)$$

where γ^{ij} is defined to be the inverse of γ_{ij}. This equation can be solved explicitly for ψ by quadrature if we choose as the last gauge condition

$$\phi = \tfrac{1}{2}\psi. \qquad (3.10)$$

This choice allows equation (3.9) to be integrated (in analogy with equation (2.7)) to give

$$\psi = \frac{1}{4}\frac{1}{\partial_+{}^2}[\partial_+\gamma^{ij}\partial_+\gamma_{ij}]. \qquad (3.11)$$

Our purpose in these manipulations is to express the Lagrangian entirely in terms of two fields, corresponding to the physical degrees of freedom of a graviton. The matrix γ_{ij} has the correct number of components since it is unimodular and symmetric. It may be parametrized in the form

$$\gamma_{ij} = -\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} e^{\rho} & 0 \\ 0 & e^{-\rho} \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= -\begin{bmatrix} \cosh\rho + \sinh\rho\cos2\theta & -\sinh\rho\sin2\theta \\ -\sinh\rho\sin2\theta & \cosh\rho - \sinh\rho\cos2\theta \end{bmatrix}. \qquad (3.12)$$

Then ρ and θ may be regarded as the two independent fields.

The remaining components of $g_{\mu\nu}$ that need to be related to γ_{ij} are g_{--} and g_{-i}. It is equivalent, and somewhat more convenient, to derive formulas for g^{++} and g^{+i}. g_{--} and g_{-i} can then be inferred using the relations

$$g_{-i} = -e^{3\psi/2}\gamma_{ij}g^{+j}, \qquad (3.13)$$

$$g_{--} = e^{-\psi}\gamma^{ij}g_{-i}g_{-j} - e^{\psi}g^{++}, \qquad (3.14)$$

which follow from equations (3.6-8,10) and the fact that $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. Constraint equations determining g^{+i} are obtained from $R_{+i} = 0$. Remarkably, these equations also can be solved by quadrature to yield

$$g^{+i} = e^{-\psi/2}\frac{1}{\partial_+}\Big[\gamma^{ij}e^{-3\psi/2}\frac{1}{\partial_+}\{e^{\psi}[\tfrac{1}{2}\partial_+\gamma^{k\ell}\partial_j\gamma_{k\ell} + \partial_\ell(\gamma^{k\ell}\partial_+\gamma_{jk})$$

$$+ \partial_\ell\psi\gamma^{k\ell}\partial_+\gamma_{jk} + \frac{1}{2}\partial_j\psi\partial_+\psi - \frac{3}{2}\partial_+\partial_j\psi]\}\Big]\,. \qquad (3.15)$$

The constraint equation for g^{++} is determined similarly from $R_{+-} = 0$. Once again quadratures are possible giving the result

$$g^{++} = -e^{-\psi}\frac{1}{\partial_+}\Big\{\frac{3}{2}e^{2\psi}\gamma_{ij}\partial_+(g^{+i}g^{+j}) + \frac{1}{2}e^{\psi}g^{+i}\partial_i\psi$$

$$+ (\frac{7}{2}\partial_+\psi\gamma_{ij} + 2\partial_+\gamma_{ij})e^{2\psi}g^{+i}g^{+j}$$

$$+ e^{-\psi/2}\frac{\partial_i}{\partial_+}[\frac{1}{2}e^{\psi/2}\gamma^{ij}\partial_j\psi$$

$$+ e^{3\psi/2}(\frac{3}{2}\partial_+\psi g^{+i} + \partial_+ g^{+i} + \gamma^{ij}\partial_+\gamma_{jk}g^{+k})]$$

$$+ e^{-\psi/2}\frac{1}{\partial_+}[e^{\psi}(\partial_+\psi\partial_-\psi + 3\partial_+\partial_-\psi - \frac{1}{2}\partial_+\gamma_{ij}\partial_-\gamma^{ij})]\Big\}\,. \qquad (3.16)$$

