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New Solutions for Charged Spheres in General Relativity

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Exact solutions of the Einstein–Maxwell field equations are obtained for the case of static and spherically symmetric distribution of charged matter. The solutions are obtained through the extension of a method originally used for neutral configurations and are conveniently matched to the Reissner–Nordstrom exterior metric. The physical plausability of the solutions is discussed and it is shown that some of them reduce in appropriate limits to known neutral or charged solutions.

1. INTRODUCTION

It is well known that the Reissner–Nordstrom metric is the unique static, spherically symmetric, and asymptotically flat solution of Einstein–Maxwell coupled equations. The problem of finding analytically exact static interior solutions for the Reissner–Nordstrom metric has focused the interest of many researchers [1–6] because such solutions can describe the equilibrium configurations of collapsing distributions of charged matter whose collapse is countered by the coulombian repulsion due to the electric charge.

In this paper we present new solutions for charged spheres by adapting a method first used to reproduce known solutions representing neutral spheres [7]. The solutions are given in terms of a generating function which is conveniently adjusted to the corresponding functions of the Reissner–Nordstrom metric in order to match both metrics. We illustrate the method by considering first the case of charged incoherent

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matter obtaining a family of solutions which generalizes previous solutions admitting a conformal Killing vector [8]. For a particular case we find that the charge density and the energy density bear a constant relation through the distribution. As a second example we find the uniformly charged analogue of the Tolman V solution [9]. A third solution for a neutral sphere satisfying physically plausible conditions is also presented. The paper is organized as follows. In Section 2 we present the field equations and the method used to obtain the solutions. In Section 3 the matching conditions and the examples are given.

2. FIELD EQUATIONS AND THE GENERATING FUNCTION

We are concerned with static, spherically symmetric solutions of Einstein–Maxwell combined equations. It is convenient to use standard Schwarzschild coordinated $x^\alpha = (t, r, \theta, \phi)$, with respect to which the line element assumes the form

$$ds^2 = e^\nu\, dt^2 - e^\lambda\, dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{1}$$

where ν and λ are functions of the radial coordinate.

The energy-momentum tensor for the charged matter can be expressed as

$$T^\alpha{}_\beta = (\rho + p)\, U^\alpha U_\beta - p\delta^\alpha{}_\beta + \tfrac{1}{4}\pi(F^{\alpha\lambda}F_{\lambda\beta} + \tfrac{1}{4}\delta^\alpha{}_\beta F_{\mu\nu}F^{\mu\nu}) \tag{2}$$

where ρ, p, and U^α are, respectively, the energy density, the isotropic pressure, and the unit timelike four-velocity of the fluid, and $F^{\alpha\beta}$ is the Maxwell field tensor.

The resulting Einstein–Maxwell equations are

$$8\pi\rho + Q^2/r^4 = e^{-\lambda}(\lambda'/r - 1/r^2) + 1/r^2 \tag{3}$$

$$8\pi p - Q^2/r^4 = e^{-\lambda}(\nu'/r + 1/r^2) - 1/r^2 \tag{4}$$

$$8\pi p + Q^2/r^4 = e^{-\lambda}[\nu''/2 + \nu'^2/4 - \lambda'\nu'/4 + (\nu' - \lambda')/2r] \tag{5}$$

where we have defined

$$Q(r) = 4\pi \int_0^r \sigma r^2 e^{\lambda/2}\, dr \tag{6}$$

which represents the total electric charge obtained within a sphere of radius r, and σ is the charge density.

Hereafter a prime denotes differentiation with respect to r. We need the Bianchi identity $T^{\alpha}{}_{1;\alpha}=0$ which, as can be recalled, is not independent of the field equations. A simple calculation yields

$$p' + \tfrac{1}{2}(\rho + p)\nu' = \left[\frac{1}{8\pi r^4}\right] d(Q^2)/dr \tag{7}$$

Furthermore, Eq. (3) can be integrated to give

$$e^{-\lambda} = 1 - 2m(r)/r + Q^2/r^2 \tag{8}$$

where we introduced the mass function $m(r)$ defined as

$$m(r) = \int_0^r (4\pi\rho r^2 + QQ'/r)\, dr \tag{9}$$

Substitution of Eqs. (7) and (8) into Eq. (4) gives

$$8\pi p + 1/r^2 - Q^2/r^4$$
$$= r[1 - 2m(r)/r + Q^2/r^2]\{1/r^3 - 2(p' - Q^{2\prime}/8\pi r^4)/[r^2(\rho + p)]\} \tag{10}$$

Defining a function $G(r)$ by

$$G \equiv -\frac{r[1 - 2m(r)/r + Q^2/r^2]}{8\pi p + 1/r^2 - Q^2/r^4} \tag{11}$$

Eq. (10) can be rewritten in the form

$$8\pi p' + \frac{(r^3 + G)(r^2 + G')}{G(G - r^3)}\, 8\pi p$$
$$+ \frac{(r^3 + G)}{r^3 G(G - r^3)}\,(r^3 + G'r - 2G - Q^2 r - (GQ^2/r^4)' r^3] + \frac{2Q^{2\prime}}{(G - r^3) r} = 0 \tag{12}$$

