General Relativity and Gravitation, Vol. 24, No. 5, 1992

Physical Interpretation of Vacuum Solutions of Einstein's Equations. Part I. Time-independent Solutions

W. B. Bonnor[1]

Received 17 December 1991

This article is a review of interpretations which have been given to some well known solutions of the vacuum equations. Special attention is paid to those of Schwarzschild, Curzon and Kerr, and it is argued that the bizarre topologies they have been endowed with are physically unrealistic. Among others discussed are the two-centres solution of Bach and Weyl, the NUT solution, and solutions for an infinite line-mass, both static and rotating.

1. INTRODUCTION

Relativists have not been diligent in interpreting solution of Einstein's equations. Thus in the book of exact solutions [1] one finds many whose physical meaning is unknown, or only partially understood.

One reason for this may be that interpretation is difficult and uncertain. Another reason is the following. The observational verification of general relativity, now and in the near future, is likely to depend on a very small number of exact solutions: those of Schwarzschild and possibly Kerr, cosmological models, and perhaps some gravitational wave metrics. Some workers feel it is a waste of time to try to interpret metrics which have no prospect of observational verification.

Yet it seems to me that we cannot claim to understand general relativity unless we can determine the physics behind the exact solutions we

[1] Queen Mary and Westfield College, Mile End Road, London E1 4NS, UK

0001-7701/92/0500-0551$06.50/0

know. Of course, it may be—though it seems a heresy to say so—that behind some of the solutions there is no physics, or wrong physics. If this is the case it is important to know it, and so find out the physical limits of the theory.

To me the essence of interpretation is to understand the *sources* in the exact solutions. This is certainly so in the classical field theories of gravitation and electromagnetism. In the vacuum metrics of general relativity sources appear as singularities, and singularities in Einstein's equations are notoriously difficult to handle. Much of this article will be about singularities and what they are likely to represent. I shall also in some cases allow the filling in of the singular region with matter.

My intention in this work, and in Part II to follow, is to describe the known physical interpretations of some of the more important exact vacuum metrics, i.e. solutions of

$$R_{ik} = 0. \tag{1}$$

(The cosmological constant is set equal to zero throughout.) Therefore it is a review article. There are many relevant papers which should have been mentioned but have not, and I wish to apologise to disappointed colleagues. The omissions have been caused partly by my ignorance and partly by the need to keep this article to a readable length. Where research papers have been reported in textbooks I have usually given the textbook reference. I emphasise that this article is not a history of interpretations.

Two omissions should be explicitly mentioned. First, I have paid scant attention to a considerable literature on static plane metrics: these are only briefly mentioned. Secondly, I have left out altogether solutions referring solely to topological defects such as strings and domain walls.

The Schwarzschild solution is discussed in Section 2. Section 3 is about static axially symmetric metrics, including brief descriptions of the singularity structure of the Curzon solution, and the Bach–Weyl two-centres solution. In Section 4 I turn to stationary axially symmetric solutions, paying particular attention to the Kerr, NUT and van Stockum metrics. Section 5 considers how far understanding can be improved if one allows in-filling of singular regions with reasonable matter. There is a conclusion in Section 6.

I use units such that $c = 1$, $G = 1$, and signature $---+$. General relativity is abbreviated as GR. The following letters always denote constants: a, b, e, h, k, m, m_1, m_2, r_0, z_1, A, B, C, R_0, ϵ, θ_0, ϕ_0, σ, σ_0. By a physical singularity I shall normally mean a region where the scalar $K = R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ is infinite.

2. THE SCHWARZSCHILD SOLUTION

If one solves (1) for a static, spherically symmetric space-time one expects to obtain a metric referring to the field outside a single static, spherical particle. This is the Schwarzschild solution. I shall not attempt to give a complete description of all the efforts that have been made to interpret the Schwarzschild solution. A good account is given in [2]. Here I confine myself to some of the more important aspects.

The Schwarzschild metric in its standard form is

$$ds^2 = -(1-2mr^{-1})^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + (1-2mr^{-1})dt^2, \quad (2)$$
$$r > 0, \qquad \pi \geq \theta \geq 0, \qquad 2\pi \geq \phi \geq 0, \qquad \infty > t > -\infty.$$

In this g_{tt} is zero at $r = 2m$, which I shall call the *Schwarzschild surface.* A test particle moving on a radial geodesic takes infinite coordinate time to reach $r = 2m$ from a finite radial coordinate whereas it can be calculated that such a particle falls from $r = r_0$ to $r = 0$ in finite proper time. Hence the coordinate t is unsuitable near $r = 2m$, and metric (2) has a coordinate singularity there, though not a physical one.

Although (2) has a coordinate singularity at $r = 2m$ it is a valid solution of the vacuum equations for $r > 2m$ and $r < 2m$. For $r < 2m$ something very strange happens: the coefficients g_{rr} and g_{tt} both change sign so t becomes a space-like and r a time-like coordinate. The metric is neither static nor spherically symmetric. This is the region of final approach to the black hole which is supposed to exist at $r = 0$; notice that this (physical) singularity is space-like. Evidently the object being described is very far from the static spherical particle which we may have expected to find when we set out to obtain (2).

For $r > 2m$ (2) is physically sensible and is used with success to predict the GR tests of the solar system. However, sometimes the isotropic form of the metric is used. This is obtained by transforming to a new radial coordinate ρ by

$$r = \rho\left(1 + \frac{m}{2\rho}\right)^2, \qquad \rho > 0, \qquad (3)$$

which takes (2) into

$$ds^2 = -\left(1 + \frac{m}{2\rho}\right)^4 (d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta d\phi^2)$$
$$+ \left(1 - \frac{m}{2\rho}\right)^2 \left(1 + \frac{m}{2\rho}\right)^{-2} dt^2. \qquad (4)$$

There is still trouble at the Schwarzschild surface, which is now at $\rho = \frac{1}{2}m$, but the black hole region $r < 2m$ of (2) does not occur because (3) gives no real values of ρ for $r < 2m$. In fact, if we assume $\rho > 0$ the region $r > 2m$ of (2) is covered twice by (4), once by $\rho > \frac{1}{2}m$ and once for $\frac{1}{2}m > \rho > 0$; but the region $r < 2m$ is not covered at all. The simple transformation (3) illustrates an important and frustrating feature of GR (and one which bedevils the interpretation of solutions): locally equivalent solutions of Einstein's equations may have different topologies.

A transformation similar to (3), namely

$$r = \rho^2 + 2m, \qquad \infty > \rho > -\infty,$$

was proposed by Einstein and Rosen [3], and leads to the metric

$$ds^2 = -4(\rho^2+2m)d\rho^2-(\rho^2+2m)^2(d\theta^2+\sin^2\theta d\phi^2)+\rho^2(\rho^2+2m)^{-1}dt^2. \quad (5)$$

This has no spatial singularity, but it still has the Schwarzschild surface, now at $\rho = 0$. Like the isotropic form (4) it covers $r > 2m$ twice and $r < 2m$ not at all.

Let us revert to (2). There are several different transformations that regularise the Schwarzschild surface [2]. One of these yields the Kruskal metric which, according to current conventional wisdom, gives the best description of Schwarzschild space-time. It is the maximal analytic extension; it is geodesically complete, which means that time-like and null geodesics drawn from any non-singular point either reach spatial or null infinity, or end on a physical singularity. This is not the case in (2) because, as we saw, infalling geodesics cannot cross $r = 2m$ in finite t.

