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Smoothing and Extending Cosmic Time Functions

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Abstract

Any continuous time function on a C^k space-time V (i.e., a scalar on V that increases along any causal curve) can be approximated by smooth C^k time functions. A time function defined on a (bounded) subset of a stably causal V can be extended to a time function on the whole of V.

§(1): *Introduction*

In the heroic era of global general relativity some relativists worked their way through the (only partially ordered) set of causality conditions and came to a halt at the comfortable station "stable causality." The virtue of stable causality is that it is a reasonable notion for almost the weakest causality requirement and, on the other hand, one can work with it also in concrete calculations because of a theorem of Hawking [3] stating that it is equivalent to the existence of a global "cosmic" time function. The very elegant proof of Hawking modifying a method of Geroch [1] constructs a continuous scalar t that increases along any causal curve.

In my thesis [6] I gave (i) a smoothing procedure ("regularization") for t, i.e., one might write the metric globally as $ds^2 = -f^2 dt^2 + g_{\alpha\beta} dx^\alpha dx^\beta$, where f, $g_{\alpha\beta}$ have the same differentiability class as g_{ab} (see Section 3); (ii) another way of constructing t, which is much more laborious than Hawking's but explicit, and flexible enough so that it can be used to extend a time function given on a (compact) subset of the space time (Section 4).

Meanwhile I have learned that these procedures treating (null) cones and their relations to families of level surfaces of a scalar t are not only useful for proving plausible improvements of Hawking's theorem. For example, in the case of a potential field these methods can help to find suitably adapted coordinates and to understand the "causal" behavior of particles under the influence of the potential; see [5].

The basic principle of smoothing a time function is the fact that certain approximations of continuous functions f by smooth ones (f_n) also approximate bounds fulfilled by f and/or derivatives of f; e.g., if $f(x + \xi) - f(x) \geqslant K \cdot \xi$ for all ξ, then for any $\epsilon > 0$ and for suitably large n, $df_n/dx \geqslant K - \epsilon$. Therefore the gradient vector of a suitable smooth approximation t_n of a time function t will lie in an arbitrarily small neighborhood of the light cone. If t in addition is "uniformly" timelike in some sense (see Section 1), we can find a smooth t_n with a timelike gradient and t_n is then the desired smooth time function as constructed in Hawking's proof and is automatically uniform and can be smoothed without further "preparation"; on the other hand, Hawking's definition requires that the level surfaces (t = const) remain spacelike under a slight variation of the metric which widens the null cones. It is compatible with *our* definition that a surface (t = const) is tangent to a null direction in a single point, hence we need the possibility of uniformization to show that the existence of a time function implies stable causality.

The basic idea of constructing extensions of time functions is to take a metric $\hat{g}$ that has light cones wider than those of g (as g is stably causal we always can find such a $\hat{g}$) and use $\hat{g}$-null horizons (hence g-spacelike hypersurfaces) as level surfaces for t.

§(2): *The Zoology of Spacelike Hypersurfaces*

In notation and conventions this paper mainly follows the book of Hawking and Ellis [4]; V denotes a space-time on a smooth, Hausdorff, paracompact, connected manifold possessing a time-oriented C^2-Lorentz metric of signature $(-+++)$.

On the set of Lorentz metrics on a manifold one can introduce a partial ordering: $g < \hat{g}$ if for any vector $v^a \neq 0$ with $g_{ab}v^a v^b \leqslant 0$ it holds that $\hat{g}_{ab}v^a v^b < 0$. The causal/timelike futures/pasts and the domains of dependence $J^\pm, I^\pm, D^\pm$, D are denoted and defined as usual. If it is not obvious with respect to which Lorentz metric the future is taken, this metric g is explicitly denoted: $J^+(.;g)$.

$\tilde{J}^+(p;g) := \cap J^+(p;\hat{g})$, where the intersection has to be taken over all $\hat{g} > g$.

Sometimes an auxiliary Euclidean metric e_{ab} is used. We distinguish two types of distances (with respect to g and to e):

$$F_W(A,B) := \sup_\gamma \left\{ \int_\gamma ds \,\middle|\, \gamma = \widehat{xy}, x \in A, y \in B, \gamma \text{ causal curve in } W \right\}$$

$$\Delta_W(A,B) := \sup_{x \in A} \left\{ \inf_\gamma \int_\gamma d\sigma \,\middle|\, \gamma = \widehat{xy}, y \in B, d\sigma \text{ is the line element corresponding to } e_{ab}, \gamma \text{ curve in } W \right\}$$

F and Δ denote the maximal g (or e) distance that an element x of A can have from B.

For the definition of causality conditions (chronology, strong/stable causality, global hyperbolicity) see [4].

$\bar{A}, \mathring{A}, \partial A$ denote the closure, the interior, and the boundary of A.

$A \subset\subset B \Leftrightarrow \bar{A} \subset \mathring{B}$.

Definition 2.1

A is an *achronal* set if for all $p \in A$: $I^+(p) \cap A = \emptyset$

A is an *acausal* set if for all $p \in A$: $J^+(p) \cap A = \{p\}$.

A is a *stably acausal* set if there exists a $\hat{g} > g$ on V and for all $p \in A$: $J^+(p; \hat{g}) \cap A = \{p\}$.

A is a *partial Cauchy surface* if A is acausal and the domain of dependence $D(A)$ is a neighborhood of A.

A is *nontimelike (spacelike; stably spacelike,* a *slice)* if the metric g induces a causal structure on some neighborhood U of A such that A is an achronal (acausal, stably acausal) set (a partial Cauchy surface) in U.

A set is called a *hypersurface* in V if it is an embedded three-dimensional submanifold (with or without edge).

A set A is called a *boundary* if there exists a $W \subset V$ such that $A = \partial W$. A boundary is called *time oriented* if any causal curve crossing over A from W to $V \backslash W$ is future directed.

A family of sets A_i is a *covering* if any $p \in V$ belongs to some A_i.

A *covering* is *simple* ("decomposition") if any $p \in V$ belongs to exactly one A_i.

A *simple covering* is *parametrized* if the A_i are the level surfaces of a continuous function τ: $V \to \mathbb{R}$ or $V \to S^1$.

A *time function* τ is a continuous function τ: $V \to \mathbb{R}$ that increases along any causal curve γ in V. The corresponding level surfaces $\{\tau = \alpha\}$ are denoted by S_α; they form a parametrized simple covering. If τ is increasing along timelike curves and not decreasing along causal curves, τ is a *semi-time-function*.

A *stable time function* τ has stably spacelike level surfaces.

