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# A Note on Asymptotically Flat Spaces. II[1]

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*Abstract*

After brief reviews of the Geroch and spin-coefficient formalism approaches to null infinity, we present a dictionary which translates between the two formalisms.

## §(1): *Introduction*

There are now (at least) two frequently used methods of studying the asymptotic structure of space-time: (1) the method of spin coefficients [1] with the related Penrose [2a, b] conformal techniques, and (2) Geroch's [3] reformulation of the conformal approach with the Ashtekar [4] method of abstraction and formalization to problems on null infinity $\mathscr{I}$.

Though the two methods are obviously equivalent, the words, ideas, and formulas which arise appear to be very different, so much so that workers using the different formalisms frequently find it difficult to communicate with each other.

It is the purpose of this note to try to bridge this gap by presenting what is in essence a dictionary translating between the two formalisms. The basic tool used in this note is the introduction of three independent vectors and three independent one-forms on $\mathscr{I}$ which can be thought of as the pull-backs to $\mathscr{I}$ of a null tetrad system from the full four-space.

In Sections 2 and 3 we summarize, respectively, the Geroch results and the results from the spin-coefficient point of view and give their relationships. Finally in Section 4 we briefly describe the asymptotic symmetry of $\mathscr{I}$.

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We emphasize that this note is not intended as a review [5] of either formalism but as an aid in going from one to the other. It is assumed the reader is familiar with one of the formalisms. Furthermore, we point out that we have taken certain liberties with notation; most of the time we use the abstract index notation, but on occasions we switch to component or form notation without carefully distinguishing between them.

## §(2): *The Geroch Approach*

Following Geroch [3], we let $(\widetilde{M}, \widetilde{g}_{ab})$ be a space-time. By a simple asymptote of $(\widetilde{M}, \widetilde{g}_{ab})$ we mean a manifold $M$ with boundary $I$ with a smooth metric $g_{ab}$, a smooth scalar $\Omega$ together with an imbedding of $\widetilde{M}$ into $M$ (by which we identify $\widetilde{M}$ with its image in $M$, namely, $M - I$) such that

1. on $\widetilde{M}, g_{ab} = \Omega^2 \widetilde{g}_{ab}$;
2. on $I, \Omega = 0, n_a \equiv \nabla_a \Omega \neq 0$;
3. the restriction of $n^a \equiv g^{ab} n_b$ to $I$; i.e., $\underline{n}^a = i^* n^a$ is complete and the manifold of the orbits of $\underline{n}^a$ is diffeomorphic to $S^2$. $I$ has topology of $S^2 \times R$;
4. there exists a neighborhood $U_I$ of $I$ in $M$ such that $\widetilde{g}_{ab}$ satisfies the vacuum Einstein equations on $U_I \cap \widetilde{M}$.

If $(M, g_{ab}, \Omega)$ constitute a simple asymptote so does $(M, \omega^2 g_{ab}, \omega\Omega)$ for all smooth positive $\omega$. One thus has the freedom of additional conformal transformations. Asymptotes so related are called equivalent.

The fundamental equation (which is derived from the transformation law of the Ricci tensor under conformal rescalings) from which most of the basic results follow is

$$\Omega S_{ab} + 2\nabla_a n_b - f g_{ab} = \Omega^{-1} L_{ab} \equiv \Omega^{-1} \widetilde{S}_a{}^c g_{bc} = 0 \tag{1}$$

with

$$S_{ab} \equiv R_{ab} - \tfrac{1}{6} R g_{ab}, \qquad \widetilde{S}_{ab} \equiv \widetilde{R}_{ab} - \tfrac{1}{6} \widetilde{R} \widetilde{g}_{ab} \tag{2a}$$

$$f = \Omega^{-1} n_a n^a \tag{2b}$$

Equation (1) is to hold on $U_I$. From (1) and the Bianchi identities on $R_{abcd}$ the following equations (also on $U_I$) can be derived [3]:

$$S_{ab} n^b + \nabla_a f = 0 \tag{3a}$$

$$\nabla_{[a} S_{b]c} = -K_{abcd} n^d \tag{3b}$$

$$\nabla^m K_{abcm} = 0 \tag{3c}$$

$$\nabla_{[a} K_{bc]de} = 0 \tag{3d}$$

where $K_{abcd} = \Omega^{-1} C_{abcd}$.

Equation (1) restricted to $I$ implies $n_a n^a = 0$, i.e., $I$ is a null hypersurface.

Furthermore if we introduce on $M$ a null tetrad system $n^a, m^a, \overline{m}^a, l^a$ such that $n_a l^a = m_a \overline{m}^a = 1$ with all other contractions vanishing, then it is implied that $n^a$, $m^a$, and $\overline{m}^a$ are tangent to $I$.

The next step should be to define an abstract manifold $\mathcal{I}$ which is a diffeomorphic copy of $I$ and build the intrinsic structure in it. However, we do not lose generality if we simply identify $\mathcal{I}$ with $I$ and think of the pull-back operation to $\mathcal{I}$ as being the restriction to $I$. The action of the restriction operator $i^*$ on any tensor can be obtained if we give the action of $i^*$ on the tetrad basis (and the related forms) by

$$\begin{aligned} i^*n^a &= \underline{n}^a, & i^*l_a &= \underline{l}_a \\ i^*m^a &= \underline{m}^a, & i^*m_a &= \underline{m}_a \\ i^*\overline{m}^a &= \underline{\overline{m}}^a, & i^*\overline{m}_a &= \underline{\overline{m}}_a \\ & & i^*n_a &= 0 \end{aligned} \tag{4}$$

where the underlined quantities $\underline{n}^a, \underline{m}^a$, etc. (which are the same as $n^a, m^a$, etc. on $I$) reminds us that they should be thought of as living intrinsically on $\mathcal{I}$.