The transformation from Schwarzschild coordinates (r, θ, ϕ, t) to Kruskal coordinates (u, θ, ϕ, v) is given in the standard textbooks, e.g. [2]. The Kruskal metric is

$$ds^2 = 32m^3r^{-1}e^{-r/2m}(dv^2 - du^2) - r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (6)$$

where r is given implicitly by

$$u^2 - v^2 = (2m)^{-1}(r - 2m)e^{r/2m}.$$

v is a time-like coordinate and u is space-like. The Schwarzschild surface transforms into $u = \pm v$ and is quite regular in the new coordinates. The big surprise is that the physical singularity at $r = 0$ transforms into two singular space-like lines

$$v = \pm(1 + u^2)^{1/2}.$$

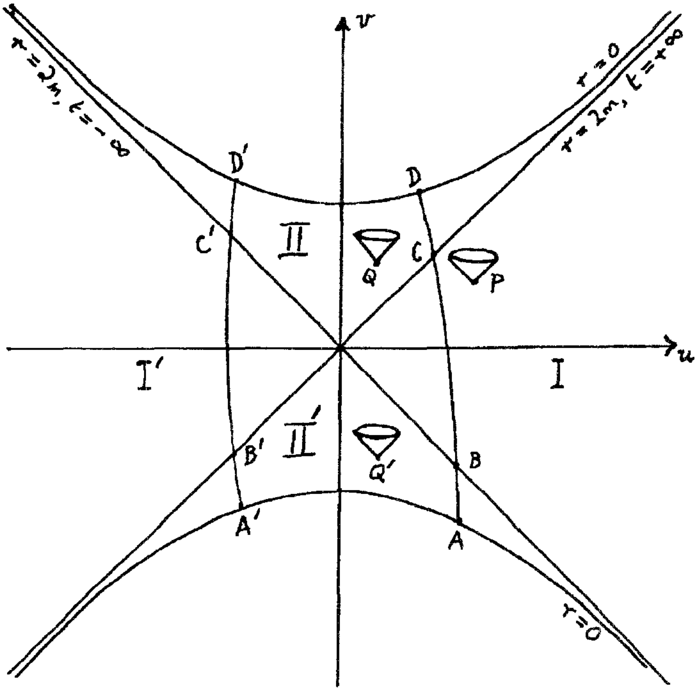

Fig. 1. The Kruskal diagram.

The Schwarzschild space-time in Kruskal coordinates can be pictured as in Figure 1 (θ, ϕ omitted). The space-time does not exist above and below the singular lines $r = 0$,[2] but extends indefinitely in both horizontal directions. A null cone is drawn schematically at a point P: it illustrates the neat feature of Kruskal coordinates that $u = \pm v + C$, $\theta = \theta_0$, $\phi = \phi_0$ is a null geodesic, so in the diagram every null geodesic is parallel to one of the two straight lines $r = 2m$.

Four regions are shown in Fig. 1. Of these I and II correspond to $r > 2m$ and $2m > r > 0$ in Schwarzschild coordinates, and the join between them is now smooth. Regions I′, II′ duplicate I and II so that the Kruskal manifold is twice as big as that supporting (2).

There is an important difference between II and II′ arising because v has the time sense shown by the arrow. A test particle at Q in II must hit

[2] Recently Lynden-Bell and Katz [4] have suggested a method of extending the Schwarzschild metric through $r = 0$. This leads, roughly speaking, to an infinity of Kruskal diagrams, one above the other.

the upper singularity $r = 0$ as it proceeds, always within the upper null cone like the one shown. This is the reason for the black hole property that a test particle once within $2m > r > 0$ cannot escape the singularity.

Now consider a test particle at Q′ in II′. It too must proceed within the upper null cone at every point in its trajectory, and so must ultimately cross $r = 2m$ outwards. So we have the strange situation that a test particle in II cannot escape, whereas one in II′ not merely escapes, but is expelled! Moreover, whereas the test particle in II is being attracted to the upper singularity, as one would expect, the one in II′ is apparently being repelled from the lower singularity.

A further queer property of the Kruskal diagram concerns the two regions $r > 2m$, namely I and I′ in Fig. 1. Observers O and O′ in the two regions are totally disconnected and cannot communicate. The diagram tells us that every spherically symmetric particle has two distinct worlds outside its Schwarzschild surface. It also turns out, following a detailed examination [5] of the transformation $(r, \theta, \phi, t) \rightarrow (u, \theta, \phi, v)$ that there is a curious time reversal in I′ relative to I. Consider a geodesic such as ABCD. It transpires that as a test particle proceeds along BC the Schwarzschild time t increases (i.e. in I t increases with v). However, along the portion B′C′ of the geodesic A′B′C′D′ the Schwarzschild time decreases (in I′ t decreases with v).

It seems to me that the Kruskal interpretation, though mathematically complete, does not make physical sense. A possible escape from its difficulties is described in Section 5.

A related transformation of the Schwarzschild metric, that of Novikov [2], throws some light on the Kruskal diagram, without removing all its obscurities. (For a clear and critical review of the Kruskal and Novikov representations see Ref. 5). The metric in Novikov coordinates (R, θ, ϕ, τ) is

$$ds^2 = -S dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) + d\tau^2,$$

where S, r are complicated functions of R and τ. In these coordinates the curves

$$R = R_0, \qquad \theta = \theta_0, \qquad \phi = \phi_0, \qquad \tau = s$$

are geodesics, τ being the proper time along them. Now imagine test particles being thrown out of the central mass at different speeds (less than the speed of escape) and suppose each carries a clock which records τ. Attach a specific number R to each test particle as it is ejected and let it keep this number throughout its trajectory. Arrange that the particles are ejected so that each arrives at its summit when its clock reads $\tau = 0$.

This scheme gives a hypothetical way of setting up a Novikov coordinate system. It assumes that particles can be thrown out of a black hole.

The central mass in the Novikov diagram (as in the Kruskal one) is represented by two singular lines, and it is clear from the foregoing why this is so. One singular line represents the ejection of test particles forming the coordinate system, and the other represents their return to the central mass. Thus the fact that there are two singular lines says nothing about the mass, but is forced on us by the coordinate system chosen.

The Novikov diagram is like the Kruskal one but distorted (see Ref. 2); in particular, it has four regions like I, I′, II, II′ of Fig. 1. It may therefore be that in the Kruskal representation also the presence of two singularities is a coordinate effect.

Before leaving the maximal analytic extension of the Schwarzschild manifold I shall refer to the solution for a static spherically symmetric charged particle, although this is not a vacuum space-time. Its metric is

$$\begin{aligned} ds^2 = &- \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &+ \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 \end{aligned} \tag{7}$$

and the ranges of the coordinates are the same as in (2). If $e^2 > m^2$ the only singularity is at $r = 0$. If $e^2 < m^2$, g_{tt} has two zeros, and coordinate singularities like that at the Schwarzschild surface occur. The maximal analytic extension of this manifold is much more complicated than that of Schwarzschild. It has an infinity of pairs of singular lines $r = 0$, an infinity of pairs of regions like (I, I′) (II, II′) of Fig. 1, and others besides [6]. It is hard to believe that this extension has any relevance to charged masses in the real world. I shall refer to this again in Section 5.