A *uniform time function* is stable and fulfils an "anti-Lipschitz property": For any compact set $C \subset V$ there exists a constant $K_C > 0$ such that for all α, β

$$K_C \cdot \Delta_C(S_\alpha, S_\beta) \leqslant |\alpha - \beta|$$

(Roughly speaking, τ increases more quickly than at a certain minimum rate along any causal curve.)

A *smooth time function* τ on a C^k (C^∞) manifold is a C^k (C^∞) function.

Remark. Any simple covering by acausal hypersurfaces can be locally parametrized and hence corresponds to a local time function. Globally this does not hold: Cut out of the Minkowski space V^2 the two lines $\{x = 3 | 1 \leqslant t \leqslant 2\}$, $\{x = -3 | -2 \leqslant t \leqslant -1\}$ and identify the edges $\{x = 3 + 0 | 1 < t < 2\}$ with $\{x =$

$-3 - 0|-2 < t < -1\}$ and vice versa. The acausal sets $\{t = \text{const}\}$ cannot be parametrized. A covering by slices is a time function if it is parametrized by $\mathbb{R}$ (not by S^1), but a space-time with a covering by slices parametrized by S^1 might still possess a time function: In Minkowski space V^2 identify $\{t = 0\}$ and $\{t = 3\}$ and remove $\{t = 1 | x \leqslant +2\}$, $\{t = 2 | x \geqslant -2\}$; the slices $\{t = \text{const}\}$ have a periodic parameter, but the space time is stably causal and hence possesses a time function.

Theorem 2.1. (Covering by slices and causality.)

(i) If V can be covered by *partial Cauchy surfaces* then V is *strongly causal.*

(ii) If V can be covered by *time-oriented spacelike boundaries*, then V is *stably causal* and V can be simply covered by time-oriented spacelike boundaries.

(iii) If V is a *simply connected* manifold and can be covered by *slices*, then V is *stably causal*; if the covering is simple, then there exists a time function possessing these slices as level surfaces.

Remarks and Counterexamples

(iv) If V is covered by partial Cauchy surfaces, V is not necessarily stably causal; example: V^2 is Minkowski space, $\{t = \pm 1; x \geqslant 0\}$ and $\{t = 0; x \leqslant 1\}$ removed, all pairs of points $(-2; x)$, $(+2; x)$ identified; the lines $\{t = \text{const}\}$ are partial Cauchy surfaces, but V^2 is not stably causal.

(v) If V is strongly causal, there does not necessarily exist a covering by slices; in [6, p. 39 f.] an example of a strongly causal space with no slice at all is given (the result of a discussion with Geroch).

(vi) If a covering by slices exists, one generally cannot find a simple covering by slices. (This is a *conjecture* of mine based on the example after Lemma 2.2. and similar arguments.)

(vii) In multiply connected space-times a simple covering by time-oriented spacelike boundaries is not necessarily the family of level surfaces of a time function. Example: Remove $\{|t| \leqslant 1; x = 0\}$ from Minkowski space V^2; the lines $\{t = a\}$ for $|a| > 1$ and $\{t = a; x < 0\} \cup \{t = -a; x > 0\}$ for $|a| \leqslant 1$ form a simple covering but cannot be continuously parametrized.

(viii) If one drops the assumption that V is time-orientable, curious things might happen. For example (Möbius strip), take the unit square $\{|x| \leqslant 1, |t| < 1\}$ and identify $(t, 1)$ with $(-t, -1)$ for all $|t| < 1$. One obtains a space-time not stably causal but possessing a Cauchy surface $\{t = 0\}$. In fact, V can be covered by Cauchy surfaces, but any two of them intersect. On the other hand there exists a simple covering by slices $\{|t| = \text{const}\}$ which with one exception are spacelike boundaries. In general, for stably causal non-time-oriented space-times one can find something like a time-function, namely a τ: $V \to [0, \infty[$ such that $\pm\tau$ is a monotone continuous parameter along any causal curve γ if one switches the sign when γ meets $\{\tau = 0\}$; see [6, p. 31].

(ix) The existence of a semi-time-function is a causality property stronger than chronology and weaker than stable causality but not comparable with either causality or strong causality: Minkowski space V^2, all pairs of points $(t - 1; x + 1)$ and $(t + 1; x - 1)$ identified, possesses a semi-time-function $\tau = t + x$ but is not causal; the example of (iv) is strongly causal but does not possess a semi-time-function τ [as τ must be continuous by definition, τ would have to fulfill: $\tau(0.5; 0.5) \leqslant \tau(1.5; -0.5) < \tau(-1.5; -0.5) \leqslant \tau(-0.5; 0.5) \leqslant \tau(0.5; 0.5)$].

Sketch of the Proof of Theorem 2.1

(i) The first part is obvious, as the domain of dependence of a partial Cauchy surface S is a neighborhood of any $p \in S$.

(ii) Lemma 2.1 shows that one can find a covering by countably many "collars" of spacelike boundaries.

Lemma 2.2 gives the essential idea of how one can obtain a simple covering by spacelike boundaries from an arbitrary covering (for a more explicit construction, cf. [6, p. 37]). In fact it would be sufficient to get a locally finite covering of V by collars. Then there exists a widening $\hat{g}$ of g such that all these time-oriented spacelike boundaries are $\hat{g}$-spacelike boundaries.

Any space-time that can be covered by $\hat{g}$-spacelike boundaries is $\hat{g}$-causal (hence g-stably causal), for if a causal curve γ leaves a region A at a point $p \in \partial A$, then γ cannot reenter A if ∂A is time-oriented; i.e., no closed $\hat{g}$-causal curve meets the (arbitrarily chosen) point p.

(iii) Part (iii) is a consequence of statement (ii) and Lemmas 2.2. and 2.4.

Lemma 2.1. (Collaring of slices.) Let S be a slice. Then there exists a simple covering of a neighborhood of S by a one-parameter family ("collar") of stably spacelike slices S_α $(-1 < \alpha < 1)$. If S is a partial Cauchy surface (or, respectively, a time-oriented spacelike boundary), then the S_α have this property too and their domains of dependence $D(S_\alpha)$ equal $D(S)$. For any $\epsilon > 0$ we can require $F(S, S_0) < \epsilon$.

Proof. By definition, there is a neighborhood U of S in which [(U, g) taken as space-time] S is an acausal set. There the uniformization procedure of Lemmas 3.4 and 3.5 for a family S_τ can be also applied to a single S.

Lemma 2.2. (Removal of crossings.) Let S_1, S_2 be time-oriented boundaries ($S_i = \partial A_i$) and q some fixed point on S_2, $q \notin S_1$. Then there exists a time-oriented boundary S_2' such that $q \in S_2'$ and $S_1 \cap S_2' = \phi$.