Note that though there is no meaning to $i^*l^a$, terms of the form $l^a n_b$ are allowed but pull back to zero. Roughly speaking an arbitrary tensor which depends at least in one slot on $l^a$ is either nonintrinsic to $\mathcal{I}$ or trivial when restricted to $\mathcal{I}$.

The basic idea of Geroch was to express or describe the asymptotic structure of $\widetilde{M}$ in terms of the pulled-back fields of $M$ on $\mathcal{I}$. There are essentially two types of structure: (a) the universal or geometrical structure which is common to all simple asymptotes, and (b) the dynamic structure which describes the gravitational fields.

Before proceeding further we point out that a natural restriction of the permissible choices of $\Omega$ (in the equivalence class of asymptotes) greatly simplifies the analysis. In the following, this choice is always made, namely, $\Omega$ is chosen so that $f = 0$ at $I$ [or using Eq. (1) $\nabla_a \nabla_b \Omega = 0$ at $I$]. Geometrically this means $n^a$ is covariantly constant on $I$. This limits the rescalings to those $\omega$ such that $\pounds_{\underline{n}} \underline{\omega} = 0$.

The two basic universal fields of $\mathcal{I}$ are $\underline{n}^a$ and $\underline{g}_{ab} = i^* g_{ab}$. By applying $i^*$ to (1) and using $\underline{f} = 0$ we have

$$\pounds_{\underline{n}}\, \underline{g}_{ab} = 0 \tag{5}$$

From $\underline{n}_a = \underline{g}_{ab}\underline{n}^b = 0$ one can conclude that $\underline{g}_{ab}$ is a degenerate metric with signature $(0, +, +)$. Note that if we write $g_{ab} = 2l_{(a}n_{b)} + 2m_{(a}\overline{m}_{b)}$ then $\underline{g}_{ab} = 2\underline{m}_{(a}\underline{\overline{m}}_{b)}$.

Condition (3) tells us that $\mathcal{I}$ has the structure of the bundle $S^2 \times R$ with $S^2$, the base space $B$, being the manifold of orbits of $\underline{n}^a$ and $R$ representing the integral curves of $\underline{n}^a$. If one denotes the projection $\pi$: $\mathcal{I} \to B$, equation (5) is telling us that $\underline{g}_{ab}$ is the lift of the positive definite metric $h_{ab}$ that lives on $B$, i.e., $\underline{g}_{ab} = \pi^* h_{ab}$.

Under the allowed conformal transformations

$$\underline{g}'_{ab} = \underline{\omega}^2 \underline{g}_{ab}, \qquad \underline{n}'^a = \underline{\omega}^{-1} \underline{n}^a \tag{6}$$

the tensor

$$\Gamma^{ab}{}_{cd} = \underline{n}^a \underline{n}^b \underline{g}_{cd} \tag{7}$$

is gauge invariant. The studies of the symmetries of $\mathscr{I}$ is based on solutions to $\mathcal{L}_\xi \Gamma^{ab}{}_{cd} = 0$. We will return to this in the last section.

One of the advantages of the particular choice of $\omega$'s is that it allows one to define in a very simple way the pull-back of the covariant derivative operator $\nabla_a$. It is easy to show [3] that $D_a \equiv i^* \nabla_a$ is unique and lives intrinsically on $\mathscr{I}$.

From $\nabla_a g_{bc} = 0$ and $\nabla_a n^b = 0$ [from equation (1) on $I$] we have

$$D_a \underline{g}_{bc} = 0 \quad \text{and} \quad D_a \underline{n}^b = 0 \tag{8}$$

and for covectors $\alpha_a$ such that $\alpha_a n^a = 0$ at $I$ we have (from $\nabla_a \alpha_b = \nabla_{[a}\alpha_{b]} + \frac{1}{2}\mathcal{L}_\alpha g_{ab}$)

$$D_a \underline{\alpha}_b = D_{[a}\underline{\alpha}_{b]} + \tfrac{1}{2}\mathcal{L}_{\underline{\alpha}}\, \underline{g}_{ab} \tag{9}$$

where $\underline{\alpha}^c$ is any vector such that $\underline{\alpha}_c = \underline{g}_{cd}\underline{\alpha}^d$. If further $\underline{\alpha}_a$ satisfies $\mathcal{L}_{\underline{n}} \underline{\alpha}_a = 0$ then equation (9) says, in effect, that the covariant derivative of vectors orthogonal to $\underline{n}^a$ which are Lie transported along $\underline{n}^a$, is by the metric connection of the base space $B$.

We can obtain the full action of the intrinsic connection by operating $D_a$ on the covectors $l_a$, $m_a$, and $\overline{m}_a$. In order to do this it is useful at this point to introduce coordinates on $\mathscr{I}$ in the following fashion:

(a) choose $\zeta$ and $\bar{\zeta}$ as the (complex) stereographic coordinates on $B$ with the result that $\zeta$ and $\bar{\zeta}$ are constant on the null generators (the integral curves of $\underline{n}^a$) on $\mathscr{I}$ and (b) choose an arbitrary cross section of $\mathscr{I}$, then "slide" it up and down by the integral curves of $\underline{n}^a$. This yields cross sections $u$ = const such that $\mathcal{L}_{\underline{n}} u = 1$. The coordinates on $\mathscr{I}$ are thus $(u, \zeta, \bar{\zeta})$.