The Kruskal version of Schwarzschild has not been accepted by all relativists. From quite a copious literature I mention three lines of heresy.

1. The Schwarzschild surface is a real singularity and the coordinates should end there. (See, for example Refs. 7–9). Certainly something physical happens at $r = 2m$, so this is an arguable case.

2. Spherical symmetry is a mathematical abstraction and no real object can be precisely spherically symmetric; therefore we must beware of properties that depend crucially on the spherical symmetry. This interesting view was put forward in [8]. In its favour is the following: the Schwarzschild space-time has been proved to be the only static vacuum one with a non-singular horizon, so the slightest depature from spherical symmetry will destroy the Schwarzschild surface at $r = 2m$.

3. All the static spherical bodies we know have boundaries outside their Schwarzschild surfaces, so one should use (2) as an exterior metric matched at $r = r_0 > 2m$ to a suitable static interior. This will be mentioned again in Section 5.

These three theories would be falsified if a black hole were definitely observed.

3. STATIC AXIALLY SYMMETRIC METRICS

These can be expressed in Weyl form

$$ds^2 = -e^{-2\mu}[e^{2\nu}(dz^2 + dr^2) + r^2 d\phi^2] + e^{2\mu} dt^2, \tag{8}$$

where μ and ν are functions of z, r and the coordinate ranges are

$$\infty > z > -\infty, \qquad r \geq 0, \qquad 2\pi \geq \phi \geq 0, \qquad \infty > t > -\infty.$$

One has in mind a background of cylindrical polars as the spatial coordinates. μ satisfies Laplace's equation

$$\frac{\partial^2 \mu}{\partial z^2} + \frac{\partial^2 \mu}{\partial r^2} + \frac{1}{r}\frac{\partial \mu}{\partial r} = 0, \tag{9}$$

and ν is determined (up to an additive constant) by the field equations once μ is prescribed. For weak static fields μ can be interpreted as an approximate Newtonian potential of the gravitational field. This gives us a guide to the physical meaning of exact Weyl solutions, though it has to be used with caution, as we shall see.

The singularities of (9) will refer to sources of the gravitational field, but (8) may also have singularities representing stresses on the z-axis. The latter is free of singularities provided

$$\lim_{r \to 0} \nu = 0.$$

Singularities of this type are called conical singularities. They arise from a topological defect angle, as in cosmic strings.

The Curzon solution.
The simplest Weyl solution for an isolated system has

$$\mu = -mR^{-1}, \qquad \nu = -\tfrac{1}{2}m^2 r^2 R^{-4}, \qquad R = +(z^2 + r^2)^{1/2}. \tag{10}$$

This is called the Curzon metric. μ in (10) is the Newtonian potential for a spherical particle, but the Curzon solution is different from that of

Schwarzschild (it has fewer Killing vectors). Its far-field is that of a mass at $R = 0$ with multipoles on it.

It was discovered twenty-five years ago that the Curzon metric has a strange singularity structure [10], quite different from that of the Schwarzschild solution. It has no horizon (g_{tt} does not vanish for $R > 0$) but it has a curvature singularity at $R = 0$. This is naked. However, as first shown by Gautreau and Anderson [11] it has also at $R = 0$ what is known as a directional singularity. This means that the limit of $R_{abcd}R^{abcd}$ ($=: K$) depends on the direction of approach to the singularity. In fact $K \to 0$ as $R \to 0$ along the z-axis, but $K \to \infty$ as $R = 0$ is approached from any other straight line direction. (For the behaviour of K along curved approaches see Ref. 12). This suggests that the Weyl coordinates might be extensible through the singularity.

This extension has recently been carried out by Scott and Szekeres [13]. They interpret the Curzon singularity as a ring on which some timelike geodesics terminate. Others pass through the ring and here the extended Curzon solution can be matched to Minkowski space-time. The latter is another world not contemplated in the ordinary view of the Curzon metric. Strange phenomena, reminiscent of those occuring in the Kruskal diagram, are envisaged: for example, an observer falling from the Minkowski region through the ring sees a material object created out of nothing.

None of this arises, of course, if we stick to the region $R > R_0 > 0$. Interior solutions for $R_0 > R > 0$, matched to the Curzon metric are known [14,15], so there are non-singular space-times which are asymptotically Curzon.

The γ-metric

There is a useful Weyl metric referring to an isolated body, which generalises the Curzon metric. It is (8) with

$$\begin{aligned}\mu &= \frac{m}{2a}\log\frac{R_1+R_2-2a}{R_1+R_2+2a}\,,\\ \nu &= -\frac{m^2}{2a^2}\log\left[\frac{4R_1R_2}{(R_1+R_2)^2-4a^2}\right]\,,\end{aligned}$$

where $R_{1,2} = +[(z \pm a)^2 + r^2]^{1/2}$.

The Curzon metric is obtained by letting $a \to 0$ and keeping m finite. The γ-metric gives Schwarzschild if one puts $m = a$; this shows that in Weyl coordinates the Newtonian potential μ corresponding to Schwarzschild is that of a finite rod with mass density 1/2 and length $2m$. Weyl coordinates represent Schwarzschild space-time for only the region outside the Schwarzschild surface.

The γ-metric was discovered by Darmois [16]. It has been rediscovered in various forms and investigated many times since [11,17–21]. It has a directional singularity if $m > 2a$, but not for $m < 2a$ [11]. At infinity the metric represents an isolated body with monopole and higher mass moments. An interior solution for it has been given [22].

Solution for a circular disc

The γ-metric is easily derived if one uses (8) in prolate spheroidal coordinates. Zipoy [18] used instead oblate spheroidals and obtained a metric which he regarded as having a ring singularity and a double sheeted topology, somewhat in the style of the Scott–Szekeres interpretation of the Curzon metric. However, Bonnor and Sackfield [23] showed that this complicated topology was unnecessary and that Zipoy's metric could represent the field of a disc in 3-space with euclidian topology.

Other solutions for circular discs are known. These include several for counter-rotating discs [24–26], and another [27] which was used to study the cosmic censorship hypothesis.

Static solution for two articles

What is probably the most perspicacious of all exact solutions in GR was discovered in 1922 by Bach and Weyl [28]. Equation (9) which generates the metric (8) is linear, so solutions may be superposed. Let us take

$$\mu = -m_1[r^2 + (z-a)^2]^{-1/2} - m_2[r^2 + (z-b)^2]^{-1/2}. \qquad (11)$$

This evidently refers to two Curzon particles on the axis of symmetry, one at $z = a$ and the other at $z = b$. But this is a static solution: how can there be two separate masses at rest?

The theory answers this question in a beautiful way. When one works out the function ν corresponding to (11) one finds that there is a singularity along the z-axis between the particles, representing a stress holding them apart.[3] But wait, we know that in GR a stress generates a gravitatlonal field; why does this not show up in (11)? The answer, mysteriously wonderful, is that the matter carrying the stress is of precisely the sort (not realisable in nature) that has zero active gravitational mass, and so will exert no gravitational field [29]. The force exerted by the stress on each particle is, in the lowest approximation, $m_1m_2(a-b)^{-2}$, as expected [30].

[3] Alternatively, by a different choice of arbitrary constant one can make the stresses stretch from each particle to infinity, i.e. if $a > b$, the singular lines can be on $\infty > z > a$ and $b > z > -\infty$.