Proof. Let $\partial A_{1,\alpha}$ be a collar of ∂A_1 not containing q; then take $S_2' = \partial(A_2 \cap A_{1,\alpha})$ where $\alpha = -\frac{1}{2}$ if $q \in A_1$, $\alpha = +\frac{1}{2}$ if $q \in V \backslash A_1$.

Example. This procedure does not work if S_1, S_2 are arbitrary slices. Let η_{ab} be the Minkowski metric on $\mathbb{R}^3(t, x, y)$ and $v^a := (1; -y; x)$, $g_{ab} := \eta_{ab} +$

$(v^c v_c)^{-1} v_a v_b$. The circle $\gamma_1 = \{x^2 + y^2 = 1; t = 0\}$ is a null curve; take a spacelike surface intersecting γ_1 in exactly one point and let γ_2 be its edge (which "encircles" γ_1). Let (V^3, g) be the space-time $\mathbb{R}^3 \backslash \{\gamma_1, \gamma_2\}$; all slices in V^3 have as "ideal edge" γ_1 or γ_2, none of them both γ_1 and γ_2. We can find p, q such that for any pairs of slices S_p, S_q containing p or q, respectively, it holds that $S_p \cap S_q \neq \phi$ but $S_p \neq S_q$.

Lemma 2.3. (Hawking [2]; slices in simply connected space-times.) Let V be a simply connected space and S be a slice in V. Then S is a partial Cauchy surface and a time-oriented spacelike boundary.

Proof. (See also [2].) For any hypersurface without edge it holds that a homotopy deformation of a curve (keeping the end points fixed) changes the number of crossings by only an even number. In particular, two points p, q locally separated by S (i.e., joined by a $\widehat{pq}$ crossing S once) cannot be joined by a curve γ without crossing S as γ is homotopic to $\widehat{pq}$ in a simply connected V; that is, S is a boundary and S is an acausal set.

Lemma 2.4. (Parametrization of simple coverings.) Let V be a simply connected space and $\mathcal{S}$ a simple covering by slices. Then there exists a time function τ such that the family of level surfaces $\{\tau = \text{const}\}$ is $\mathcal{S}$.

Proof. Let $\{\gamma_n\}$ be a sequence of nonextendible causal curves such that the paths of the γ_n form a dense subset of V. We parametrize $\mathcal{S}$ by recursion: Let the γ_i $(i < k)$ be those already considered, i.e., they parametrize the subset $\mathcal{S}_{k-1} := \{S \in \mathcal{S} \mid S \cap \gamma_i \neq \phi \text{ for some } i < k\}$ of $\mathcal{S}$ and the connected subset $V_{k-1} := \cup\{S \in \mathcal{S}_{k-1}\}$ of V by a parameter τ. Let l be the smallest number $\geqslant k$ such that $\gamma_l \cap V_{k-1} \neq \phi$; if $l \neq k$ we reorder a finite section of the sequence $\{\gamma_n\}$ in the following way: γ_l becomes γ_k and γ_m becomes γ_{m+1} for all $k \leqslant m < l$. As the S's are boundaries (see Lemma 1.3.), any curve $\widehat{pq}$ intersects S_α if $\tau(p) \leqslant \alpha \leqslant \tau(q)$; therefore $\gamma_k^0 := \gamma_k \cap V_{k-1}$ is connected, it might have values $\tau \in]a; b[$; we may parametrize $\gamma_k^\pm := J^\pm(\gamma_k^0) \cap \gamma_k$ by $]a - 1; a]$ and $[b; b + 1[$ if these parts of γ_k are not empty. Any $S \in \mathcal{S}$ with $S \cap \gamma_k^\pm \neq \phi$ does not belong to $\mathcal{S}_{k-1}$ and is met by a causal curve γ_k only once, hence it is uniquely parametrized. The new set V_k is again connected. $\mathcal{S} = \cup \mathcal{S}_n$, $V = \cup V_n$.

§(3): *A Construction to make Time Functions Uniform and Regular*

Theorem 3.1. Let V be a space-time possessing a time function τ, and ϵ be an arbitrary number >0. Then there exists a uniform time function $\hat{\tau}$ on V; for all $x \in V$ it holds that $|\hat{\tau}(x) - \tau(x)| < \epsilon$, for all α it holds that $F(\hat{S}_\alpha, S_\alpha) < \epsilon$.

If V is a C^k manifold, $\hat{\tau}$ can be assumed to be a C^k function.

Corollary. Any space-time V possessing a time function is stably causal (as $\hat{\tau}$ admits a widening of the null cones).

Corollary. On any stably causal space-time V one can (globally) write the metric in the form (signature of $g_{\alpha\beta}$: +++)

$$ds^2 = -f(x^a)^2 dt^2 + d\sigma_t^2$$

where $d\sigma_t^2$ is a positive definite (t-dependent) line element on the hypersurfaces t = const.

Corollary. Any stably causal space-time V (globally) admits a timelike nonrotating congruence (e.g., $\nabla\tau$).

Proof. The proof of the theorem is a consequence of the following seven lemmas. The requirement $F(\hat{S}_\alpha, S_\alpha) < \epsilon$ is not explicitly considered there, but it can be obviously fulfilled by choosing sufficiently small sets U in Lemmas 3.5 and 3.7.

Definition 3.1. (l.c. Sets; see also Figure 2)

A set $U \subset V$ is called a local causality set ("l.c. set") if the following conditions hold:

(i) U is homeomorphic to the closed unit cube of $\mathbb{R}^n$, hence compact, and U has a coordinate system (t, x^α), where t is a (local) time function and the lines x^α = const are timelike.

(ii) U is g-geodetic convex.

(iii) The local causality on U is the global causality, i.e., $q \in J^+(p), p, q \in U$ implies that there exists a causal curve $\widehat{pq}$ that lies in U.

In Lemma 3.4 we use a triad of l.c. sets U, U', $\tilde{U}$, for which we assume the following properties:

(iv) $\tilde{U} \subset\subset U' \subset\subset U$.

(v) U' and $\tilde{U}$ have timelike boundaries that are smooth except for two cone points (past and future end points).

In Lemma 3.6 we further require that for a given $\hat{g} > g$ ($\hat{g}$ being a Lorentz metric with respect to which a given uniform time function τ is still a time function) the following holds:

(vi) A g-causal vector v^a ($g_{ab}v^a v^b \leqslant 0$) after parallel transport [with respect to the flat connection given by the coordinates (t, x^α)] remains $\hat{g}$-causal with U.