Using these coordinates the one-forms $\underline{l}, \underline{\overline{m}}, \underline{m}$ could have been chosen proportional to $du$, $d\zeta$, and $d\bar{\zeta}$, respectively and the vectors $\underline{n}^a, \underline{\overline{m}}^a, \underline{m}^a$ proportional to $\partial/\partial u$, $\partial/\partial\bar{\zeta}$, and $\partial/\partial\zeta$. Specifically we would have

$$\underline{m} = \frac{d\bar{\zeta}}{\sqrt{2}P}, \qquad \underline{\overline{m}} = \frac{d\zeta}{\sqrt{2}P}, \qquad l = du \tag{10a}$$

$$\underline{m}^a \frac{\partial}{\partial x^a} = \sqrt{2}P \frac{\partial}{\partial \zeta}, \qquad \underline{\overline{m}}^a \frac{\partial}{\partial x^a} = \sqrt{2}P \frac{\partial}{\partial \bar{\zeta}}, \qquad \underline{n}^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u}$$

with

$$\underline{g}_{ab}\, dx^a\, dx^b = \frac{d\zeta d\bar{\zeta}}{P^2} \tag{10b}$$

[For the unit sphere $P = \frac{1}{2}(1 + \zeta\bar{\zeta})$.]

Since $\underline{m}_a$ satisfies $\pounds_n \underline{m}_a = 0$ and $\underline{m}_a \underline{n}^a = 0$, the *universal* part of the intrinsic connection is given by $\underline{D}_a \underline{m}_b = \pi^*(\hat{\nabla}_a \hat{m}_b)$ where $\hat{\nabla}_a$ is the metric connection of $B$ and $\underline{m}_a = \pi^* \hat{m}_a$.

If one now gives the action of $D_a$ on an $\underline{\alpha}_c$ such that $\underline{\alpha}_c \underline{n}^c \neq 0$ then we will have the general action of $D_a$. In order to simply express this action we let $\underline{\alpha}_c = \underline{l}_c$ of equation (10a) and set

$$D_a \underline{l}_b \equiv D_b \underline{l}_a = \underline{\gamma}_{(ab)} \tag{11}$$

from $\underline{l}_a \underline{n}^a = 1$ and $D_a \underline{n}^b = 0$ we have

$$\underline{\gamma}_{ab} \underline{n}^b = 0 \tag{12}$$

$\underline{\gamma}_{ab}$ can be decomposed into a trace-free part and a trace part

$$\underline{\gamma}_{ab} = \sigma^0_{ab} + \tfrac{1}{2} \underline{g}_{ab} \underline{g}^{mn} \underline{\gamma}_{mn} \tag{13}$$

where $\underline{g}^{mn}$ satisfies $\underline{g}^{mn} \underline{g}_{am} \underline{g}_{bn} = \underline{g}_{ab}$, i.e., it is defined up to multiples of $\underline{n}^a$ and hence by (12) defines the trace uniquely. The trace of $\underline{\gamma}_{ab}$ is pure gauge and can be made to vanish by an appropriate choice of $\underline{\omega}$. $\sigma^{\circ}_{ab}$, which is the radiation data for the asymptotically flat space and constitutes the "nonuniversal" part of the connection, $D_a$, on $\mathscr{I}$, can be written

$$\sigma^{\circ}_{ab} = \bar{\sigma}^{\circ} \underline{m}_a \underline{m}_b + \sigma^{\circ} \underline{\bar{m}}_a \underline{\bar{m}}_b \tag{14}$$

In the spin-coefficient formalism $\sigma^{\circ}$ is known as the asymptotic shear.

We will now collect some of the relevant equations from $M$ that are pulled back by $i^*$ to $\mathscr{I}$.

From the fact that the unphysical Weyl tensor $C_{abcd}$ vanishes on $I$ (see Ref. 3, Theorem II) we have on $I$

$$\begin{aligned} C_{abcd} &= 0 \\ R_{abcd} &= g_{a[c} S_{d]} - g_{b[c} S_{d]a} \end{aligned} \tag{15}$$

(There is no implication that $K_{abcd} \equiv \Omega^{-1} C_{abcd}$ vanishes on $I$.) From (15) we have that the pull-back to $\mathscr{I}$ of $R_{abcd}$ involves only the Ricci tensor, i.e., $S_{ab}$, or more accurately $S_a{}^b$, namely,

$$\underline{S}_a{}^b = i^* S_a{}^b \tag{16}$$

From $\nabla_{[a} \nabla_{b]} k_c = \tfrac{1}{2} R_{abc}{}^d k_d$ one has

$$D_{[a} D_{b]} \underline{k}_c = \tfrac{1}{2} \underline{R}_{abc}{}^d \underline{k}_d \tag{17a}$$

with

$$\underline{R}_{abc}{}^d = \underline{g}_{c[a} \underline{S}_{b]}{}^d + \underline{S}_{c[a} \underline{\delta}^d_{b]} \tag{17b}$$

and

$$\underline{S}_{ab} = \underline{g}_{bc} \underline{S}_a{}^c = \underline{S}_{ba}. \tag{17c}$$

Note that the pull back of $R_{abc}{}^d$ is the curvature tensor of the connection associated with $D_a$.