At this point it is possible to obtain the equations of motion for γ_{ij} by taking $R_{ij} = 0$ and using the equations obtained above to eliminate the other variables. However, it is quicker and more concise to make the substitutions directly in the Lagrangian so as to obtain an expression analogous to equation (2.10). This procedure is valid for the same reasons as in the Yang-Mills case—performing the eliminations on the Lagrangian turns out to be equivalent to performing them on the Hamiltonian. The mathematics is cumbersome, but not as terrible as one might expect. In particular, the dependence on g^{++} and g_{--} drops out without having to make explicit use of equation (3.16). Dropping some total derivatives one obtains the final result

$$\mathcal{L} = e^{\psi/2}\{\gamma^{ij}\partial_i\partial_j\psi - \frac{3}{4}\gamma^{ij}\partial_i\psi\partial_j\psi + \gamma^{ik}\partial_i\gamma^{j\ell}\partial_j\gamma_{k\ell}$$

$$- \frac{1}{2}\gamma^{ij}\partial_i\gamma^{k\ell}\partial_j\gamma_{k\ell}\} + e^{\psi}\{4\partial_+\partial_-\psi - \partial_+\gamma^{ij}\partial_-\gamma_{ij}\}$$

$$+ e^{-3\psi/2}\gamma^{ij}\frac{1}{\partial_+}R_i\frac{1}{\partial_+}R_j\,, \qquad (3.17)$$

where

$$R_i = \tfrac{1}{2}e^{\psi}\{\partial_+\gamma^{jk}\partial_i\gamma_{jk} + \partial_i\psi\partial_+\psi - 3\partial_+\partial_i\psi\} + \partial_k(e^{\psi}\gamma^{jk}\partial_+\gamma_{ij}). \quad (3.18)$$

Requiring the action formed from this Lagrangian to be stationary with respect to variations of the fields θ and ρ defined in equation (3.12) gives equations of motion involving these fields only. Clearly, the equations that would be obtained in this way are rather complicated.

While equation (3.17) is undeniably rather formidable, and perhaps not very suitable for explicit calculations of complicated Feynman graphs, we would like to make the point that it might have been much worse. Given the complexity of the formulas implied by $R_{\mu\nu} = 0$, we find it remarkable that there is a gauge in which the unphysical field components can be explicitly related to the physical ones by quadrature, rather than by intractable integral equations.

In principle, the Lagrangian of equation (3.17) could be the starting point for quantizing gravity in the light cone gauge. One would have to prove explicitly the Lorentz invariance of the S matrix. Although it seems likely that such a proof could be obtained, we have not attempted to construct it because of the technical difficulties associated with normal ordering, in particular.

§(4): *EXACT CLASSICAL WAVE SOLUTION OF THE EINSTEIN EQUATIONS*

It is of some physical interest to find exact classical solutions of the empty space Einstein equations (3.2). Some examples of wave solutions are known, but they only begin to scratch the surface of the class of all possible solutions [*10*]. It seems to us that the light cone gauge should be well suited for investigating the possibilities because the restriction,

$$\partial_+ g_{\mu\nu} = 0, \quad (4.1)$$

corresponding to waves travelling in the positive z direction, tends to eliminate a large number of terms from the equations of motion when used in conjunction with equation (3.5). We will set up the appropriate equations in this section, and only look very superficially for particular solutions.

For the present purpose it is convenient to maintain equation (3.5) and the definition of equations (3.6,7), but to replace equations (3.8,10) by the gauge condition

$$g_{+-} = 1. \quad (4.2)$$

Given these choices and equation (4.1) one finds that the condition $R_{++} = R_{+i} = R_{+-} = 0$ are trivially satisfied. The equations $R_{ij} = 0$ take the form

$$\gamma_{ij}\partial_\ell(\partial_k\psi\gamma^{k\ell}) - \tfrac{1}{2}\partial_i\gamma^{k\ell}\partial_j\gamma_{k\ell} - \partial_\ell\{\gamma^{k\ell}(\partial_i\gamma_{jk} + \partial_j\gamma_{ik} - \partial_k\gamma_{ik})\}$$

$$+ \gamma^{k\ell}\gamma^{mn}\partial_\ell\gamma_{im}(\partial_n\gamma_{kj} - \partial_k\gamma_{nj}) = 0. \qquad (4.3)$$