A similar solution applies to two Schwarzschild particles in Weyl coordinates. These solutions must follow because GR is not only a field theory of gravitation but also contains its equations of motion. That these results can be achieved in this way shows that some, at least, of the singularities in GR have remarkable physical validity.

The C-metric

This metric

$$ds^2 = -A^{-2}(x+y)^{-2}(F^{-1}dy^2 + G^{-1}dx^2 + B^{-2}Gd\phi^2 - B^2Fdt^2), \qquad (12)$$

where

$$F = -1 + y^2 - 2mAy^3, \qquad G = 1 - x^2 - 2mAx^3,$$

was discovered by Levi-Civita in 1918, and its history has been described in [31] and [32]. Some recent references are listed in [33]. The metric's appearance gives no indication of its physical meaning beyond the fact that it is static and can be taken as axially symmetric. Searching for its interpretation was rather like detective work in a mystery story. Although static in the form (12) the C-metric has an extension which is time-dependent, and which refers to the field of accelerated particles [33]. Therefore it will be described in detail in part II of this work.

Infinite line-mass (ILM)

Even simpler than form (10) for μ is the cylindrically symmetric potential

$$\mu = 2\sigma \log r$$

originally considered by Levi-Civita. In Newtonian theory this is the gravitational potential of an infinite uniform line-mass (ILM), σ being the mass per unit length. The corresponding Weyl metric is [34]

$$ds^2 = -r^{8\sigma^2 - 4\sigma}(dz^2 + dr^2) - C^{-2}r^{2-4\sigma}d\phi^2 + r^{4\sigma}dt^2; \qquad (13)$$

this contains two significant arbitrary constants σ, C whereas the Newtonian solution contains only one. C refers to the deficit angle, and cannot be removed by scale transformations. Both σ, C are presumably fixed by the internal composition of the ILM.

K is infinite at $r = 0$ for all C, σ except $\sigma = 0$, $\sigma = \frac{1}{2}$ (see below) and nowhere else. Thus metric (13) has a singularity along the axis $r = 0$, and we can tentatively take this as referring to an infinite line source. There is no horizon.

For small σ it seems reasonable to regard (13) as describing an ILM with mass σ per unit coordinate length. There are two reasons for this.

First, the time-like geodesics, interpreted as test particle paths, are what one would expect from the Newtonian analogue. However, circular time-like geodesics exist only for

$$\tfrac{1}{4} > \sigma > 0. \tag{14}$$

(Note that $\sigma = 1$ means about $10^{28}\,\mathrm{g\ cm^{-1}}$). Secondly, one can, for σ satisfying (14), match (13) with a perfect fluid interior lying within a boundary $r = \text{const.}$ [35,36]: this gives a satisfactory global solution. When $\sigma = \frac{1}{4}$ the circular geodesics become null, suggesting a limiting case. Metric (13) with $\sigma = \frac{1}{4}$, $C = 1$ is a transform of one of Kinnersley's type D metrics [37] (his Case IVB with his $C = 0$).

When $\sigma = \frac{1}{2}$ the metric (13) is flat so clearly σ does not for all its values refer to the mass per unit length of an ILM. In [34] we showed that the strength of the gravitational field diminished as σ increased from $\frac{1}{4}$ to $\frac{1}{2}$, and we were inclined to agree with Lathrop and Orsene [38] that $\sigma = \frac{1}{4}$ is the greatest positive mass per unit length for an ILM, and that for $\sigma > \frac{1}{4}$ metric (13) must refer to something else. However, there has recently appeared [39] a solution for an ILM, with incompressible fluid interior, which permits values of $\sigma > \frac{1}{4}$. Such an ILM would not allow circular geodesics. This puzzle is so far unresolved.

Semi-infinite line-mass (SILM)

This was discussed in some detail in [34]. In general the metric contains only one arbitrary constant σ and takes the form

$$ds^2 = -X^{-2\sigma}\{(X/2R)^{4\sigma^2}(dz^2 + dr^2) + r^2 d\phi^2\} + X^{2\sigma} dt^2, \tag{15}$$

where

$$X = R + \epsilon(z - z_1), \qquad R = +[(z - z_1)^2 + r^2]^{1/2}, \qquad \epsilon = \pm 1.$$

$\sigma \log X$ is the Newtonian potential of a SILM of line-density σ lying along the z-axis from z_1 to ∞ (if $\epsilon = -1$) and from z_1 to $-\infty$ (if $\epsilon = +1$). As in the case of an ILM it seems reasonable to assume that (15) gives the space-time of a SILM of line-density σ if σ is small. However, for $\sigma = \frac{1}{2}$ the metric is flat; in fact it is a uniformly accelerated metric.

For $\sigma = 1$ the metric (15) acquires an extra arbitrary constant C:

$$ds^2 = -(CX)^{-2}\{(X/2R)^4(dz^2 + dr^2) + r^2 d\phi^2\} + C^2 X^2 dt^2. \tag{16}$$

This admits four Killing vectors whereas in general (15) admits two. There is evidently something unusual about this case and in [34] I interpreted it not as a SILM at all but as an infinite hollow cylinder with an applied gravitational field parallel to its axis. If in (16) we put $C = (4m)^{-1/4}$, $\epsilon = +1$, $z_1 = 0$ we obtain a transform of one of Kinnersley's type D metrics [37], namely his Case IVB with his $C = -\frac{1}{2}$ and $m = \frac{1}{4}k^4$.

For $\sigma > 1$ I was unable to interpret (15), but clearly it does not refer to a SILM. For some values of $\sigma < 0$ one can interpret (15) as a SILM with negative mass density, but it turns out that when $\sigma = -\frac{1}{2}$ (15) is a transform of Taub's plane metric (see below).

Superposition of two or more SILMs of different σ can be used [40] to interpret some well known vacuum metrics of Ehlers and Kundt [41] which have not so far been explained.

The ILM and SILM metrics illustrate a common difficulty in interpretation. A continuous change in a parameter (in these cases σ) may produce sudden and inexplicable changes in physical meaning.

Metrics for an infinite plane

Let us submit the SILM metric (15) to the coordinate transformation

$$r = Z\rho, \qquad 2\epsilon(z - z_1) = Z^2 - \rho^2, \qquad \phi = \phi, \qquad t = t; \tag{17}$$

the result is

$$ds^2 = -Z^{8\sigma^2-4\sigma}(Z^2+\rho^2)^{1-4\sigma^2}(dZ^2+d\rho^2) - Z^{2-4\sigma}\rho^2 d\phi^2 + Z^{4\sigma}dt^2, \tag{18}$$

and we assume $Z > 0$, $\rho \geq 0$. The whole of the Weyl space-time associated with (15) maps on to the half-space $Z > 0$, the singularity $\epsilon(z - z_1) < 0$, $r = 0$ going into the plane $Z = 0$. The transformation (17) opens out the SILM singularity and spreads it over a plane.

This illustrates one of the difficulties in interpreting space-times. Should one regard (15) or (18) as more realistic? Or are they both equally valid, (15) as a representation of a SILM, and (18) as a representation of an infinite plane (in general non-uniform)?