The tensors $\underline{R}_{abc}{}^d, \underline{S}_{ab}, \underline{S}_a{}^b$ and $S = \underline{S}_a{}^a$ can be explicitly expressed in terms of the connection by replacing in (17a) $\underline{k}_a$ by $\underline{l}_a, \underline{m}_a$, and $\underline{\overline{m}}_a$, respectively, and multiplying, respectively, by $\underline{n}^d, \underline{\overline{m}}^d$, and $\underline{m}^d$ and adding (using $\underline{\delta}_a^b = \underline{l}_a\underline{n}^b + \underline{m}_a\underline{\overline{m}}^b + \underline{\overline{m}}_a\underline{m}^b$) obtaining

$$\tfrac{1}{2}\underline{R}_{abc}{}^d = \underline{n}^d D_{[a}D_{b]}\underline{l}_c + \underline{m}^d D_{[a}D_{b]}\underline{\overline{m}}_c + \underline{\overline{m}}^d D_{[a}D_{b]}\underline{m}_c$$

or (17d)

$$\underline{R}_{abc}{}^d = 2\underline{n}^d D_{[a}D_{b]}\underline{l}_c + \mathcal{R}_{abc}{}^d$$

where the second term is the curvature tensor of the base space and the first term can be expressed [from (11)] in terms of $\underline{\gamma}_{ab}$. By taking the trace on $a$ and $d$ in (17d), using (17b), we obtain

$$\underline{S}_{ab} = -2\dot{\overset{\circ}{\sigma}}_{ab} + \tfrac{1}{2}\mathcal{R}\underline{g}_{ab} \tag{17e}$$

and

$$\underline{S}_a^a = S = -\dot{\underline{\gamma}} + \tfrac{1}{2}\mathcal{R} \tag{17f}$$

with $\underline{\gamma} = \underline{\gamma}_{ab}\underline{g}^{ab}$, the curvature scalar of the base space and dot meaning $\underline{n}^a D_a$. Also defining $\underline{g}^{ab} \equiv 2\underline{m}^{(a}\underline{\overline{m}}^{b)}$ we can write

$$\underline{S}_a{}^b = \underline{S}_{ac}\underline{g}^{cb} + A_a\underline{n}^b \tag{17g}$$

with

$$A_a = 2\underline{g}^{bc}D_b\overset{\circ}{\sigma}_{ac} - D_a\underline{\gamma} - \tfrac{1}{2}\mathcal{R}\underline{l}_a \tag{17h}$$

We now consider the pull-back of the tensor fields $K^{abcd}$ and ${}^*K^{abcd} = \frac{1}{2}\epsilon^{abef}K_{ef}{}^{cd}$ to $\mathcal{I}$. More specifically we consider

$$\begin{aligned} K^{ba} &= K^{ab} = -K^{ambn}n_m n_n \\ {}^*K^{ba} &= {}^*K^{ab} = -{}^*K^{ambn}n_m n_n \end{aligned} \tag{18}$$

both of which allow pull-backs $\underline{K}^{ab}$ and ${}^*\underline{K}^{ab}$. One now has the relationships [from (3b), (3c), and (3d)] between $D_a, \underline{S}_a{}^b, \underline{K}^{ab}$, and ${}^*\underline{K}^{ab}$,

$$D_{[a}\underline{S}_{b]}{}^c = \epsilon_{abm}{}^*\underline{K}^{mc} \tag{19a}$$

$$D_m\underline{K}^{am} = 0 \tag{19b}$$

$$D_m{}^*\underline{K}^{am} = 0 \tag{19c}$$

where $\epsilon_{abc}$ satisfies $\epsilon^{abc}\epsilon_{abc} = 3!$ and $\epsilon^{abc} \equiv i^*(\epsilon^{abcd}n_d)$. Knowledge of $D_a$ determines $\underline{S}_a{}^b$ (17) which in turn determines ${}^*\underline{K}^{ab}$ from (2.19a). The $\underline{K}^{ab}$ and

$*\underline{K}^{ab}$, both of which come from tensors on $M$ which were duals of each other, are related by

$$\underline{g}_{am}\underline{K}^{mb} = -\epsilon_{amp}\underline{n}^p{*\underline{K}}^{mb}$$
$$\underline{g}_{am}{*\underline{K}}^{mb} = \epsilon_{amp}\underline{n}^p\underline{K}^{mb}$$

They, however, do not determine each other completely; $\underline{K}^{ab}$ contains information about the longitudinal modes of the field, while $*\underline{K}^{ab}$ contains information only about the radiation coded in the $D_a$ or in the $\overset{\circ}{\sigma}_{ab}$ of (13). (See next section.)

Up till now we have only used the gauge condition leading to $\underline{f} = 0$ with $\mathfrak{L}_{\underline{n}}\underline{\omega} = 0$. Most of the objects discussed (and in particular $\underline{S}_{ab}$) have a rather complicated transformation law under a change in the conformal factor. There is, however, a unique tensor [3] $\rho_{ab}$ (depending only on the choice of conformal factor) such that

$$\underline{N}_{ab} \equiv \underline{S}_{ab} - \rho_{ab} \tag{20}$$

is gauge invariant, i.e., $\underline{N}'_{ab} = \underline{N}_{ab}$. $\underline{N}_{ab}$, which is known as the news tensor, has the following properties:

$$\underline{N}_{ab}\underline{g}^{ab} = 0, \quad \underline{N}_{ab}\underline{n}^b = 0, \quad \underline{N}_{ab} = \underline{N}_{ba}$$
$$D_{[a}\underline{N}_{b]c} = \epsilon_{abm}{*\underline{K}}^{mn}\underline{g}_{nc} \tag{21}$$
$$\underline{N}_{ab} = \bar{N}\underline{m}_a\underline{m}_b + N\underline{\bar{m}}_a\underline{\bar{m}}_b$$