Lemma 3.1. (Uniform Increase; see also Figure 1.) Let $\hat{g} > g$, τ be a $\hat{g}$ time function and U a l.c. set. Then the following conditions are equivalent:

$$\exists K, K' > 0, \qquad \forall \alpha, \beta:$$

(1) $K \cdot \Delta_U(S_\alpha, S_\beta) \leqslant |\alpha - \beta|$
(2) $K' \cdot F_U(S_\alpha, S_\beta) \leqslant |\alpha - \beta|$

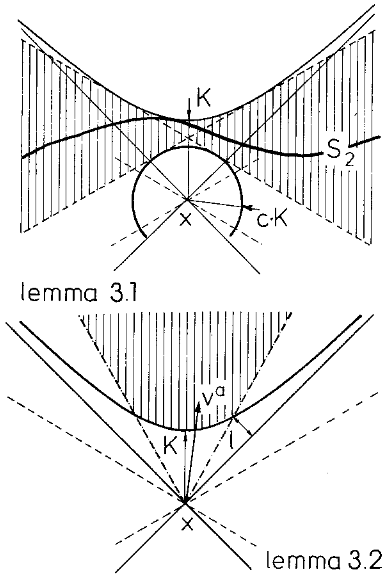

Fig. 1. Proof of Lemmas 3.1 and 3.2. The broken lines are $\hat{g}$-null-lines; the upper hatched area is the domain in which S_2 has to lie if both $x \in S_1$ and $F(x, S_2) = K$. The lower hatched area is the domain into which the vector V^a of Lemma 3.2 must point.

If τ is smooth (at least C^1), then these conditions are equivalent to $\exists K''$, $K''' > 0$:

(3) $e^{ab}\nabla_a\tau\nabla_b\tau \geqslant K''^2$ in U
(4) $-g^{ab}\nabla_a\tau\nabla_b\tau \geqslant K'''^2$ in U

(e^{ab}, Δ_U, and F_U were introduced in Section 2)

Proof. Let V be the Minkowski space

$$g\colon ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

$$\hat{g}\colon d\hat{s}^2 = -\theta^2 dt^2 + dx^2 + dy^2 + dz^2 \quad \theta > 1$$

$$e\colon d\tilde{s}^2 = +dt^2 + dx^2 + dy^2 + dz^2$$

And let S_1 and S_2 be two $\hat{g}$ slices in V. Then it holds that for $c^2 := (\theta^2 - 1)/(\theta^2 + 1)$

$$c \cdot \Delta(S_1, S_2) \leqslant F(S_1, S_2) \leqslant \Delta(S_1, S_2) \leqslant c^{-1}F(S_1, S_2)$$

(cf. Figure 1.) If τ is smooth one has

$$c^2 e^{ab}\nabla_a\tau\nabla_b\tau \leqslant -g^{ab}\nabla_a\tau\nabla_b\tau \leqslant e^{ab}\nabla_a\tau\nabla_b\tau \leqslant -c^{-2}g^{ab}\nabla_a\tau\nabla_b\tau$$

and

$$\theta^{-1}(\partial_t\tau)(x) \leqslant \lim_{\tau\to\tau(x)} \frac{|\tau(x)-\tau|}{F(\{x\}, S_\tau)} \leqslant (\partial_t\tau)(x)$$

The proof for Minkowski space can be easily extended to l.c. subsets of V. ∎

Lemma 3.2. (Uniformly timelike vectors.) Let $\hat{g} > g$, τ be a $\hat{g}$ time function and U be a l.c. set in V. Let v^a be a vector field such that the 3-spaces g-orthogonal to v^a are $\hat{g}$-spacelike and such that $g_{ab}v^av^b \leqslant -K$ for some $K > 0$; then there exists an $l > 0$ such that for any vector field w^a with $e_{ab}w^aw^b < l^2$ it holds that $v^a + w^a$ is a g-causal vector.

Proof. (By an argument similar to that of Lemma 3.1.) In Minkowski space l can be chosen as $K[(\theta - 1)/(2\theta + 2)]^{1/2}$. (cf. Figure 1.) ∎

Lemma 3.3. (Addition of time functions.)

Let τ', τ'' be two semi-time-functions and a, b be positive numbers;

(i) then $\tau := a\tau' + b\tau''$ is a semi-time-function.

(ii) In any l.c. set in which τ' is a time function, τ is a time function too.

(iii) In any l.c. set in which τ' is uniform and τ'' is stable, τ is a uniform time function. (Stability of τ holds if both τ' and τ'' are stable; the anti-Lipschitz property holds for τ if it holds for at least one of the two τ', τ''.)

(iv) τ is not necessarily uniform if τ' is uniform; example: Minkowski space V^2, $\tau' = t$, $\tau'' = (x + t)^{1/3}$; we might even choose τ'' to be a time function $(\sin x + t)^{1/3}$ and τ is still not stable at the origin (0; 0). (I am indebted to Steven Harris for pointing out this fact to me.)

(v) The statements above become wrong if we consider functions a, b. But for slightly varying coefficients we can obtain some results: Let τ' be a time function with τ'' being uniform in some l.c. set U (anti-Lipschitz constant K) and let f be a Lipschitz continuous function taking values in [0; 1[, the Lipschitz constant L being smaller than

$$K \cdot (1 - \max_U f)/\max_{x\in U} |\tau'(x) - \tau''(x)|$$

Then $\tau = f \cdot \tau' + (1 - f) \cdot \tau''$ is a time function in U that fulfills an anti-Lipschitz property.

Proof. Let $p, q \in U$, $q \in J^+(p)$. Then $\tau(q) - \tau(p) = [1 - f(p)] . [\tau''(q) - \tau''(p)] + f(p) \cdot [\tau'(q) - \tau'(p)] + [f(q) - f(p)] \cdot [\tau''(q) - \tau'(q)] \geqslant (1 - \max f) \cdot K \cdot \Delta(p, q) + 0 - L \cdot \Delta(p, q) \cdot \max |\tau'' - \tau'|$. ∎

Lemma 3.4. (Local uniformization; see Figure 2). Let G be an open set such that its domain of dependence $D(\bar{G})$ is contained in the open interior of some l.c. set U. For any time function τ we can find a metric $\tilde{g} \geqslant g$ and

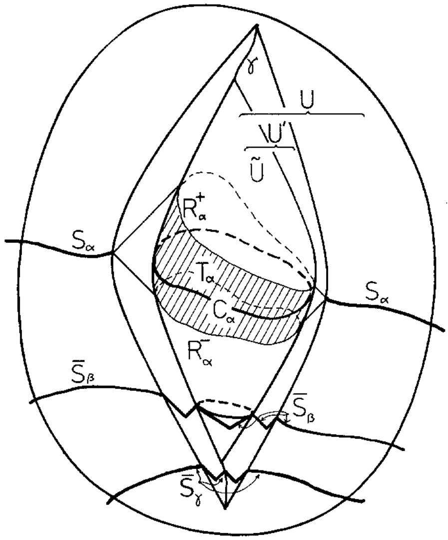

Fig. 2. The triad U, U', $\tilde U$ of Lemma 3.4.

a $\tilde g$-time-function $\tilde\tau$ such that $\tilde g = g$, $\tilde\tau = \tau$ in $V\backslash U$ with $\tilde\tau$ being uniform in G, $\tilde g > g$ in G. Furthermore for any prescribed $\epsilon > 0$ one can arrange $\tilde\tau$ to satisfy $|\tilde\tau(x) - \tau(x)| < \epsilon\ \forall x \in V$.