Equation (4.3) may be regarded as three simultaneous equations for the unknown functions ρ, θ, and ψ defined in equations (3.6,12). Once a solution of these equations is known, g_{i-} may be found by solving the equation that arises from $R_{i-} = 0$,

$$\partial_i\partial_-\psi - e^{-\psi}\partial_j(e^{\psi}\gamma^{jk}\partial_-\gamma_{ik}) - \tfrac{1}{2}\partial_i\gamma^{k\ell}\partial_-\gamma_{k\ell} \qquad (4.4)$$

$$- \partial_j\{e^{-\psi}\gamma^{jk}(\partial_i g_{k-} - \partial_k g_{i-})\} + e^{-\psi}\gamma^{jk}\gamma^{\ell m}\partial_j\gamma_{i\ell}(\partial_m g_{k-} - \partial_k g_{m-}) = 0.$$

Given a solution of this equation as well, the final step is to determine g_{--} from the equation corresponding to $R_{--} = 0$,

$$2\partial_-^{\,2}\psi + (\partial_-\psi)^2 - \tfrac{1}{2}\partial_-\gamma^{k\ell}\partial_-\gamma_{k\ell} - e^{-\psi}\partial_i[\gamma^{ij}(2\partial_- g_{j-} - \partial_j g_{--})] \qquad (4.5)$$

$$+ e^{-2\psi}\gamma^{ij}\gamma^{k\ell}\partial_j g_{k-}(\partial_\ell g_{i-} - \partial_i g_{\ell-}) = 0.$$

The equations (4.2-4) for the metric tensor are much simpler than they would be without the restriction to the light cone gauge and x^+-independent functions. They could be studied in a systematic fashion to obtain large classes of solutions. However, this is far afield from our original purposes in this matter, so we will merely illustrate the possibilities with a simple example. Specifically, we consider $\rho = \theta = 0$, corresponding to $\gamma_{ij} = -\delta_{ij}$. In this case equation (4.3) reduces to the requirement that ψ is harmonic in the transverse variables

$$\nabla^2\psi(\mathbf{x},x^-) = 0. \qquad (4.6)$$

Equations (4.4,5) become

$$\partial_i\partial_-\psi + \partial_j\{e^{-\psi}(\partial_i g_{j-} - \partial_j g_{i-})\} = 0, \qquad (4.7)$$

and

$$2\partial_-^{\,2}\psi + (\partial_-\psi)^2 + e^{-\psi}(2\partial_i\partial_- g_{i-} - \nabla^2 g_{--})$$

$$+ e^{-2\psi}\partial_i g_{j-}(\partial_j g_{i-} - \partial_i g_{j-}) = 0, \qquad (4.8)$$

respectively. A further simplification is achieved by assuming

$\partial_-\psi = 0$. We then find the solution:

$$g_{ij} = -e^{\psi(\mathbf{x})}\delta_{ij}, \tag{4.9a}$$

$$g_{i-} = \partial_i\phi(\mathbf{x},x^-), \tag{4.9b}$$

$$g_{--} = 2\partial_-\phi(\mathbf{x},x^-) + \chi(\mathbf{x},x^-), \tag{4.9c}$$

where

$$\nabla^2\psi = \nabla^2\chi = 0. \tag{4.10}$$

The arbitrary function ϕ may be eliminated by making a change of coordinate system of the type

$$y^+ = x^+ + \phi(\mathbf{x},x^-). \tag{4.11}$$

Therefore our solution can be written in the form

$$ds^2 = 2dx^+dx^- + \chi dx^-dx^- - e^{\psi}d\mathbf{x}\cdot d\mathbf{x}. \tag{4.12}$$

This is a modest generalization of a solution found by Peres [*11*], which corresponds to the special case $\psi = 0$.

§(5): *SCALAR-TENSOR GRAVITATION*

In reference [*1*] it was shown that the small-slope expansion of dual models does not give the ordinary gravitation theory discussed in the preceding sections, but scalar-tensor theory instead. The Lagrangian may be written in the form

$$\mathcal{L} = \sqrt{g}\left(\frac{1}{16\pi G}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right). \tag{5.1}$$

Regarding the experimental status of the scalar field we may note the following. Stringent limits exist on the possible couplings of a massless scalar field to ordinary matter. If a refinement of the theory gives mass to the scalar it is 'safe'. Alternatively, one could imagine a circumstance in which the scalar field did not couple directly to matter at all, in which case it would only represent a modification of the ordinary Einstein theory at the loop level. In the field theory described by equation (5.1) loops are problematical because it is not renormalizable [*12*]. However, dual models with a small (but non-zero) slope probably are renormalizable. In using a dual model for a unified theory of weak, electromagnetic, and gravitational interactions along the lines suggested in reference [*1*], one expects loops to represent corrections at most of order α, the fine structure constant.