The tensor $\rho_{ab}$ can be written

$$\rho_{ab} = \tfrac{1}{2}\mathcal{R}\underline{g}_{ab} + \overset{\circ}{\rho}_{ab} \tag{22}$$

with $\overset{\circ}{\rho}_{ab}\underline{g}^{ab} = 0$. In the *special case* of the conformal factor having been chosen so that $\mathcal{R}$ = const then $\overset{\circ}{\rho}_{ab} = 0$. Using (22) and (17f) we have

$$\underline{N}_{ab} = -2\dot{\overset{\circ}{\sigma}}_{ab} - \overset{\circ}{\rho}_{ab} \tag{23}$$

so that in the special case of constant $\mathcal{R}$ or in particular in a Bondi frame (where $\underline{g}_{ab}$ is the metric of a unit sphere) with $\mathcal{R} = 1$ we have

$$\underline{N}_{ab} = -2\dot{\overset{\circ}{\sigma}}_{ab} \tag{24a}$$

with the news satisfying [4, 6]

$$N = -2\dot{\overset{\circ}{\sigma}} \tag{24b}$$

To conclude this section we mention that Ashtekar [4] has reformulated the material of this section so that it is intrinsic to $\mathscr{I}$ with no need for $M$ or the pull-back operation $i^*$.

## §(3): *The Spin-Coefficient Formalism*

In this section we will first briefly review the basic ideas of the spin-coefficient (S.C.) formalism and some of the results for asymptotically flat space-time all expressed in the physical space-time $(\tilde{M}, \tilde{g}_{ab})$. We will then indicate how the spin-coefficient formalism appears in the rescaled manifold and how the spin-coefficient quantities are related to the fields on $\mathcal{I}$ of the previous section.

The basic quantities of the S.C. formalism are the null tetrad field $\tilde{l}_a, \tilde{n}_a, \tilde{m}_a, \tilde{\bar{m}}_a$ satisfying $\tilde{l}_a\tilde{n}^a = -\tilde{m}_a\tilde{\bar{m}}^a = 1$ with all other products vanishing. $\tilde{m}_a$ and $\tilde{\bar{m}}_a$ are complex conjugates of each other. The metric tensor takes the form

$$\tilde{g}_{ab} = 2\tilde{l}_{(a}\tilde{n}_{b)} - 2\tilde{m}_{(a}\tilde{\bar{m}}_{b)} \tag{25}$$

The metric connection is expressed in terms of the 12 complex spin coefficients, (the covariant derivatives of each of the tetrad vectors contracted on all indices by the different tetrad vectors), e.g.,

$$\tilde{\sigma} = \tilde{\nabla}_a\tilde{l}_b \cdot \tilde{m}^a\tilde{m}^b, \qquad \tilde{\rho} = \tilde{\nabla}_a\tilde{l}_b \cdot \tilde{m}^a\tilde{\bar{m}}^b \tag{26}$$

etc. (see Ref. 1) many of which have simple geometric meaning. Instead of the Weyl tensor $\tilde{C}_{abcd}$ one has its five complex tetrad components:

$$\begin{gathered}\psi_0 = -\tilde{C}_{abcd}\tilde{l}^a\tilde{m}^b\tilde{l}^c\tilde{m}^d, \qquad \psi_1 = -\tilde{C}_{abcd}\tilde{l}^a\tilde{m}^b\tilde{\bar{m}}^c\tilde{m}^d \\ \psi_2 = -\tilde{C}_{abcd}\tilde{l}^a\tilde{m}^b\tilde{\bar{m}}^c\tilde{n}^d \\ \psi_3 = -\tilde{C}_{abcd}\tilde{l}^a\tilde{n}^b\tilde{\bar{m}}^c\tilde{n}^d, \qquad \psi_4 = -\tilde{C}_{abcd}\tilde{n}^a\tilde{\bar{m}}^b\tilde{n}^c\tilde{\bar{m}}^d\end{gathered} \tag{27}$$

The vacuum Einstein equations are then differential equations relating the tetrad vectors, spin-coefficients, and $\psi$'s.

In the discussion of asymptotically flat space-times, a special choice of coordinate system is frequently made and referred to as Bondi coordinates. This coordinate system is not unique; the lack of uniqueness forms what was originally called the Bondi-Metzner-Sachs (coordinate) group [8]. One chooses a one-parameter set of null surfaces labeled by $u$; on the surface, the (null) geodesics are labeled by the (complex) stereographic coordinate $\zeta$ and $\bar{\zeta}$, the affine length $r$ along the geodesics being the last coordinate. In terms of these coordinates the tetrad has the form

$$\begin{aligned}\tilde{l}_a\, dx^a &= du \\ \tilde{l}^a \frac{\partial}{\partial x^a} &= \frac{\partial}{\partial r} \\ \tilde{n}^a \frac{\partial}{\partial x^a} &= \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^A \frac{\partial}{\partial x^A} \\ \tilde{m}^a \frac{\partial}{\partial x^a} &= \omega \frac{\partial}{\partial r} + \zeta^A \frac{\partial}{\partial x^A}\end{aligned} \tag{28}$$

with $A = (\zeta, \bar{\zeta})$. With simple assumptions on the behavior of the $\psi$'s and assuming the wave fronts ($u$ and $r$ constant) are topologically spheres the Einstein equations can be integrated asymptotically. The results, to leading terms, can be summarized as follows:

(a) the Weyl tensor components are

$$\begin{aligned} \psi_0 &= \psi_0^0 r^{-5} + O(r^{-6}) \\ \psi_1 &= \psi_1^0 r^{-4} + O(r^{-5}) \\ \psi_2 &= \psi_2^0 r^{-3} + O(r^{-4}) \\ \psi_3 &= \psi_3^0 r^{-2} + O(r^{-3}) \\ \psi_4 &= \psi_4^0 r^{-1} + O(r^{-2}) \end{aligned} \tag{29}$$

(b) the spin-coefficient $\sigma$ (the shear of $\tilde{l}_a$) is

$$\tilde{\sigma} = \sigma^\circ r^{-2} + O(r^{-4}) \tag{30}$$

(c) for the tetrad vectors we have

$$\hat{l}^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial r}, \qquad \tilde{n}^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u} - \frac{\partial}{\partial r} + O(r^{-1})$$

$$\hat{m}^a \frac{\partial}{\partial x^a} = \frac{\sqrt{2}P}{r} \frac{\partial}{\partial \zeta} + \frac{\bar{\omega}^\circ}{r} \frac{\partial}{\partial r} + \cdots$$

(d) the metric has (for leading terms) the form

$$ds^2 = du^2 \left(1 - \frac{\psi_2^0}{r} + \cdots\right) + 2\, du\, dr - r^2 \frac{d\zeta d\bar{\zeta}}{P^2} + \cdots \tag{31}$$

with

$$P = \tfrac{1}{2}(1 + \zeta\bar{\zeta})$$

(e) the leading Weyl tensor terms are given by

$$\begin{aligned} \psi_4^0 &= -\ddot{\bar{\sigma}}^0 \\ \psi_3^0 &= \eth\dot{\bar{\sigma}}^0 \\ \psi_2^0 - \bar{\psi}_2^0 &= \bar{\eth}^2 \sigma^0 - \eth^2 \bar{\sigma}^0 + \bar{\sigma}^0 \dot{\sigma}^0 - \sigma^0 \dot{\bar{\sigma}}^0 \end{aligned} \tag{32}$$

(f) with the evolution equations

$$\begin{aligned} \dot{\psi}_0^0 &= -\eth\psi_1^0 + 3\sigma^0 \psi_2^0 \\ \dot{\psi}_1^0 &= -\eth\psi_2^0 + 2\sigma^0 \psi_3^0 \\ \dot{\psi}_2^0 &= -\eth\psi_3^0 + \sigma^0 \psi_4^0 \end{aligned} \tag{33}$$

where a dot denotes $\partial/\partial u$ and $\eth$ [9] is essentially $\partial/\partial\zeta$. Knowledge of $\sigma(u, \zeta, \bar{\zeta})$, the asymptotic shear, allows, from (32), the calculation of $\psi_4^0$, $\psi_3^0$, and Im $\psi_2^0$, while further knowledge of Re $\psi_2^0$, $\psi_1^0$, and $\psi_0^0$ at *one* value of $u$ gives the Weyl tensor to order $r^{-6}$.

Equations (29)-(33) summarize the basic properties of the asymptotic region of $(\widetilde{M}, \widetilde{g}_{ab})$.

Before constructing the asymptote associated with $(\widetilde{M}, \widetilde{g}_{ab})$ [7] we point out a slight technical (or notational) problem which must be overcome before we translate from one formalism to the other: the Geroch formalism of Section 2 uses a signature $(- + + +)$ while the spin-coefficient formalism of this section uses $(+ - - -)$. Since in the spin-coefficient formalism the tetrad vectors are considered to be the basic set of variables (the metric being defined from them) it is easiest to arrange the change of signature by changing some of the tetrad variables. In particular it can be accomplished, for example, by leaving all the contravariant components of the tetrad vectors unchanged but changing the signs of all the covariant components. We will accomplish this in a slightly different way which not only changes the signature but reverses the sign of the $l^a$ vector (making it past rather than future pointing and hence keeps $l^a n_a = 1$), namely, by introducing

$$\begin{aligned} \widetilde{l}'^a &= -\widetilde{l}^a, & \widetilde{l}'_a &= \widetilde{l}_a \\ \widetilde{n}'^a &= \widetilde{n}^a, & \widetilde{n}'_a &= -\widetilde{n}_a \\ \widetilde{m}'^a &= \widetilde{m}^a, & \widetilde{m}'_a &= -\widetilde{m}_a \end{aligned} \tag{34}$$

We thus have $\widetilde{l}' \cdot \widetilde{n}' = \widetilde{m}' \cdot \overline{\widetilde{m}}' = 1$ and $\widetilde{g}'_{ab} = 2\widetilde{l}'_{(a}\widetilde{n}'_{b)} + 2\widetilde{m}'_{(a}\overline{\widetilde{m}}'_{b)}$. Noting that covariant differentiation and the curvature tensor in the form $R^a{}_{bcd}$ are unaffected by the signature change it is easy, using (34), to see how all the tetrad variables change under the signature change and reversal of $l^a$, e.g.,

$$\psi'_4 = -\psi_4$$

To now construct the asymptote we must choose a conformal factor $\Omega$ that vanishes at the boundary ($r = \infty$) and a coordinate system which includes the boundary. With $g_{ab} = \Omega^2 \widetilde{g}'_{ab} = -\Omega^2 \widetilde{g}_{ab}$ and the choice

$$\begin{aligned} l^a &= \Omega^{-2}\widetilde{l}'^a = -\Omega^{-2}\widetilde{l}^a, & l_a &= \widetilde{l}'_a = \widetilde{l}_a \\ n^a &= \widetilde{n}'^a = \widetilde{n}^a, & n_a &= \Omega^2 \widetilde{n}'_a = -\Omega^2 \widetilde{n}_a \\ m^a &= \Omega^{-1}\widetilde{m}'^a = \Omega^{-1}\widetilde{m}^a, & m_a &= \Omega\widetilde{m}'_a = -\Omega\widetilde{m}_a \end{aligned} \tag{35}$$