Proof. The proof consists of nine steps. First a few definitions. We shall use a triad U, U', $\tilde U$, as described in Definition 3.1. such that $G \subset\subset \tilde U$. $R_\alpha^\pm := \partial\tilde U \cap \partial J^\pm(S_\alpha \cap V\backslash U')$; T_α is the strip of $\partial\tilde U$ lying between R_α^- and R_α^+; $\chi(\alpha) := \min\{\Delta(x,y) | x \in R_\alpha^-, y \in R_\alpha^+\}$; $\chi(\tau)$ is defined for $\tau \in [\tau_-, \tau_+]$, where $R_{\tau-}^-$ and $R_{\tau+}^+$ are the two cone-points of $\partial\tilde U$; $\chi(\alpha)$ measures the "width" of T_α.

(A) It holds that

$$\chi := \inf\{\chi(\tau) | \tau_- \leqslant \tau \leqslant \tau_+\} > 0$$

(the $R_\alpha^\pm$ are compact sets continuously depending on τ, hence $\chi = 0$ would imply the existence of a null curve joining two points on some $\mathcal{S}_\tau$).

(B) In any strip T_τ we can find a slice C_τ that is stably spacelike. $\{\mathring{T}_\tau$ is a globally hyperbolic three-dimensional Lorentz space since the $R_\tau^\pm$ are nowhere timelike (see [7]); global hyperbolicity is a stable property (see [1]) and equivalent to the existence of a stably spacelike Cauchy surface C.$\}$

(C) If for two values $\alpha < \beta$ the corresponding C_α, C_β intersect, we can find new C'_α, C'_β fulfilling the requirements of step B for C_τ ($\tau = \alpha, \beta$) and $C'_\alpha \subset I^-(C'_\beta)$ (construction by exchanging some parts):

$$C_\alpha^\pm := C_\alpha \cap J^\pm(C_\beta), \qquad C_\beta^\pm := C_\beta \cap J^\pm(C_\alpha)$$

$$C'_\alpha := C_\alpha^- \cup C_\beta^-, \qquad C'_\beta := C_\alpha^+ \cup C_\beta^+$$

As $T_\alpha \subset I^-(T_\beta)$, $T_\beta \subset I^+(T_\alpha)$ we have $C'_\alpha \subset T_\alpha$, $C'_\beta \subset T_\beta$. C'_α, C'_β do not cross over but are still "in contact" along $H := C_\alpha \cap C_\beta$. This can be removed by a slight shifting of C' in a small neigborhood of the compact set H (see also Lemma 2.2).

(D) There exists a finite sequence τ_n with corresponding C_n ($n = 0, 1, \ldots, N$), C_0 and C_N being the cone points of $\partial\tilde{U}$; for $k < l$ it holds that $\tau_k < \tau_l$, $C_k \subset I^-(C_l)$; for any $\alpha \in [\tau_-, \tau_+]$ at least two C_m, $C_n (\tau_m < \alpha < \tau_n)$ are contained in T_α. (This step is obvious.)

(E) One can find a metric $\bar{g}$ on U: $\bar{g} \geqslant g$, $\bar{g} = g$ on $U \backslash U'$, $\bar{g} > g$ on $\tilde{U}$ such that all C_n are stably spacelike also with respect to $\bar{g}$. (This step is obvious.)

(F) One can assume (possibly after a suitable finite supplementation of the set $\{C_n\}$) that the surfaces C_τ of linear interpolation between the C_n are $\bar{g}$-spacelike. (Let v^a be a smooth timelike vector field on $\partial\tilde{U}$ and t be the parameter along its integral curves. In the coordinate representation $C_n \mathrel{\overset{=}{*}} \{t = f_n(p)$; $p \in S^2$ (the orbit space of the integral curves)$\}$ we can interpolate: Let $\tau_k < \tau < \tau_{k+1}$; then

$$C_\tau \mathrel{\overset{=}{*}} \{t = f_k(p) + [f_{k+1}(p) - f_k(p)](\tau - \tau_k)/(\tau_{k+1} - \tau_k); p \in S^2\}$$

The maximal distance between the C_n is assumed to be so small that the v^a-Lie-shifted tangents to the C_k remain $\bar{g}$-spacelike in the strip between C_{k-1} and C_{k+1}.)

(G) Now we can introduce a uniform time function $\bar{\tau}$ on $\tilde{U}$ by fixing its level surfaces: $\bar{S}_\tau := \partial J^-(C_\tau; \bar{g}) \cap \tilde{U}$. The $\bar{S}_\tau$ are spacelike, $\bar{S}_\alpha \cap \bar{S}_\beta = \phi$ for $\alpha \neq \beta$; $\bar{S}_\alpha \cap \partial\tilde{U} = C_\alpha$; there exists an $M \in \mathbb{R}$ such that $\Delta_{\tilde{U}}(\bar{S}_\alpha, \bar{S}_\beta) \leqslant M \cdot \Delta_{\partial\tilde{U}}(C_\alpha, C_\beta)$, which implies the uniformity–see Lemma 3.1. (In Minkowski space, $M = 1$.)

(H) Finally, we have to link $\bar{\tau}$ in $\tilde{U}$ and τ in $V \backslash U'$. In order to get a well-defined function also near the cone points of $\partial\tilde{U}$, we join the corresponding cone points of $\partial\tilde{U}$ and $\partial U'$ by a smooth timelike curve $\gamma(t)$ with a parameter which at the endpoints continuously joins to the values of τ on $\partial U'$ and of $\bar{\tau}$ on $\partial\tilde{U}$; $C_\alpha := \{\gamma(\alpha)\}$. The semispacelike surfaces

$$\bar{S}_\tau := \begin{cases} S_\tau & \text{in } V \backslash U' \\ \bar{S}_\tau & \text{in } \tilde{U} \\ \partial J^-((S_\tau \backslash U') \cup C_\tau) & \text{in } U' \backslash \tilde{U} \end{cases}$$

correspond to a semi-time-function $\bar{\tau}$ on the whole of V. We can find a function f fulfilling the conditions in Lemma 3.3.(v) such that $f = 0$ in G and $f = \text{const} > 0$ in $V \backslash \tilde{U}$. According to Lemma 3.3. (ii) and (v), $\tilde{\tau} = f \cdot \tau + (1 - f)\bar{\tau}$ fulfils all the properties required.