In case $\sigma = -\frac{1}{2}$ (18) reduces, via an obvious transformation to

$$ds^2 = \zeta^{-1/2}(-d\zeta^2 + d\tau^2) - \zeta(dx^2 + dy^2), \qquad \zeta > 0, \tag{19}$$

a metric originally due to Taub, which in [1] is called the general plane symmetric vacuum solution. In fact I showed [42] that this solution has no fewer than four different manifestations in the literature—SILM with $\sigma = -\frac{1}{2}$, plane, ILM with $\sigma = -\frac{1}{2}$, and a Robinson–Trautman solution. Each form offers a different physical interpretation.

Although Horský (see, for example, Ref. 43) has studied a transform of (19) and argued that it describes the gravitational field of a uniform, infinite plane, this does not seem to me established. In favour of this interpretation is the nature of the four Killing vectors: one is time-like but the three space-like ones are easily recognisable as characterising plane symmetry. On the other hand we find [42,44,45] that test particles are repelled by the singularity at $\zeta = 0$ so if (19) represents a plane the plane has negative mass, and there seems to be no corresponding solution for a plane of positive mass. Consider now the family of test-particle paths orthogonal to $\zeta = 0$, all with the same value of x, but each with a different value of y. One finds that the proper distance between neighbouring paths of coordinate separation dy is $[\zeta(\tau)]^{1/2}dy$, where $\zeta(\tau)$ is the ζ coordinate of the particle at time τ. Thus the proper separation changes with time. If $\zeta = 0$ represents a plane, one would expect the trajectories to remain parallel.

In [42] I argued that the SILM interpretation of (19) is the most realistic.

4. STATIONARY AXIALLY SYMMETRIC METRICS

An appropriate form of the metric is

$$ds^2 = -e^{-2\mu}[e^{2\nu}(dz^2 + dr^2) + r^2 d\phi^2] + e^{2\mu}(dt + \omega d\phi)^2, \tag{20}$$

which reduces to the static Weyl metric (8) when $\omega = 0$. The vacuum field equations reduce to two for μ and ω,

$$\frac{\partial^2 \mu}{\partial z^2} + \frac{1}{r}\frac{\partial \mu}{\partial r} + \frac{\partial^2 \mu}{\partial r^2} = -\frac{1}{2}\frac{e^{4\mu}}{r^2}\left[\left(\frac{\partial \omega}{\partial z}\right)^2 + \left(\frac{\partial \omega}{\partial r}\right)^2\right], \tag{21}$$

$$\frac{\partial}{\partial z}\left(\frac{e^{4\mu}}{r}\frac{\partial \omega}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{e^{4\mu}}{r}\frac{\partial \omega}{\partial r}\right) = 0, \tag{22}$$

and, as in the static case, ν is determined by quadrature up to an additive constant when μ and ω are prescribed.

Equation (20) extends the Weyl metric to steadily spinning sources. For an isolated axially symmetric body (or system of bodies) the conditions at spatial infinity are known from the weak-field approximation to be

$$\mu \sim -mR^{-1}, \qquad \omega \sim 2hr^2R^{-3}, \qquad R = +(z^2 + r^2)^{1/2}, \tag{23}$$

m being the mass and h the angular momentum. An intensive search over many years for solutions of (21) and (22) satisfying (23) has produced only a few metrics for a massive spinning body.

Papapetrou [46] obtained a class of solutions of (21) and (22) by putting

$$\frac{\partial \mu}{\partial z}\frac{\partial \omega}{\partial z} + \frac{\partial \mu}{\partial r}\frac{\partial \omega}{\partial r} = 0. \tag{24}$$

It is then found that

$$e^{-2\mu} = a \cosh \psi + b \sinh \psi$$

where ψ satisfies Laplace's equation, and ω can be obtained from (24). This solution therefore depends on one harmonic function plus some constants. It turns out, however, that there is no member of this class satisfying the boundary conditions (23) with $mh \neq 0$. Thus the Papapetrou class contains no solution referring to an isolated spinning mass. The same applies to some related classes which can be generated from the Papapetrou class (Ref. 1, p.204).

The Kerr solution

A metric for a steadily spinning isolated axi-symmetric body was first obtained by Kerr. It is convenient to write it in Boyer–Lindquist coordinates [2]:

$$\begin{aligned} ds^2 = {} & -\Sigma\Delta^{-1}dr^2 - \Sigma d\theta^2 - \Sigma^{-1}\sin^2\theta[(r^2+a^2)d\phi - a dt]^2 \\ & + \Sigma^{-1}\Delta(dt - a\sin^2\theta d\phi)^2, \end{aligned} \tag{25}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2mr + a^2.$$

r here is like the radial coordinate of the Schwarzschild metric, to which (25) reduces if $a = 0$. For large r (25) has the correct form to represent a spinning mass m with angular momentum ma. (25) has two Killing vectors, one time-like (but not hypersurface-orthogonal) and the other space-like, referring to axial symmetry.

The Kerr solution exhibits the dragging of inertial frames expected from early weak-field calculations in GR. A gyroscope held at fixed r would according to (25) precess relative to the distant stars, and the frame-dragging requires a correction to Kepler's law for circular geodesics [47]. The nearer the centre the stronger the dragging until, at a value of r given by

$$r_1 = m + (m^2 - a^2\cos^2\theta)^{1/2},$$

called the static limit, an observer cannot remain at rest relative to the fixed stars: the dragging is too strong and he must orbit the centre in the sense of the spinning body. Mathematically, the reason is that as r

falls below r_1, g_{tt} changes sign and the world-line $r = r_0$, $\theta = \theta_0$, $\phi = \phi_0$ becomes space-like.

For still smaller r the horizon is reached at

$$r_+ = m + (m^2 - a^2)^{1/2},$$

which is real if $m^2 \geq a^2$. This is like the Schwarzschild horizon and is taken to indicate the unsuitability of Boyer–Lindquist coordinates for small r.

The region between r_1 and r_+ is called the ergosphere. It is possible to extract energy from the Kerr black hole by throwing certain particles into the ergosphere. This is called the Penrose process.

To extend the Kerr solution inside the horizon r_+ it is necessary to use different coordinates, and the system usually chosen is Kerr–Schild coordinates. A good account of the extensions of Kerr space-time is given in Ref. 6, Ch. 5. They differ according as $a^2 \gtreqless m^2$. In all cases there is a physical singularity at $r = 0$, $\cos\theta = 0$, which is interpreted as a ring, not a point. The extension takes place through the ring and r ranges from$+\infty$ to $-\infty$. Allowing negative values of r leads to closed time-like lines since $g_{\phi\phi}$ can become positive.

The simplest of the three cases is $a^2 > m^2$ because then Δ does not vanish. This case has a single ring singularity. For $a^2 \leq m^2$ Δ can vanish and the maximal analytic extension is extremely complicated, containing an infinite number of sigularities, and resembling the Reissner–Nordström solution for an electric charge with $m^2 \geq e^2$ (see Section 3). Once again, I question whether this picture represents real physics. This issue will be taken up again in Section 5.

The Kerr solution is remarkable for three reasons. The first is that Kerr managed to find it. His methods were quite unlike those generally available at the time, and even today the derivation of the Kerr metric is not easy. (The mere verification of it is of course straightforward.) The second is that the Kerr solution with $a^2 \leq m^2$ is the only stationary, axially symmetric, vacuum and asymptotically flat solution with a non-singular horizon. This means that the singularity at $r = 0$ would not be naked (i.e. visible to outside observers), and popular wisdom has it that the Kerr solution represents the end-point of gravitational collapse of a rotating star to a black hole.