we have

$$g_{ab} = 2l_{(a}n_{b)} + 2m_{(a}\overline{m}_{b)}$$

If we introduce the new coordinate $\hat{r} = r^{-1}$ and choose $\Omega = \hat{r}$, we have, from (29)-(34) on $I$, i.e., at $\hat{r} = 0$, the following:

$$ds^2 = 2du\, d\hat{r} + \frac{d\zeta\, d\bar{\zeta}}{P^2} \tag{36}$$

$$l_a\,dx^a = du, \qquad l^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial \hat{r}}$$

$$m_a\,dx^a = \frac{d\bar{\zeta}}{\sqrt{2}P}, \qquad m^a \frac{\partial}{\partial x^a} = \sqrt{2}P \frac{\partial}{\partial \zeta} \tag{37}$$

$$n_a\,dx^a = d\hat{r}, \qquad n^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial u}$$

$$\sigma = \sigma^0 = \nabla_a l_b \cdot m^a m^b \tag{38}$$

and finally

$$\begin{aligned}
\psi_0^0 &= +\Omega^{-1} C_{abcd} l^a m^b l^c m^d \\
\psi_1^0 &= -\Omega^{-1} C_{abcd} l^a m^b \bar{m}^c m^d \\
\psi_2^0 &= -\Omega^{-1} C_{abcd} l^a m^d \bar{m}^c n^d \\
\psi_3^0 &= -\Omega^{-1} C_{abcd} \bar{m}^a n^b l^c n^d \\
\psi_4^0 &= +\Omega^{-1} C_{abcd} \bar{m}^a n^b \bar{m}^c n^d
\end{aligned} \tag{39}$$

[Note the sign change in the first and last term compared with (27).]

We now see that $K_{abcd}$ of Section 2 restricted to $I$ is equivalent to the leading terms of the physical $\psi$'s, i.e., to $\psi_0^0$, $\psi_1^0$, $\psi_2^0$, $\psi_3^0$, and $\psi_4^0$. The pull-back of (36), (37) give $g_{ab}$, $\underline{l}_a$, $\underline{m}_a$, $\underline{\bar{m}}_a$, $\underline{n}^a$, $\underline{m}^a$, $\underline{\bar{m}}^a$, while the pull-back of (38) is equivalent to (11), ($\overline{13}$), and (14). The trace part in (13) which was gauge there is now determined uniquely because the conformal factor was chosen as $\Omega = r^{-1}$ with the $r$ an affine length.

From the definitions of $K^{ab}$ and $^*K^{ab}$ in (18) using (39) we have

$$\underline{K}^{ab} = 2\ \mathrm{Re}\ \psi_2^0 \underline{n}^a \underline{n}^b + 2\psi_3^0 \underline{n}^{(a}\underline{m}^{b)} + 2\bar{\psi}_3^0 \underline{n}^{(a}\underline{\bar{m}}^{b)} - \psi_4^0 \underline{m}^a \underline{m}^b - \bar{\psi}_4 \underline{\bar{m}}^a \underline{\bar{m}}^b \tag{40a}$$

$$^*\underline{K}^{ab} = -2\ \mathrm{Im}\ \psi_2^0 \underline{n}^a \underline{n}^b + 2i\psi_3^0 \underline{n}^{(a}\underline{m}^{b)} - 2i\bar{\psi}_3^0 \underline{n}^{(a}\underline{\bar{m}}^{b)} - i\psi_4^0 \underline{m}^a \underline{m}^b + i\bar{\psi}_4^0 \underline{\bar{m}}^a \underline{\bar{m}}^b \tag{40b}$$

from which we see that $^*\underline{K}^{ab}$ is determined from $\sigma_{ab}$ while $\underline{K}^{ab}$ is not.

Equations (40), with (31), (32), and (35) and

$$\underline{N}_{ab} = \underline{S}_{ab} - \rho_{ab} = -2(\dot{\bar{\sigma}}^0 \underline{m}_a \underline{m}_b + \dot{\sigma}^0 \underline{\bar{m}}_a \underline{\bar{m}}_b)$$

$$\sigma_{ab} = \bar{\sigma}^0 \underline{m}_a \underline{m}_b + \sigma^0 \underline{\bar{m}}_a \underline{\bar{m}}_b$$

constitute the main dictionary items between the S.C. and Geroch versions of asymptotically flat spaces.

A freedom which still exists in the spin-coefficient version would be a different choice of Bondi coordinates. This would manifest itself on $\mathscr{I}$ as a new set of cuts $u' = u + \alpha(\zeta, \bar{\zeta})$, new coordinates $\zeta'$ and $\bar{\zeta}'$ for the generators, new vectors $\underline{n}'^a$, $\underline{m}'^a$, $\underline{\bar{m}}'^a$, $l'_a$, $\underline{m}'_a$, $\bar{m}'_a$ and a new $\Omega'$ as is discussed in the next section.

## §(4): *Asymptotic Symmetries of $\mathcal{I}$*

The last topic we want to discuss is how the B.M.S. group arises in both formalisms.