(I) In order to obtain a $\tilde{\tau}$ with $|\tilde{\tau} - \tau| < \epsilon$ one can modify the described construction in the following way. Replace the pair U', $\tilde{U}$ by a finite sequence $U_k (U_0 = U', U_1 = \tilde{U}, U_N \neq \phi, U_{N+1} = \phi)$ fulfilling (iv) and (v) of Definition 3.1 such that all cone-points are outside $D(\bar{G})$ (the smooth parts of ∂U_k for $k \geqslant 2$ might intersect G) and such that any $p \in U_k \backslash U_{k+1}$ can be joined with $V \backslash U_k$ by a future-directed as well by a past-directed causal arc, along each of them τ changes only by an amount smaller than $\epsilon/3$ (i.e., no time function can differ more than $2\epsilon/3$ from τ within $\tilde{U}$ if it coincides with τ on every ∂U_k). If one uses $\cup \partial U_k$, $k = 1, \ldots, N$ instead of $\partial \tilde{U}$ (then the C_τ consist of N connected components) and constructs the C_τ such that $|\tau(x) - \alpha| < \epsilon/3$ for any $x \in C_\alpha$, then $|\bar{\tau} - \tau| < \epsilon$ holds on U'. ∎

Lemma 3.5. (Global uniformization.) If there exists a time function τ on V then there exists a uniform time function $\hat{\tau}$ such that $|\tau - \hat{\tau}| < \epsilon$ on V.

Proof. Take a (countable, locally finite) covering of V with sets G_n fulfilling the requirements in the statement of Lemma 3.4. We can construct a sequence (g_k, τ_k) by recursion: First we set $g_0 := g$ and $\tau_0 := \tau$ and then define (g_k, τ_k) to be (g, τ) as constructed in the preceding lemma with g, τ, G, ϵ replaced by $g_{k-1}, \tau_{k-1}, G_k, \epsilon \cdot 2^{-k}$. Evidently, $\{g_n, \tau_n\}$ converges in the compact-open topology since for any compact set $A \subset V$ after finitely many steps the sequency $\{g_n, \tau_n\}$ becomes constant on A. Let $x \in G_k$ and $x \in U'_l (l > k)$ then τ_l is the combination of two g_k-semi-time-functions hence of g-stable time functions $\tau_{l-1}, \bar{\tau}_l$ one of which (τ_{l-1}) is anti-Lipschitz; according to Lemma 3.3 (v) τ_l is uniform. Therefore lim τ_n is a uniform time function. ∎

Lemma 3.6. (Local regularization), Let τ be a uniform time function on V, $\epsilon > 0$, and U, $\tilde{U}$ be l.c. sets (see Definition 3.1). Then one can find a $\tilde{\tau}$, defined on $\tilde{U}$, which (i) is a C^∞ function of the coordinates, (ii) is a uniform time function, and for which (iii) $|\tau(x) - \tilde{\tau}(x)| < \epsilon$.

Proof. (Convolution with mollifiers.) $\|x^a\|$ denotes the Euclidean norm of coordinate values:

$$\delta_n := \begin{cases} 0, & \|x^a\| \geq n^{-1} \\ \exp\left[\dfrac{-n^2}{(n^2 - \|x^a\|^2)}\right] \cdot \left\{\displaystyle\int \exp\left[\dfrac{-n^2}{(n^2 - \|\xi^a\|^2)}\right] d^4\xi^a\right\}^{-1}, & \|x^a\| < n^{-1} \end{cases}$$

$$\tau_n(x^a) = \int \delta_n(x^a - \xi^a)\tau(\xi^a)\, d^4\xi^a$$

where the integration is formally taken over $\mathbb{R}^4$; for all $n > [\min\{\|x^a - y^a\| \,|\, x \in \tilde{U}, y \notin U\}]^{-1}$ the supp (δ_n) is contained in a (coordinate) domain, where τ is defined, hence τ_n is defined on $\tilde{U}$. The following holds:

(A) $\tau_n \to \tau$ (uniform convergence in $\tilde{U}$) as

$$\sup_x \lim_n |\tau_n(x) - \tau(x)| = \sup \lim \left| \int [\tau_n(x^a) - \tau(\xi^a)]\, \delta_n(x^a - \xi^a)\, d^4\xi^a \right|$$
$$\leq \lim_n \left[\max\{|\tau(x^a) - \tau(\xi^a)|; \|x^a - \xi^a\| \leq n^{-1}\}\right] = 0$$

(since τ is continuous).

(B) $\tau_n \in C^\infty(\tilde{U})$.

(C) τ_n is a time function:

$$\tau_n(y^a) - \tau_n(x^a) = \int \delta_n [y^a - \xi^a - (y^a - x^a)] \tau(\xi^a + (y^a - x^a)) d^4 \xi^a$$

$$- \int \delta_n(x^a - \xi^a) \tau(\xi^a) d^4 \xi^a = \int \delta_n(x^a - \xi^a)[\tau(\xi^a + (y^a - x^a)) - \tau(\xi^a)] d^4 \xi^a$$

If $y \in J^+(x)\backslash\{x\}$, i.e., $y^a - x^a$ is a g-causal vector in x hence is a $\hat{g}$-causal vector in any $\xi \in \tilde{U}$ [see Definition 3.1 (vi)], we can further estimate

$$\cdots \geqslant K \int \delta_n(x^a - \xi^a) \| y^a - x^a \| d^4 \xi^a = K \| y^a - x^a \|$$

(K refers to Lemma 3.1; Δ is replaced by the Euclidean coordinate distance, g by $\hat{g}$). ∎

Lemma 3.7. (Global regularization.) Let τ be a uniform time function on a C^k space-time V, and $\epsilon > 0$, then one can find a uniform time function $\hat{\tau}$ that is a C^k function on V and $|\tau(x) - \hat{\tau}(x)| < \epsilon$ for all $x \in V$. $(1 \leqslant k \leqslant \infty)$.