The third reason why the Kerr solution is remarkable is that so far, in spite of many attempts, no entirely convincing stationary interior solution, with realistic matter, has been discovered. (For reviews of interior solutions see Refs. 48,49.) This distinguishes it from the Schwarzschild solution, for which several physically reasonable interiors are known.

Other solutions for spinning bodies

After Kerr's discovery many workers were stimulated to look for other metrics for isolated spinning bodies, and some have been found. Other two-parameter solutions are given in [50,51] and three-parameter solutions in [52–55]; the extra parameter refers to higher multipole moments. The maximal analytic extensions of these solutions have not been given.

Mention should also be made of a class of stationary metrics due to Debever and to Plebański and Demiański [1]. These contain the cosmological constant (which can be put zero) and six parameters. Two of the latter refer to electric and magnetic charges, leaving four dynamical parameters, apparently referring to mass, NUT parameter (see below), angular momentum and acceleration. The precise physical meaning of this metric, and its relation to others, for example the Kinnersley Case II metrics referred to below, has not been studied as far as I know.

The double Kerr solution

In 1980 Kramer and Neugebauer [56] published a vacuum solution referring to two Kerr bodies. This generalises the static Weyl solution for two non-spinning particles (Section 3). The feature of special interest is the gravitational action on one body caused by the spin of the other, which has no counterpart in Newton's theory. It appears that the spin-spin interaction can balance the gravitational attraction of two positive masses, and keep them apart, without the need of a strut between them [57–59]. It has been stated that this effect would be detectable if we knew G more accurately [60].

The NUT solution

This was discovered by Newman, Unti and Tamburino. It is a member of Papapetrou's class, but it is more simply written in a different coordinate system as follows [1]:

$$\begin{aligned} ds^2 &= -U^{-1}dr^2 - (r^2+a^2)(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad + U(dt + 4a\sin^2\tfrac{1}{2}\theta d\phi)^2, \\ U &= (r^2+a^2)^{-1}(r^2 - 2mr - a^2). \end{aligned} \tag{26}$$

a is called the NUT parameter. If $a = 0$ it reduces to the Schwarzschild metric, but if $a \neq 0$ (as we now assume) it is very different. There are singularities along the symmetry axis at $\theta = 0$ and $\theta = \pi$; the former of these may be removed by transforming to cartesian coordinates but this does not get rid of the one at $\theta = \pi$. The latter has been treated in two different ways.

Misner [61] showed that the rotation axis could be made completely regular by introducing two coordinate patches, one at the north pole and the other at the south. However, to do this consistently one has to introduce a periodic time coordinate. Thus every observer on a coordinate time-line moves on a closed path. The hypersurfaces $r = \text{const.}$ have the topology of S^3, instead of $R \times S^2$. The invariants of the Riemann tensor depend only on r so these hypersurfaces are homogeneous. The coordinate r is allowed to run from $-\infty$ to $+\infty$. In later work [62] Misner showed that his interpretation of the NUT metric was connected with earlier work of Taub [63]. In this, r is interpreted as a time-like coordinate, T, and the pace-time is a sort of cosmological model, homogeneous on the hypersurfaces $T = \text{const.}$

In 1969 I gave a different interpretation of the NUT metric [64]. I accepted $\theta = \pi$ as a genuine singularity to be regarded physically as a semi-infinite massless source of angular momentum, and supposed there to be a source of mass centred on $r = 0$. Support for this picture came from a model studied by Sackfield [65].

A related interpretation draws an analogy from Dirac's theory of magnetic monopoles [66,67]. The source of the NUT metric is taken to be an ordinary mass together with a gravitomagnetic monopole (sometimes called mass and dual mass, or dyon). Both are centred on $r = 0$ and the singularity on $\theta = \pi$ is the analogue of a Dirac string. This accounts neatly for the homogeneity of the hypersurfaces $r = \text{const.}$ However, whereas in electromagnetic theory the string has no physical effects, the singularity along $\theta = \pi$ in the NUT solution does affect space-time [68].

Stationary solutions with cylindrical symmetry

If we suppose that μ, ν, ω in (20) are functions of r only, the equations (21) and (22) can be completely solved [1], apparently giving the space-time for an infinitely long rotating line source. The metric has physical singularities at $r = 0$ and $r = \infty$ or both; these have not been investigated in detail. However, the singularity at $r = 0$ can be covered by a regular region containing matter, and I shall describe the solution in this case.

The complete solution, including an interior metric for rotating dust, correctly matched at a boundary $r = r_0$, was obtained in 1937 by van Stockum [69]. It has some unexpected features [70]. The exterior metric has three cases I,II,III according as the mass per unit coordinate length σ satisfies $\sigma < \sigma_0$, $\sigma = \sigma_0$, $\sigma > \sigma_0$ respectively, where[4] $\sigma_0 \sim 0.13$ relativistic unit ($= 2 \times 10^{27}\,\text{g cm}^1$).

When $\sigma < 1$ the metric is locally flat at infinity; for $\sigma = 1$ the algebraic invariants of the Riemann tensor are constants, the metric has an extra

[4] In [70] the formula for the mass per unit coordinate length, there denoted by m, contained a parameter q. In the results described here we put $q = 0$.

Killing vector and the exterior space-time is homogeneous. For $\sigma > 1$ these algebraic invariants are zero at $r = 0$ but infinite at $r = \infty$. It thus appears that for $\sigma \geq 1$ the exterior metric does not describe an infinite rotating source alone, even though it can be matched to an interior describing one. In what follows we shall assume $\sigma < 1$.

The criterion for an axially symmetric space-time with genuine steady rotation (as distinct from coordinate effects) is that the time-like Killing vector shall *not* be hypersurface-orthogonal (h.s.o.) It turns out that in Case I the time-like Killing vector *is* h.s.o. and by a coordinate transformation the exterior can be brought to the Levi-Civita static form for an ILM (Section 3).[5] The diagonalisation of the metric in Case I was also achieved in [71]. Case II has a h.s.o. Killing vector, but it is null. Case III has no h.s.o. time-like or null Killing vector.

The result in Case I is surprising. One would expect the rotating matter to drag the inertial frames outside the body, as happens in the Kerr solution. The rotation has some effect on the exterior: see footnote 5.

In Case III the exterior contains closed time-like lines. Other space-times in GR, notably those of Kerr and Gödel, contain closed time-like lines but in these the physical nature of the sources is obscure. Case III has a well-understood source, namely a cylinder of rotating dust. Thus we have a specification for a time-machine in GR, though admittedly there are technical difficulties in making a cylinder of infinite length! The mass per unit length required is about $10^{28}\,\mathrm{g\,cm^{-1}}$, a few orders of magnitude greater than figures mentioned for cosmic strings.

Embacher [72] studied rotating hollow cylinders and in the low mass case found a similar result, namely that the cylinder did not drag the inertial frames outside. See also [73].

The Kinnersley stationary metrics

In his paper [37] on metrics of Petrov type D, Kinnersley exhibited a subset (his Case II) of six about which he wrote laconically "We propose that all the metrics of Case II represent spinning particles and correspond to the six different ways we can pick a velocity four-vector and an angular-momentum vector orthogonal to it." It would be very interesting to see a complete investigation of these solutions, and their relation to other known metrics such as, for example, those in [74].