From the Geroch point of view the kinematical arena is constructed with the universal fields that live on $\mathcal{I}$. Therefore the asymptotic symmetries of $\mathcal{I}$ arise from considering the diffeomorphisms that preserve the conformally invariant universal fields. More specifically, the Lie algebra of infinitesimal symmetries $\mathfrak{L}$ is generated by vector fields $\xi^a$ that are solutions of the equation

$$\mathcal{L}_\xi \Gamma^{ab}{}_{cd} = 0 \Longleftrightarrow \mathcal{L}_\xi \underline{g}_{ab} = 2K\underline{g}_{ab}; \mathcal{L}_\xi \underline{n}^a = -K\underline{n}^a \tag{41}$$

for some scalar field $K$.

The solutions of $\mathcal{L}_\xi \underline{g}_{ab} = 0, \mathcal{L}_\xi \underline{n}^a = 0$ of the form $\xi^a = \alpha\underline{n}^a, \mathcal{L}_{\underline{n}}\alpha = 0$ are called supertranslations and it is not difficult to show [3] that the set of supertranslations $\mathcal{S}$ is indeed an ideal in the Lie algebra $\mathfrak{L}$.

Finally $\mathfrak{L}/\mathcal{S}$ can be obtained by considering the solutions of

$$\underline{n}^a\xi_a = 0, \qquad D_{(a}\xi_{b)} = K\underline{g}_{ab}, \qquad \mathcal{L}_{\underline{n}}\xi_a = 0 \tag{42}$$

since this determines a $\xi^a$ such that $\xi_a = \underline{g}_{ab}\xi^b$ up to addition of an arbitrary element of $\mathcal{S}$.

Equation (42) tells us that $\xi_a$ is the lift of a form living in the base space. Therefore $\mathfrak{L}/\mathcal{S}$ is isomorphic with the Lie algebra of conformal Killing fields of $S^2$, i.e., $\mathfrak{L}/\mathcal{S}$ is isomorphic to the Lorentz group Lie algebra.

Now as we pointed out in Section 3 there is another way to generate the asymptotic symmetries of $\mathcal{I}$.

Since the choice of a coordinate system adopted for $\mathcal{I}$ is not unique we can ask what is the freedom in the choice of coordinates and its effect on the vectors and forms. For simplicity we restrict ourselves to the infinitesimal transformations $x^a \to x'^a = x^a + \xi^a(x)$. From the freedom of initial slice we have the first possibility $u' = u + \alpha(\zeta, \bar{\zeta})$ with now $\alpha(\zeta, \bar{\zeta})$ infinitesimal or

$$\xi^a = \alpha\underline{n}^a$$

The second type of freedom arises by considering *special* conformal transformations in the *base* space, of which there is a six-parameter set since $B$ is topologically $S^2$. By a *special* conformal transformation is meant a conformal rescaling of $\underline{B}$ generated by solutions of the conformal Killing equation $\hat{\nabla}_{(a}\eta_{b)} = Kh_{ab}$ where $\hat{\nabla}_a$ is the base space metric connection and $\underline{\omega} = 1 + K$. Metrics related by the special conformal transformation are transformable into each other by the coordinate transformation $x'^a = x^a + \eta^a$ with $\eta^a$ the conformal Killing vector. This coordinate transformation can be lifted to $\mathcal{I}$ in the following fashion. Since the base space metric is changed by $h_{ab} \to \underline{\omega}^2 h_{ab}$ we must have

$$\underline{n}'^a = \underline{\omega}^{-1}\underline{n}^a \quad \text{or} \quad \frac{\partial}{\partial u'} = \underline{\omega}^{-1}\frac{\partial}{\partial u}$$

which implies

$$u' = \omega u + \beta(\zeta, \bar{\zeta}) = (1 + K)u + \beta(\zeta, \bar{\zeta})$$

The demand that the original slice go into itself makes $\beta = 0$. The conformal Killing fields $\eta^a$ can be lifted to the $u'$ = const slices giving

$$\xi^a = \eta^a + uK\underline{n}^a$$

Note that in either case

$$\xi^a = \alpha(\zeta, \bar{\zeta})\underline{n}^a$$

or (43)

$$\xi^a = \eta^a + uK\underline{n}^a$$

one could think of these transformations in the active sense. They generate what is called the Bondi-Metzner-Sachs group, with the first expression being the supertranslations and the second the homogeneous Lorentz transformations (which maps $u = 0$ into itself). Note that (43) constitutes the general solution to the equation for asymptotic symmetries $\mathfrak{L}_{\underline{\xi}}\Gamma^{cd}_{ab} = 0$.

## References


1. Newman, E., and Penrose, R. (1962). *J. Math. Phys.*, **3**, 566.
2. (a) Penrose, R. (1963). *Phys. Rev. Lett.*, **10**, 66.
   (b) Penrose, R. (1968). In *Battelle Recontres*, B. De Witt and J. A. Wheeler (eds.), Benjamin, New York.
3. Geroch, R. (1976). In *Asymptotic Structure of Space-Time*, P. Esposito and L. Witten (eds.), Plenum, New York.
4. Ashtekar, A. (1981). *J. Math. Phys.*, **22**, 2885.
5. Newman, E., and Tod, K. (1980). In *General Relativity and Gravitation*, Vol. II, A. Held (ed.), Plenum, New York.
6. Ramaswamy, S., and Sen, A. (1981). *J. Math. Phys.*, **22**, 2612.
7. Ludwig, G. (1981). *Gen. Rel. Grav.*, **13**, 291.
8. Penrose, R. (1972). *Group Theory in Non-linear Problems*, A. O. Barut (ed.), Reidel, Dordrecht.
9. Ko, M., and Newman, E. (1976). In *Asymptotic Structure of Space-Time*, P. Esposito and L. Witten (eds.), Plenum, New York.