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The Cosmological Term and a Modified Brans–Dicke Cosmology

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Abstract

Adding the cosmological term Λ, which is assumed to be variable in this paper, to the Brans-Dicke Lagrangian, we try to understand the meaning of the term and to relate it to the mass of the universe. We also touch upon the Dirac large-number hypothesis, applying the results obtained from the application of our theory to a uniform cosmological model.

§(1): *Introduction*

After the cosmological constant was introduced into cosmology by Einstein, its real significance was studied by various cosmologists (for example, [1]), but no satisfactory results of its meaning have yet been reported. Zel'dovich [2] has tried to visualize the meaning from the theory of elementary particles because the constant corresponds to the vacuum energy. Actually Linde [3] has argued that the cosmological term[1] arises from spontaneous symmetry breaking and suggested that the term is not a constant but is a function of temperature. Also Dreitlein [4], though he regards the term as a constant, connects the mass of the Higgs scalar boson with both the term and the gravitational constant. In cosmology, the term may be understood by incorporation with Mach's principle, which suggests the acceptance of the Brans-Dicke Lagrangian as a realistic case [5]. The investigation of particle physics within the context of the Brans-Dicke Lagrangian (for example, [6]) stimulates us to study the term with a

[1] Throughout this paper, we call the constant the cosmological term.

modified Brans-Dicke Lagrangian from cosmology and elementary particle physics.

§(2): *Fundamental Equations*

We assume the cosmological term is an explicit function of a scalar field ϕ, as was proposed by Bergmann and Wagoner [7], and start with the usual variational principle of general relativity using a Brans-Dicke Lagrangian modified by $\Lambda(\phi)$:

$$0 = \delta \int \left[\phi(R - 2\Lambda(\phi)) + \frac{16\pi}{c^4} L_m - \frac{\omega}{\phi} \phi_{,i}\phi^{,i} \right] (-g)^{1/2} d^4x \tag{2.1}$$

Here R is the scalar curvature and L_m is the Lagrangian density of matter, which is assumed not to depend explicitly on derivatives of g_{ij}, and ϕ plays a role analogous to G^{-1}. The field equation for the metric field is, then,

$$R_{ij} - \frac{1}{2} g_{ij}R + g_{ij}\Lambda = \frac{8\pi}{\phi c^4} T_{ij} + \frac{\omega}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2} g_{ij}\phi_{,k}\phi^{,k}\right) + \frac{1}{\phi}(\phi_{,i;j} - g_{ij}\Box\phi) \tag{2.2}$$

where

$$T^{ij} = \frac{2}{(-g)^{1/2}} \frac{\partial}{\partial g_{ij}} [(-g)^{1/2} L_m]$$

is the energy-momentum tensor of matter. Contraction of equation (2.2) results in

$$R - 4\Lambda = -\frac{8\pi}{\phi c^4} T + \frac{\omega}{\phi^2} \phi_{,i}\phi^{,i} + \frac{3}{\phi}\Box\phi \tag{2.3}$$

While the field equation for ϕ is obtained by varying ϕ and $\phi_{,i}$ in equation (2.1)

$$R - 2\Lambda - 2\phi \frac{\partial\Lambda}{\partial\phi} = \frac{\omega}{\phi^2} \phi_{,i}\phi^{,i} - \frac{2\omega}{\phi}\Box\phi \tag{2.4}$$

By eliminating R from equations (2.3) and (2.4),

$$\Lambda - \phi \frac{\partial\Lambda}{\partial\phi} = \frac{4\pi}{\phi c^4} T - \frac{2\omega + 3}{2\phi}\Box\phi \tag{2.5}$$

Since Brans-Dicke cosmology does not pay any attention to Λ, the relation between the matter field and the scalar field can be obtained analytically through an equation analogous to equation (2.5), although a certain degree of arbitrariness inevitably accompanies the introduction of a scalar field. Here we assume the simplest case of the coupling of the two fields as follows:

$$\Box\phi = \frac{8\pi}{(2\omega + 3)c^4} \mu T \tag{2.6}$$

where the constant μ shows how much our theory including $\Lambda(\phi)$ deviates from that of Brans and Dicke. Then a particular solution of equation (2.5) is given as

$$\Lambda = \frac{2\omega + 3}{4} \frac{1 - \mu}{\mu} \frac{\Box\phi}{\phi} = \frac{8\pi(1 - \mu)}{4\phi c^4} T \tag{2.7}$$