[5] The transformation used to achieve this introduces a periodic time coordinate. The transformed metric may be called locally, but not globally, static.

5. MATCHING WITH INTERIOR SOLUTIONS

In the Schwarzschild solution difficulties of interpretation disappear if one matches the metric (2) with a static metric satisfying

$$R_{ik} - \tfrac{1}{2}g_{ik}R = -8\pi T_{ik},$$

where T_{ik} is the energy tensor of some suitable matter. The best-known interior is that of Schwarzschild which is usually taken to refer to incompressible perfect fluid. There are others, and some of these can be used as models of stars. This procedure is available only for boundaries of coordinate radius $r_0 > 2m$.

The Curzon metric can also be matched to a realistic static interior [14,15], and the bizarre topological difficulties are then irrelevant. Work by McCrea [75] matching Curzon to a thin shell as source showed, however, that the ratio (mass/characteristic radius) has unity as upper bound. A similar result was found for some other axially symmetric sources.

As already remarked in Section 4 no completely satisfactory stationary interior for the Kerr solution has yet been found. If it could be proved that none exists this would rule out the Kerr metric as a possible exterior for a stationary astronomical body. This would not be a disaster as (21),(22) must allow many such exteriors though only a few are known.

There is a further application of the interior equations which may help to eliminate the strange topologies of the maximal analytic extensions of the Schwarzschild, Kerr, Reissner–Nordström and possibly other solutions. A good description of this is given in the book by Wald [76]. The basic idea is that these extensions are not models of single, isolated bodies in the real world. Their function is to explain complete gravitational collapse. Now to do this collapsing matter must appear somewhere in an augmented representation, and this matter covers up the non-physical parts of the maximal analytic extensions. We shall illustrate this in the simplest case, i.e. the Kruskal diagram.

If we consider a collapsing cloud of spherically symmetric matter (e.g. dust) and add it to the Kruskal diagram we get Figure 2.

During the early part of the collapse there is no singularity, $r = 0$ denotes the centre of the cloud, and ordinary Schwarzschild (region I) applies outside. When the radius becomes less than $2m$, region II appears and eventually the dust forms a black hole at $r = 0$. The non-physical regions I′ and II′ are covered up by the shaded part of the diagram which represents matter.

It is not clear whether a similar process can be applied to make sense of the maximal analytic extension of the Kerr solution. The metric for the

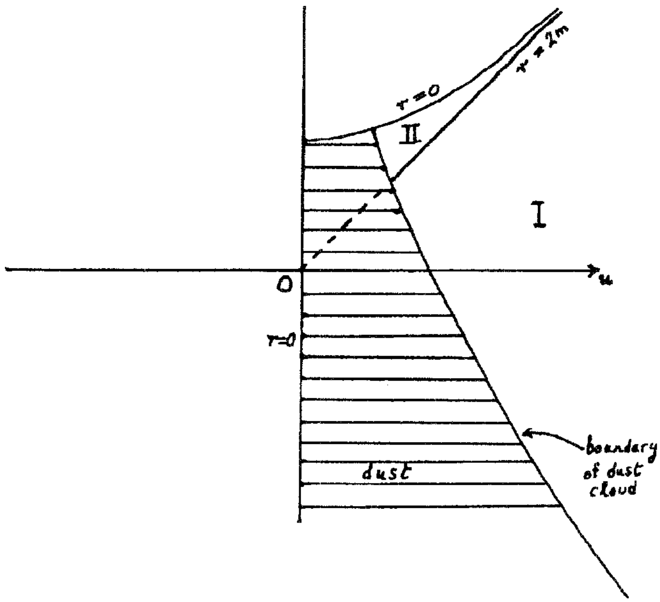

Fig. 2. Spherically symmetric dust collapsing to a singularity

collapsing matter is unknown but must be very complicated because a collapsing, rotating cloud will radiate gravitational waves. For the Reissner–Nordström metric with $m^2 \geq e^2$ it may be that a collapsing, charged cloud could cover the non-physical regions [76].

I referred in Section 3 to an interior metric for an ILM, and in Section 4 to one for a rotating ILM. These, with the corresponding exterior metrics, constitute globally regular solutions with reasonable sources. Unlike the black hole space-times, they are genuinely time-independent, and this applies for arbitrarily small cylinder radius.

6. CONCLUSION

The analytic extensions of some well known exact vacuum solutions are mathematical constructions which are physically unrealistic. For instance, the Kruskal form of the Schwarzschild space-time contains two space-like singularities and two non-communicating infinite regions. The maximal analytic extension of the Kerr metric with $m^2 > a^2$ has an infinite number of singularities and an infinity of regions like the one an ordinary observer sees.

There seem to be two ways out of the difficulty, both involving the addition of matter. Consider the Schwarzschild solution. First, one can simply insert a static core of realistic matter covering the trouble at $r \leq 2m$. This procedure is well known, and works only for bodies with coordinate radius greater than $2m$. The second way out is to suppose that the black hole forms at the end of a collapse, that the collapse requires infalling matter and that this matter covers up the unphysical parts of the maximal analytic extension. This was discussed in Section 5. It is not clear that the embarrassing parts of the maximal analytic extensions can in all cases be extinguished in this way.

Singularities in the Schwarzschild, Kerr and Curzon metrics cause much bewilderment so it is a relief to come across one which has a clear physical meaning. This is found in the two-particle solution (Section 3) where the conical singularity between the particles fulfils exactly the function required of it.

Turning to infinite sources one finds difficulties of another kind. In Sections 3 and 4 we considered metrics for infinite cylinders. The vacuum regions have no horizons, the location of the singularities seems straightforward and they can be covered by realistic matter (at least for some range of the parameters). Yet there are strange, unexplained features in the vacuum exteriors, such as the closed time-like lines in the ultrarelativistic rotating cylinder. The nature and meaning of plane-symmetric metrics does not seem to have been settled. The NUT metric continues to live up to Misner's description of it as "a counterexample to almost anything".

One of the problems in interpreting metrics arises from the coordinate freedom inherent in GR. A singularity interpreted as a plane in one coordinate system may become something quite different in another, as indeed we saw in connection with metric (19). The use of Killing vectors does not necessarily settle the ambiguity.

Another difficulty, arising in the cylindrical metrics, is that a continuous variation of parameter (e.g. σ) may bring about discontinuous changes in physical interpretation. Although it is reasonable when σ is small to take it as the mass per unit length of an infinite cylinder, one does not know what the cylindrical metrics refer to for larger σ.

REFERENCES

1. Kramer, D., Stephani, H., MacCallum, M. A. H., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge).
2. Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation* (W. H. Freeman, San Francisco).
3. Einstein, A, and Rosen, N. (1935). *Phys. Rev* **48**, 73.
4. Lynden-Bell, D., and Katz, J. (1990). *Mon. Not. R. Astron. Soc.* **247**, 651.
5. Gautreau, R. (1980). *Nuovo Cimento* **56B**, 49.