Proof. For a pair of countable locally finite coverings of V by l.c. sets $\tilde{U}_n \subset\subset U_n$ one can obtain a sequence of functions φ_n ("partition of unity") such that

$$\tilde{U}_m \subset\subset \operatorname{supp} \varphi_m \cap U_m, \qquad 0 \leqslant \varphi_m^{(x)} \leqslant 1 \text{ for all } x \in V, m \in \mathbb{N}$$

$$\varphi_m \in C^k(V)$$

If we combine local (smooth) time functions τ_m of Lemma 3.6 in the U_m,

$$\hat{\tau} := \sum_m \varphi_m \tau_m$$

we obtain a C^k function on V, but we cannot be sure that $\hat{\tau}$ really is a time function, i.e., that $\nabla\hat{\tau}$ is timelike:

$$\nabla\hat{\tau} = \sum_m \varphi_m \nabla\tau_m + \sum_m (\tau_m - \tau)\nabla\varphi_m =: v^a + w^a$$

v^a is timelike, w^a generally is not. However, we can apply Lemma 3.2 in order to show that $\nabla\hat{\tau}$ is timelike if we make w^a very small by choosing τ_m sufficiently close to τ.

For an explicit construction we introduce the following constants: $\mathbb{N}_m := \{n \in \mathbb{N} | U_n \cap U_m \neq \phi\}$; N_m is the (finite!) cardinal number of $\mathbb{N}_m$; $\psi_m := \min\{\varphi_m(x); x \in \tilde{U}_m\}$ (by assumption $\psi_m > 0$); $M_m^2 := \max\{e^{ab}\nabla_a\varphi_k\nabla_b\varphi_k;\ k \in \mathbb{N}_m, x \in U_m\}$; K_m'' is a constant of uniformity for τ on $\tilde{U}_m$ [refers to Lemma 3.1 (3) and Lemma 3.6 (C)]; l_m is the constant l of Lemma 3.2 corresponding to a $K = K_m''$; $\epsilon_m := \max\{|\tau_m - \tau|; \text{on } \tilde{U}_m\}$; τ_m is a regularization of τ according to Lemma 3.6. in $\tilde{U}_m$.

If $\epsilon_m^{-1} \geqslant \max \{N_k \cdot M_k \cdot l_k \cdot \psi_k^{-1} | k \in \mathbb{N}_m\}$ [which can be arranged according to Lemma 3.6 (A)] then Lemma 3.2 implies that $\nabla \hat{\tau}$ is timelike. ■

§(4): *Extension of Time Functions*

Theorem 4.1. Let A be a compact subset of a stably causal space-time V and τ be a stable time function defined on A. Then there exists an extension of τ onto the whole of V. [That is, $\exists\, \tilde{g} > g, \tilde{g}$ causal metric on $V; p, q \in A, q \in J^+(p; \tilde{g}) \setminus \{p\}$ implies $\tau(q) > \tau(p)$. Note that it is not required that some $\tilde{g}$-causal curve $\widehat{pq}$ lies in A.]

Proof. The stability of causality and time function enables us to extend τ onto a neighborhood of A (step A). Then we shall construct a countable family of spacelike boundaries S_α of the form $\tilde{J}^-(Q_\alpha; \bar{g})$, where $\bar{g} > g$ and Q_α contains the level surface $\{\tau = \alpha\}$ on A (step B). This extension of τ onto a dense subset of V (by constructing the level surfaces of τ) can be completed to a continuous τ on V (step C).

Remarks. The stability of the time function is essential for step A; example: Minkowski space $V^2 \setminus \{(0,0)\}, A = \{(-1,-1); (+1,+1)\}, \tau(-1,-1) = +1, \tau(+1,+1) = -1$ is a time function that obviously cannot be extended onto V^2. The condition that A has to be compact can be weakened at the cost of very tedious modifications; one has to require bounds on τ, otherwise τ might become infinite in finite regions of V (example: In the maximal analytic extension of Reissner and Nordström's vacuum solution the time of an observer in asymptotically flat parts cannot be extended onto the whole of V; any extension must become infinite before one crosses an inner horizon: "infinite blue shift").

Step A. First we introduce a one-parameter family of metrics $g_\theta : g < g_\theta < \tilde{g}$, where $g_\alpha < g_\beta$ if $\alpha < \beta; \theta \in [a; b]$, where $a := \min \{\tau(x) | x \in A\} - 1, b := \max \{\tau(x) | x \in A\} + 1$; the cones of g_θ-null directions depend continuously on θ (local continuity of null cones). By assumption, τ is a g_θ-time-function for all θ:

$$\tilde{J}_\theta^+(p) := \bigcap_{\theta < \eta} J^+(p; g_\eta)$$

$\tilde{J}_\theta^+(B)$ for compact sets B is closed ($\partial \tilde{J}^+ \subset \tilde{J}^+$) and upper semicontinuous: $q \notin \tilde{J}_\theta^+(B)$ implies that there exists a neighborhood U of B such that $q \notin \tilde{J}_\theta^+(U)$; see [7, 6]. The latter property can be shown as follows: Since B is compact and $\tilde{J}^-(q)$ is closed, there exists an $\eta > \theta$ such that $J_\eta^-(q) \cap B = \phi$; in some normal neighborhood $\tilde{U}$ of q, $\tilde{U} \cap B = \phi$ we can find a $U(q)$ such that $\tilde{J}_\theta^-(U(q)) \setminus \tilde{U} \subset J_\eta^-(q)$, roughly speaking, the g_η light rays starting at q "overtake" the g_θ light rays starting in $U(q)$ within $\tilde{U}$.

Therefore any level surface $S_\alpha = \{x \in A \mid \tau(x) = \alpha\}$ can be extended as a g_θ-spacelike set $\tilde{S}_\alpha$ having no edge within some neighborhood of A such that $\tilde{S}_\alpha \cap \tilde{J}_\theta^+(\{x \in A \mid \tau(x) > \alpha\}) = \phi$. Finitely many of such sets $\tilde{S}_\alpha$ can be chosen such that their domains of dependence cover some neighborhood U of A. Then one can choose some extension $\bar{\tau}$ of τ as a continuous function onto some compact neighborhood $\tilde{A}$ of A such that $S_\alpha \subset \partial J^-(\bar{S}_\alpha; \tilde{g})$ and the dimension of the level surfaces $\bar{S}_\alpha$ is 3.

Let σ be a continuous function on V, $\sigma \equiv a$ on A, $\sigma \equiv b$ on $V \backslash \tilde{A}$ and $\rho := \max(\sigma, \bar{\tau})$ possessing level surfaces $R_\alpha = \{\rho = \alpha\}$; it holds that $S_\alpha \subset \partial \tilde{J}_\theta^-(R_\alpha)$ for all θ [otherwise a $\hat{g}$-causal curve runs from some $p \in S_\alpha$ to some $q \in R_\alpha$, hence $\alpha = \tau(p) < \tau(q) \leqslant \rho(q)$ which contradicts $\alpha = \rho(q)$].