Since we assume Λ is a function of only ϕ, we have

$$\Box\phi = f(\phi) \tag{2.8}$$

By multiplying equation (2.2) by ϕ and taking the covariant derivative, we find

$$\frac{8\pi}{c^4} T^i_{j;i} = -\frac{1}{2}\left(R - 2\Lambda - 2\phi\frac{\partial\Lambda}{\partial\phi} - \frac{\omega}{\phi^2}\phi_{,i}\phi^{,i} + \frac{2\omega}{\phi}\Box\phi\right)\phi_{,j}$$

Equation (2.4) then ensures that the conservation law $T^i_{j;i} = 0$ holds.

Therefore equations (2.2) and (2.6)-(2.8) are the fundamental equations of our theory.

§(3): *Application to Cosmology*

We apply those equations to the homogeneous and isotropic universe. Then the metric is that given by Robertson and Walker, and the energy-momentum tensor, that of a perfect fluid,

$$ds^2 = -g_{ij}dx^i dx^j = c^2 dt^2 - a^2(t)\left\{\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\zeta^2)\right\}$$

$$T_{ij} = (p + \epsilon)\, u_i u_j + p g_{ij}$$

where $a(t)$ is the spatial scale factor, k the dimensionless curvature index, ϵ and p the total energy density and pressure of the universe, respectively, and u^i a velocity four-vector that has components of $(1, 0, 0, 0)$ in the comoving coordinates. The $(0, 0)$ component of equation (2.2) is

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}c^2 = \frac{c^2}{3}\left(\frac{8\pi}{\phi c^4}\epsilon + \Lambda\right) - \frac{\dot{a}}{a}\frac{\dot{\phi}}{\phi} + \frac{\omega\dot{\phi}^2}{6\phi^2} \tag{3.1}$$

where we assume ϕ depends only on universal time t and a dot indicates differentiation with respect to t. From equation (2.6), we obtain

$$\frac{d}{dt}(\dot{\phi}a^3) = \frac{8\pi}{(2\omega + 3)c^2}\mu(\epsilon - 3p)a^3 \tag{3.2}$$

From equation (2.7), we obtain the variable Λ as follows:

$$\Lambda = \frac{8\pi(1 - \mu)}{4\phi c^4}(3p - \epsilon) \tag{3.3}$$

The equation of state in the radiation field $3p = \epsilon$ leads to the vanishing cosmological term. Equation (3.3) yields the expression for $f(\phi)$ as

$$f(\phi) = \frac{8\pi\mu}{(2\omega + 3)c^4}(3p - \epsilon) \tag{3.4}$$

The convseration law $T^{ij}{}_{;j} = 0$ requires the well-known relation

$$\dot{\epsilon} = -3\frac{\dot{a}}{a}(p + \epsilon) \tag{3.5}$$

As an example, we consider the case of the matter field where Λ is not zero. In this field we can neglect pressure, which has less significance for the model of the universe [8]. Then equation (3.5) gives

$$\epsilon a^3 = \epsilon_0 a_0^3 \tag{3.5'}$$

where subscript 0 indicates the present value of a quantity. From equation (3.2), we obtain the following relation for the big-bang universe:

$$\dot{\phi} a^3 = Kt \tag{3.2'}$$

where

$$K = \frac{8\pi}{(2\omega + 3)c^2}\mu\epsilon_0 a_0^3$$

From equation (3.3), we have

$$\Lambda = \frac{8\pi}{4\phi c^4}(\mu - 1)\epsilon \tag{3.3'}$$

This implies that the cosmological term participates in the mass of the universe, and that may have a significance in determining the mass of an elementary particle when applied to the hadron era as studied in [4]. From equation (3.4), $f(\phi)$ is proportional to a^{-3}:

$$\frac{\dot{a}}{a} = -\frac{\dot{\phi}}{3}\frac{f'}{f} \tag{3.4'}$$

where $f' = df/d\phi$.