It should be noted that given G and the charge distribution as known functions of r, the solution of the linear differential Eq. (12) takes the general form

$$8\pi p(r) = e^{\int -B(r)\,dr}\left[p_0 + \int C(r)\, e^{\int B(r)\,dr}\, dr\right] \tag{13}$$

where p_0 is an integration constant, and $B(r)$ and $C(r)$ are

$$B(r) = (G + r^3)(G' + r^2)/G(G - r^3)$$

$$C(r) = -\frac{(G + r^3)}{Gr^3(G - r^3)}\,[r^3 + G'r - 2G - Q^2 r - (GQ^2/r^4)' r^3]$$
$$- 2Q^{2\prime}/(G - r^3) r$$

After obtaining $p(r)$, the energy density $\rho(r)$ is easily calculated from Eqs. (9) and (11), obtaining

$$8\pi\rho(r) = 1/r^2\{1 + G'(8\pi p + 1/r^2 - Q^2/r^4) + G[8\pi p' - 2/r^3 - (Q^2/r^4)'] + Q^2/r^2\} \tag{14}$$

Finally, taking into account Eqs. (7)–(9) and (11), the metric coefficients in terms of G (and Q) can be expressed as

$$e^{-\lambda} = -G/r(8\pi p + 1/r^2 - Q^2/r^4) \tag{15}$$

$$e^{\nu} = (A^2/r)e^{-\int(r^2/G)dr} \tag{16}$$

where A^2 is a constant.

3. THE SOLUTIONS

We should like to stress that any given function $G(r)$ and charge distribution generates a static and spherically symmetric solution of Einstein–Maxwell equations. The function G must satisfy some general requirement if its associated solution is to be physically meaningful. Thus, the regularity conditions at the origin $r=0$ $[m(r)/r \to 0$ and $Q^2(r)/r^2 \to 0$ as $r \to 0]$ imply that $\lim_{r\to 0} G(r)/r^3 = -1$, assuming a nondivergent pressure at the origin. For instance, if $G(r) = -r^3$ and $Q(r)=0$ one obtains Minkowski flat space-time as a trivial solution. The following choice

$$G^{\text{R.N.}}(r) = -\frac{r^3(1 - 2M/r + e^2/r^2)}{1 - e^2/r^2} \tag{17}$$

$$Q(r) = e \tag{18}$$

where M and e are constants, gives the vacuum Reissner–Nordstrom solution as can be verified.

Any interior solution must join smoothly to the Reissner–Nordstrom metric at the surface $r = r_0$ of the distribution. This requirement is satisfied provided that

$$G(r_0) = G^{\text{R.N.}}(r_0) \tag{19}$$

$$Q(r_0) = e \tag{20}$$

The last equation implies the continuity of the radial electric field assuming no charge concentration at the boundary surface.

We turn now to the examples.

3.1. Charged Dust Solution

As a first case, consider

$$G(r) = \alpha r^3 \tag{21}$$

where α is a dimensionless constant which is to be calculated in order to satisfy the boundary condition (19), thus

$$\alpha = -\frac{(1 - 2M/r_0 + e^2/r_0^2)}{1 - e^2/r_0^2} \tag{22}$$

The charge distribution will be chosen so that $p(r) = 0$, thus we have a charged incoherent matter. Making use of (12) we obtain for $Q(r)$ the expression

$$Q^2(r) = r^{(\alpha+1)/\alpha}[C + (\alpha + 1/\alpha - 1)^2 r^{(\alpha-1)/\alpha}] \tag{23}$$

where C is an integration constant to be calculated from Eq. (20).

The energy density and the metric can be obtained from Eqs. (14)–(16) as

$$8\pi\rho = 1/r^2[-2Cr^{(1-\alpha)/\alpha} - 4\alpha(\alpha+1)/(\alpha-1)^2] \tag{24}$$

$$e^{-\lambda} = -\alpha[1 - Cr^{(1-\alpha)/\alpha} - (\alpha+1)^2/(\alpha-1)^2] \tag{25}$$

$$e^{\nu} = A^2 r^{-(\alpha+1)/\alpha} \tag{26}$$

Rearrangement of Eq. (22) gives at once

$$M = (1+\alpha)/2r_0 + (1-\alpha)e^2/2r_0 \tag{27}$$

This equation relates the total mass of the distribution to the parameter α, the second term at the right being interpreted as the increase in the mass caused by the electric energy. If we restrict ourselves to cases for which $M > e$, and the boundary of the sphere lies outside the Reissner–Nordstrom gravitational radius, $r_+ = M + \sqrt{M^2 - e^2}$, then Eq. (22) shows that α (as well as C) is negative. Therefore, the electromagnetic contribution to the total mass is positive as it should be.

Incidentally, note that after setting $Q(r) = 0$, Eq. (23) gives $\alpha = -1$ and $C = 0$, then feeding back these values in Eq. (27), we obtain $M = 0$, thus, Eq. (23) precludes a static neutral dust sphere, as expected.