Step B. (See Figure 3.) Let $\{\tau_{2n}\}$ be a dense subset of $[a; b]$ and $\{p_{2n+1}\}$ be a sequence of points dense in $V \backslash A$. We shall construct a sequence C_{τ_n} of non-intersecting stably spacelike boundaries by recursion.

Let k be odd, p_k lying between C_{τ_g} and C_{τ_h}, i.e., the connected component V_k of $V \backslash \bigcup_{n<k} C_{\tau_n}$ that contains p_k has the boundary $C_{\tau_g} \cup C_{\tau_h}$ in V. We assume $p_k \in V_k \subset \tilde{J}_{\theta_g}^-(C_{\tau_g}), p_k \notin \tilde{J}_{\theta_h}^-(C_{\tau_h}), \theta_g > \theta_h$. C_{τ_h} is of the form $\tilde{J}_{\theta_h}^-(Q_h)$, where Q_h is the union of R_{τ_h}, of p_h (if h is odd), and of finitely many points q_h^i $[i < h; q_h^i \in I^+(p_i) \backslash \tilde{J}_{\theta_h}^+(R_{\tau_h} \cup p_h)]$.

Now we can choose some value τ_k: $\tau_g > \tau_k > \tau_h$ and a θ_k: $\theta_g > \theta_k > \theta_h$ such that $p_k \notin \tilde{J}_{\theta_k}^-(R_{\tau_k})$ and $S_{\tau_k} \cap \tilde{J}_{\theta_k}^-(p_k) = \phi$ [this can always be arranged since R_{τ_k} is compact and $p_k \notin \tilde{J}_{\theta_h}^-(R_{\tau_h})$ using the fact that J^+ is closed and upper semicontinuous; cf. step A].

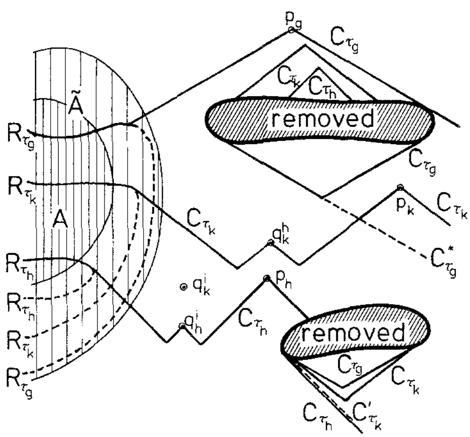

Fig. 3. The construction of C_{τ_n}. One can see there why two simpler methods do not work in general: If one would take sets of the form $C_{\tau_n}^* = J^-(Q_n)$ instead of $\tilde{J}^-(Q_n)$ one could not find an acausal boundary C_{τ_k} through p_k that does not intersect $C_{\tau_g}^*$; if one would use the same metric g_θ for all sets C'_{τ_n} instead of different g_{θ_n}'s, then C'_{τ_k} and C'_{τ_h} would have common g_θ-null-generators.

Note that S_{τ_k} is empty for $\tau_k < a + 1$ or $\tau_k > b - 1$, hence the condition $S_{\tau_k} \cap \tilde{J}^-_{\theta_k}(p_k) = \phi$ can trivially be fulfilled if A does not intersect V_k. Finally we choose for any of the q^i_h a point

$$q^i_k \in [I^+(q^i_h)\backslash \tilde{J}^+_{\theta_k}(p_k \cup S_{\tau_k})] \cap \tilde{J}^-_{\theta_g}(C_{\tau_g})$$

and a point

$$q^h_k \in [I^+(q_h)\backslash \tilde{J}^+_{\theta_k}(p_k \cup S_{\tau_k})] \cap \tilde{J}^-_{\theta_g}(C_{\tau_g})$$

Such a choice is always possible; see argument above. For $Q_k := \{p_k\} \cup R_{\tau_k} \cup \bigcup_i \{q^i_k\}$, $C_{\tau_k} := \tilde{J}^-_{\theta_k}(Q_k)$ fulfils all the requirements we assumed for a $C_{\tau_i} (i < k)$ in the beginning of our recursion step.

It remains to be shown that C_{τ_k} fits together with the C_{τ_i}. In fact, C_{τ_k} cannot intersect C_{τ_h}: $Q_k \cap Q_h = \phi$; if two generators of C_k or, respectively, of C_h intersect in a point r, then in any neighborhood of r one can find a point $r^+ \notin \tilde{J}^-_{\theta_k}(Q_k)$ [hence $r^+ \notin \tilde{J}^-_{\theta_h}(Q_k)$] and $r^+ \in \tilde{J}_{\theta_h}(Q_h)$ as the g_{θ_k} null cone in r is wider than the g_{θ_h} null cone, but this contradicts our construction of Q_k and the transitivity law for causal ordering. For the same reason C_{τ_k} does not intersect C_{τ_g}.

For an even k, we can carry out the same procedure with the compact set $R_{\tau_k} \cap \partial \tilde{J}^-_{\theta_k}(R_{\tau_k})$ (which essentially is S_{τ_k}) instead of p_k.

Step C. The C_{τ_k} of step B cover a dense subset W of V; since they do not intersect, they define a function τ on W; can τ be continuously extended onto V?

Let p be an arbitrary point and γ a timelike curve through p. $W \cap \gamma$ is a dense set of γ since the C_τ are spacelike. τ is a monotone increasing function on $W \cap \gamma$; especially, the two limits

$$\tau_\pm = \lim_{q \to p} \{\tau(q) | q \in I^\pm(p) \cap W\}$$

exist. If τ^+ were strictly greater than τ^- one could find a τ_{2n}: $\tau^- < \tau_{2n} < \tau^+$; the corresponding $C_{\tau_{2n}}$ (cf. step B) must show up in a neighborhood of p since $C_{\tau_{2n}}$ is a boundary of a set $\tilde{J}^-_{\theta_{2n}}(C_{\tau_{2n}})$ containing C_{τ_-} and is contained in $\tilde{J}^-(C_{\tau_+})$. Therefore $\tau_+ = \tau_-$. For $p \in V\backslash W$ we define $\tau(p) := \tau_-$.

In a stably causal space V the sets of the form $U = I^+(p^-) \cap I^-(p^+) [p^\pm \in \gamma \cap I^\pm(p)]$ form a basis for the neighborhoods of p, and for any time function τ it holds that on U: $\sup \tau = \tau(p^+)$, $\inf \tau = \tau(p^-)$; this implies that τ is continuous in p. ∎

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