Substituting equations (3.2′)-(3.4′) into equation (3.1), we obtain

$$\left(\frac{\dot{\phi}}{\phi}\right)^2 \left[\frac{\phi^2}{9}\left(\frac{f'}{f}\right)^2 - \frac{\phi}{3}\frac{f'}{f} - \frac{\omega}{6}\right] = -\frac{(3 + \mu)c^2}{12\mu}(2\omega + 3)\frac{f}{\phi} - kc^2\left(-\frac{c^2 f}{K}\right)^{2/3} \tag{3.1'}$$

Since, in particle physics, the terms of ϕ^n are introduced to explain the origin of the mass of an elementary particle by symmetry breaking [9], we assume the functional form of $f(\phi)$ as follows:

$$f(\phi) = -\frac{8\pi}{(2\omega + 3)c^4} \mu\epsilon_0 \left(\frac{\phi}{\phi_0}\right)^n \tag{3.6}$$

Substituting equation (3.6) into equation (3.1′), we solve in the case of $k = 0$, i.e., the flat universe. When we notice two conditions, an expanding universe and an increasing function $\phi(t)$, $n = 1$ which gives the constant cosmological term is not suitable. Then, with $n \neq 1$ and the assumption of $\phi = 0$ at $t = 0$, the solution is

$$\phi = (At)^{2/(1-n)} \tag{3.7}$$

where

$$A = \pm \frac{1-n}{2} \left[\frac{2\pi(3+\mu)\epsilon_0 \phi_0^{-n}}{(\frac{1}{3}n^2 - n - \frac{1}{2}\omega)c^2}\right]^{1/2}$$

Equation (3.4′) with equation (3.7) results in the following relation:

$$a^3 = a_0^3 \phi_0^n (At)^{2n/(n-1)} \tag{3.8}$$

Again by the two conditions, the domain of n narrows into $n < 0$. Since equations (3.7) and (3.8) satisfy equation (3.2′),

$$n = \frac{3}{16\mu} \{-(3+\mu)(2\omega+3) + 8\mu \pm [(3+\mu)^2(2\omega+3)^2 - 16\mu(2\omega+3)(1-\mu)]^{1/2}\} \tag{3.9}$$

From equation (3.3′),

$$\Lambda = \frac{2\omega + 3}{2(n-1)} \frac{1-\mu}{\mu} \frac{1}{(ct)^2} \tag{3.10}$$

Following Brans and Dicke [5] and Weinberg [10], the gravitational "constant" G is given by the weak field approximation as follows:

$$G = \frac{1}{2}\left(3 - \frac{2\omega+1}{2\omega+3}\mu\right)\phi^{-1} \tag{3.11}$$

Equations (3.11), (3.7), and (3.8) present the relations of G with the Hubble constant $H = \dot{a}/a$, as

$$\frac{\dot{G}}{G} = \frac{3}{n} H, \quad \frac{H}{G} \sim t^{(1+n)/(1-n)}, \quad \text{and} \quad G\epsilon \sim H^2 \tag{3.12}$$

When we put $n = -1$ in equations (3.8) and (3.12), the following relations are obtained:

$$a \propto t^{1/3}, \quad H = \frac{1}{3t}, \quad \frac{H}{G} \sim \text{const}, \quad \text{and} \quad \frac{\dot{G}}{G} = -3H \tag{3.13}$$

and this is the case of Dirac [11][2] although the cosmological term is not zero but is

$$\Lambda = \frac{5(3\omega + 2)}{18} \frac{1}{(ct)^2}$$

While if we put $\mu = 1$, all the equations we obtained reduce to those of Brans and Dicke and equations (3.9), (3.10), and (3.12) give $n = -3(\omega + 1)$, $\Lambda = 0$, and $\dot{G}/G = -H/(\omega + 1)$.

§(4): *Discussion*

In the present work, the cosmological term has been studied from the point of view of cosmology and elementary particle physics and its origin is partially clarified because we have established the relation connecting it with the mass of the universe. But we have failed to make Λ correspond to a mass term in particle physics because of the negative value of n. In the hadron era, the approximation of $p = 0$ holds fairly well [13] and there will be a possibility of overcoming the difficulty by taking other metrics.

Since we accept the Brans-Dicke theory, it is natural that $G\epsilon \sim H^2$ is valid, but it is interesting to notice that $n = -1$ gives the results of equation (3.13), which have been derived by Dirac concerning with the large-number hypothesis.

Thus the investigation of the cosmological term from cosmology and particle physics might give a clue to solving the problems in the large-number hypothesis and Mach's principle.

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[2] After the completion of this work, Bishop [12] gave the same result $\phi \propto a^3$ to this case in an approach differing from our own.

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