6. Hawking, S. W., and Ellis, G. F. R. (1973). *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge).
7. Bel, L. (1969). *J. Math. Phys.* **10**, 1501.
8. Cooperstock, F. I., and Junevicus, G. J. G. (1973). *Nuovo Cimento* **16B**, 387.
9. McCrea, W. H. (1964). *Astrophys. Norvegica* **9**, 89.
10. Stachel, J. (1968). *Phys. Lett.* **27A**, 60.
11. Gautreau, R., and Anderson, J. L. (1967). *Phys. Lett.* **25A**, 291.
12. Cooperstock, F. I., and Junevicus, G. J. G (1974). *Int. J. Theor. Phys.* **9**, 59.
13. Scott, S. M., and Szekeres, P. (1986). *Gen. Rel. Grav.* **18**, 557.
14. Hernandez, W. C. (1967). *Phys. Rev.* **153**, 1359.
15. Marek, J. J. J. (1967). *Phys. Rev.* **163**, 1373.
16. Darmois, G. (1927). *Mémorial des Sciences Mathematiques* (Gauthier-Villars, Paris), Fasc. 25.
17. Erez, G., and Rosen, N. (1959). *Bull. Res. Council Israel* **8F**, 47.
18. Zipoy, D. M. (1966).*J. Math. Phys.* **7**, 1137.
19. Voorhees, B. H. (1970). *Phys. Rev.* *D***2**, 2119.
20. Esposito, F. P., and Witten, L. (1975). *Phys. Lett.* **58B**, 357.
21. Papadopoulos, D., Stewart, B. W., and Witten, L. (1981). *Phys. Rev.* *D***24**, 320.
22. Stewart, B. W., Papadopoulos, D., Witten, L., Berezdivin, R., Herrera, L. (1982). *Gen. Rel. Grav.* **14**, 97.
23. Bonnor, W. B., and Sackfield, A. (1968). *Commun. Math. Phys.* **8**, 338.
24. Morgan, T., and Morgan, L.(1969). *Phys. Rev.* **183**, 1097; (1970). *Phys. Rev.* *D***2**, 2756.
25. Lynden-Bell, D., and Pineault, S. (1978). *Mon. Not. R. Astron. Soc.* **185**, 679; (1978). *ibid.* **185**, 695.
26. Lemos, J. P. S. (1989). *Class. Quant. Grav.* **6**, 1219.
27. Chamorro, A., Gregory, R., and Stewart, J. M. (1987). *Proc. Roy. Soc. Lond.* **A413**, 251.
28. Bach, R., and Weyl, H. (1922). *Math. Z.* **13**, 134.
29. Israel, W. (1977). *Phys. Rev.* *D***15**, 935.
30. Katz, A. (1968). *J. Math. Phys.* **9**, 983.
31. Kinnersley, W., and Walker, M. (1970). *Phys. Rev.* *D***2**, 1359.
32. Farhoosh, H., and Zimmerman, R. L. (1980). *Phys. Rev.* *D***21**,, 317.
33. Bonnor, W. B. (1983). *Gen. Rel. Grav.* **15**, 535.
34. Bonnor, W. B., and Martins, M. A. P. (1991). *Class. Quant. Grav.* **8**, 727.
35. Evans, A. B. (1977). *J. Phys. A* **10**, 1305.
36. Bonnor, W. B. (1979). *J. Phys. A* **12**, 847.
37. Kinnersley, W. (1969). *J. Math. Phys.* **10**, 1195.
38. Lathrop, J. D., and Orsene, M. S. (1980). *J. Math. Phys.* **21**, 152.
39. Stela, J., and Kramer, D. (1990). *Acta Physica Polonica* **B21**, 843.
40. Martins, M. A. P. (1992). Ph.d. thesis, University of London, in preparation.
41. Ehlers, J., and Kundt, W. (1962). In *Gravitation. An introduction to current research*, L. Witten, ed. (John Wiley, New York).
42. Bonnor, W. B. (1990). *Wissenschaft. Z. der Friedrich-Schiller Universitat Jena* **39**, 25.
43. Horský, J. (1968). *Czech. J. Phys.* **18**, 569.
44. Gautreau, R., and Hoffman, R. B. (1969). *Nuovo Cimento* **B61**, 411.
45. Ruban, V. A. (1986). *Soviet Astronomy* **30(2)**, 142.
46. Papapetrou, A. (1953). *Annalen Physik* **12**, 309.
47. Rindler, W., and Perlick, V. (1990). *Gen. Rel. Grav.* **22**, 1067.

48. Krasinski, A. (1978). *Ann. Phys.* **112**, 22.
49. McManus, D. (1991). *Class. Quant. Grav.* **8**, 863.
50. Gutsunaev, Ts. I., and Manko, V. S. (1989). *Class. Quant. Grav.* **6**, L137.
51. Castejón-Amenedo, J., MacCallum, M. A. H., and Manko, V. S. (1989). *Class. Quant. Grav.* **6**, L211.
52. Tomimatsu, A., and Sato, H. (1972). *Phys. Rev. Lett.* **29**, 1344.
53. Tomimatsu, A, and Sato, H. (1973). *Prog. Theor. Phys.* **50**, 95.
54. Cosgrove, C. M. (1977). *J.Phys.* **A10**, 1481.
55. Cosgrove, C. M. (1978). *J.Phys.* **A11**, 2389, 2405.
56. Kramer, D., and Neugebauer, G. (1980). *Phys. Lett.* **75A**, 259.
57. Dietz, W., and Hoenselaers, C. (1982). *Phys. Rev. Lett.* **48**, 778.
58. Bičak, J., and Hoenselaers, C. (1985). *Phys. Rev. D***31**, 2476.
59. Tomimatsu, A. (1983). *Prog. Theor. Phys.* **70**, 385.
60. Hoenselaers, C. (1986). In *Proceedings of the IV Marcel Grossmann meeting in General Relativity, Rome*, R. Ruffini, ed. (Elsevier, Amsterdam).
61. Misner, C. W. (1963). *J. Math. Phys.* **4**,, 924.
62. Misner, C. W. (1967). In *Lectures in Applied Mathematics*, J. Ehlers, ed. (American Mathematical Society, Providence, Rhode Island).
63. Taub, A. H. (1951). *Ann. Math.* **53**, 472.
64. Bonnor, W. B. (1969). *Proc. Camb. Phil. Soc.* **66**, 145.
65. Sackfield, A. (1971). *Proc. Camb. Phil. Soc.* **70**, 89.
66. Demiański, M., and Newman, E. T. (1966). *Bulletin de l'Academie Polonaise des Sciences, Ser. Sciences Phys.* **14**, 653.
67. Dowker, J. S. (1974). *Gen. Rel. Grav.* **5**, 603.
68. Zimmerman, R. L., and Shahir, B. Y. (1989). *Gen. Rel. Grav.* **21**, 821.
69. van Stockum, W. J. (1937). *Proc. R. Soc. Edin.* **57**, 135.
70. Bonnor, W. B. (1980). *J. Phys.* **A13**, 2121.
71. Som, M. M., Teixeira, A. F. F., and Idel Wolk. (1976). *Gen. Rel. Grav.* **7**, 263.
72. Embacher, F. (1983). *J. Math. Phys.* **24**, 1182.
73. Jordan, S. R., and McCrea, J. D. (1982). *J. Phys.* **A15**, 1807.
74. Plebański, J., and Demiański, M. (1976). *Ann. Phys. (NY)* **98**, 98.
75. McCrea, J. D. (1976). *J. Phys.* **A9**, 697.
76. Wald, R. M. (1984). *General Relativity* (University of Chicago Press, Chicago).