Let us consider the case in which $C = 0$ in Eq. (23). Then the boundary condition (20) establishes a fixed value for α, given by

$$\alpha = \frac{(1 \pm e/r_0)}{(1 \mp e/r_0)} \tag{28}$$

Comparison with Eq. (22) then shows that

$$M = \pm e \tag{29a}$$

Moreover, using Eq. (24) with $C = 0$ and Eq. (6), we obtain for the charge density the expression

$$8\pi\sigma = \pm\frac{4\alpha(\alpha+1)}{r^2(\alpha-1)^2} \tag{30}$$

From Eq. (24) it follows that

$$\rho = \pm\sigma \tag{29b}$$

Observe that ρ will be everywhere positive for $-1 < \alpha < 0$.

Equations (29) are known to hold for charged dust spheres in equilibrium under general requirements (see, for example, Refs. 1 and 10). If $C \neq 0$, Eq. (23) shows that Q^2/r^2 diverges as $r \to 0$ and the regularity conditions required for the general result (29) are not satisfied.

Finally, it should be pointed out that the solution described by Eqs. (23)–(26) represents a generalization of the charged dust solution admitting a conformal Killing vector found in Ref. 8; in fact, their results are recovered for the particular value $\alpha = -\frac{1}{3}$.

3.2. Charged Analogues of the Tolman Solution

We will take again $G(r)$ as given by Eq. (21) and consider that the proper charge density is constant, then Eq. (6) implies $Q(r) \sim r^3$. The appropriate junction condition at r_0 yields

$$Q(r) = e(r/r_0)^3 \tag{31}$$

The matter variables calculated from Eqs. (13) and (14) are found to be

$$8\pi p(r) = p_0 r^{-A} + \frac{B}{A-2} r^{-2} + \frac{D}{A-2} r^2 \tag{32}$$

$$8\pi\rho(r) = \left[\frac{4D\alpha}{(\alpha+1)(A+2)} - \frac{D}{A+2} - \frac{12e^2}{r_0^6}\frac{\alpha}{(\alpha+1)}\right] r^2 - \frac{B}{A-2}\left[4\alpha/(\alpha+1)+1\right] r^{-2} - p_0\left(1+\frac{2\alpha A}{\alpha+1}\right) r^{-A} \tag{33}$$

where the constants A, B, and D are given by

$$A = \frac{(\alpha+1)(3\alpha+1)}{\alpha(\alpha-1)}$$

$$B = -\frac{(\alpha+1)^2}{\alpha(\alpha-1)}$$

$$D = e^2/r_0^6 \frac{(5\alpha^2 - 6\alpha + 1)}{\alpha(\alpha-1)}$$

This solution represents the uniformly charged version of the Tolman V solution [9] [with a slight change in notation, his n corresponding to $-(\alpha+1)/2\alpha$)].

For neutral spheres (i.e., $D=0$), Tolman's results are recovered. Furthermore, if $\alpha = -\frac{1}{3}$, then $A=0$ and (28) and (29) give the charged analogues of a solution found previously in Refs. 11–13, this solution corresponding to the particular case $n=1$ of the Tolman V solution.

3.3. A New Solution

Consider the following choice for $G(r)$:

$$G(r) = \frac{r^3(a+br^2)}{a+cr^2} \tag{34}$$

where a, b, and c are constants. Observe that G satisfies the appropriate conditions at the origin for $a \neq 0$, so we expect the solution to be regular. For the sake of simplicity let us treat the neutral fluid limit, so $Q(r)=0$.

A rather cumbersome calculation gives for the matter variables the expressions

$$8\pi p(r) = a + (b-c)r^2/4 \tag{35}$$

$$8\pi\rho(r) = a + (3b+c)r^2/4 \tag{36}$$

It is clear that $a = 8\pi p_c = 8\pi\rho_c$, where the subscript c refers to the central value of the variables. On the other hand, if the pressure is to vanish at r_0, then

$$b - c = \frac{-4}{r_0^2} 8\pi p_c$$

so that Eqs. (35) and (36) become

$$8\pi p(r) = 8\pi p_c(1 - r^2/r_0^2) \tag{37}$$

$$8\pi\rho(r) = 8\pi\rho_c + \left(c - \frac{24\pi\rho_c}{r_0^2}\right) r^2 \tag{38}$$

Observe that $p(r)$ is a positive decreasing function of r satisfying the regularity condition [3] $\partial p/\partial r = 0$ at the origin. Furthermore, if $16\pi\rho_c/r_0^2 \leqslant c \leqslant 24\pi\rho_c/r_0^2$, then ρ is everywhere a nonsingular positive decreasing function of r. The dominant energy condition $\rho(r) \geqslant p(r)$ is also fulfilled. The additional requirement $dp/d\rho \leqslant 1$ implies that $c \leqslant 16\pi\rho_c/r_0^2$, so that the value of c must satisfy $c = 16\pi\rho_c/r_0^2$, thus obtaining the stiff equation of state $p = \rho$.

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