$g_{\mu\nu}$ and ϕ may be unified into a single field by choosing as part of the choice of gauge

$$\phi \propto \sqrt{g} - 1. \tag{5.2}$$

One may also alter the form of the resulting Lagrangian by a field redefinition of the type

$$\tilde{g}_{\mu\nu} = g^{\beta} g_{\mu\nu}. \tag{5.3}$$

The result of these substitutions is a Lagrangian of the form

$$\mathcal{L} = g^{-\frac{1}{2}d}(\mathcal{L}_1 - \tfrac{1}{2}\mathcal{L}_2 + c\mathcal{L}_3 + d\mathcal{L}_4), \tag{5.4}$$

where c and d are arbitrary parameters and

$$\mathcal{L}_1 = g^{\rho\lambda}\partial_\rho g^{\mu\nu}\partial_\mu g_{\lambda\nu}, \tag{5.5a}$$

$$\mathcal{L}_2 = g^{\rho\lambda}\partial_\rho g^{\mu\nu}\partial_\lambda g_{\mu\nu}, \tag{5.5b}$$

$$\mathcal{L}_3 = g^{\mu\nu}(g^{\rho\lambda}\partial_\mu g_{\rho\lambda})(g^{\sigma\tau}\partial_\nu g_{\sigma\tau}), \tag{5.5c}$$

$$\mathcal{L}_4 = \partial_\lambda g^{\rho\lambda}(g^{\sigma\tau}\partial_\rho g_{\sigma\tau}). \tag{5.5d}$$

This Lagrangian represents scalar-tensor theory because the gauge invariance has been reduced from arbitrary coordinate transformations, characteristic of a pure tensor theory, to those of unit Jacobian (provided c and d do not satisfy a certain relation). A convenient choice that appears most akin to the covariant rules for the corresponding dual model is to take $c = d = 0$. Then

$$\begin{aligned}\mathcal{L} &= g^{\rho\lambda}\partial_\rho g^{\mu\nu}(\partial_\mu g_{\lambda\nu} - \tfrac{1}{2}\partial_\lambda g_{\mu\nu}) \\ &= g^{\mu\nu}\Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\rho}{}^{\lambda}.\end{aligned} \tag{5.6}$$

The equations of motion resulting from the Lagrangian in equation (5.6) may be written compactly as

$$\partial_\rho\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\rho}{}^{\lambda}. \tag{5.7}$$

An alternative way of obtaining these equations of motion is to start from

$$\mathcal{L}' = g^{\mu\nu}(\Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\rho}{}^{\lambda} - \partial_\rho\Gamma_{\mu\nu}{}^{\rho}), \tag{5.8}$$

treating $g^{\mu\nu}$ and $\Gamma_{\mu\nu}{}^{\rho}$ as independent fields. In this case variation with respect to $g^{\mu\nu}$ gives equation (5.7), whereas variation with respect to $\Gamma_{\mu\nu}{}^{\rho}$ gives

$$\partial_\rho g^{\mu\nu} + g^{\mu\lambda}\Gamma_{\lambda\rho}{}^{\nu} + g^{\nu\lambda}\Gamma_{\lambda\rho}{}^{\mu} = 0, \tag{5.9}$$

which is just the usual statement that $g^{\mu\nu}$ has vanishing covariant derivative.

We have explored the possibilities for restricting the Lagrangian of equation (5.6) to the light cone gauge. Manipulations analogous to those of section 3 can be carried out. Suffice it to say that the formulas are of complexity comparable to that in the pure spin 2 case. Our motivation for repeating these tedious calculations was to check the Cremmer-Gervais conjecture that scalar-tensor gravitation in the light cone gauge can be formulated with a cubic interaction term only. The fact that the formula we obtain is not in this form does not in itself constitute a disproof. However, we have considered the possibilities for redefining fields — even in non-local ways — and have just about convinced ourselves that no such transformation can eliminate all the quartic and higher order interactions. As mentioned in the introduction, this means that the zero-slope limit of the closed-string theory cannot be found by applying the limit directly to the Lagrangian. One needs to include additional contact terms to represent the effects of exchanging massive states that are not present in the limiting spectrum.

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