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Relativistic Spherical Polytropes: An Analytical Approach

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Abstract

Using the technique of Padé (2, 2) approximant we present, in this paper, an approximate analytical solution to the field equations of general relativity for time-independent, spherically symmetric systems in which the pressure P and density ρ are related by a polytropic equation of state: $P = K\rho^{1+1/n}$. The boundary values of coordinate radius ξ_1, for polytropic indices $n = 0$, 1.0 (0.5) 3.0, are given in Table I. Table II contains the values of other physical parameters, $v(\xi_1)$ (mass), $\rho_c/\bar{\rho}$ (the density concentration), and $2GM/c^2R$ (the ratio of gravitational radius to the coordinate radius) for $n = 0$ and 1.

§(1): *Introduction*

Several authors have considered the effects of general relativity in a variety of astrophysical problems, for example, Tooper [1] has studied in some detail the structural features of relativistic polytropic fluid spheres in static equilibrium under their own gravitation. We find that in this and other cases of cognate interest, the method of numerical integration (Runge–Kutta method) or the variational technique provides a solution to the relativistic equilibrium equations, as in nonrelativistic situations.

No analytical solutions of the relativistic equations (1) and (2) are so far known for given n and σ, except for the case $n = 0$ [1]. Our present method, that is, Padé (2, 2) approximation technique, would enable us to obtain approximate analytical solutions of the above equations under post-Newtonian approximation [2] for any arbitrary polytropic index n and parameter σ.

[1] Work done at Azerbaijan State University, Baku, USSR (1977–79).

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In the above works, we find that the method of numerical integration (or the variational technique) provides a solution to the relativistic equilibrium equations, as in nonrelativistic situations. Our present method, that is, Padé (2, 2) approximant would enable us to obtain approximate analytical solutions of the relativistic equations (1) and (2) under the post-Newtonian approximations [2] for any arbitrary n. The aim of the present work is to introduce a new and powerful technique with which the solutions–specifically, the boundary values of physical parameters–are determinable for given n. The usefulness of Padé's technique lies mainly in its greater compactness and simplicity and the physical insight it yields.

§(2): *The Field Equations of Equilibrium and Their Approximate Analytical Solutions*

The dimensionless forms of the field equations of general relativity for a polytropic sphere in static equilibrium under its own gravitation are given by [1].

$$\frac{1-2\sigma(n+1)\,U\xi^{-1}}{1+\sigma\theta}\,\xi^2\,\frac{d\theta}{d\xi}+U+\sigma\xi\theta\,\frac{dU}{d\xi}=0 \tag{1}$$

and

$$\frac{dU}{d\xi}=\xi^2\theta^n \tag{2}$$

where symbols have their usual significance. The pair of foregoing relativistic Lane-Emden equations satisfies the following boundary conditions:

$$\theta(\xi)=1, \qquad U(\xi)=0 \quad \text{at} \quad \xi=0 \tag{3}$$

A method of constructing the relativistic Lane-Emden function would be to start with a series expansion, near the origin $\xi = 0$, of the form

$$\theta=\sum_{n=0}^{\infty} a_n\,\xi^{2n} \tag{4}$$

satisfying the boundary conditions (3) (a_n's are constants). Consequently, the solution of (2), including terms up to ξ^{11}, is found to be

$$U(\xi)=\frac{\xi^3}{3}\left[1+\frac{3Aa_1}{5}\,\xi^2+\frac{3}{7}(Aa_2+Ba_1^2)+\frac{1}{3}(Aa_3+2Ba_1a_2+Ca_1^3)\,\xi^6\right.$$
$$\left.+\frac{3}{11}(Aa_4+Ba_2^2+2Ba_1a_3+3Ca_1^2a_2+Da_1^4)\,\xi^8\right] \tag{5}$$

where

$$A = n, \quad B = \frac{n(n-1)}{2!}, \quad C = \frac{n(n-1)(n-2)}{3!}, \quad D = \frac{n(n-1)(n-2)(n-3)}{4!}$$

Substituting (2), (4), and (5) in (1) and equating the coefficients of like powers in ξ, we can successively determine the coefficients a_1, a_2, a_3, and a_4. Thus, the series including the first five terms, in post-Newtonian approximation, is written as

$$\theta(\xi) = 1 - \frac{1 + 4\sigma}{6}\xi^2 + \frac{n(3 + 10\sigma)}{6^2 10}\xi^4 - \frac{30\sigma(n^2 - 40n) + 15(8n^2 - 5n)}{6^3 10 \times 105}\xi^6$$
$$+ \frac{-28\sigma(1318n^3 + 6027n^2 - 3400n) + 70(122n^3 - 183n^2 + 70n)}{6^4 10^2 1764}\xi^8 \quad (6)$$

The nonrelativistic case [3] obtains on putting $\sigma = 0$ in (6) [or in (5)]. Let us express the series (6) as a Padé (2, 2) approximant (in the form of a rational function):

$$\theta_{22}(\xi) = \frac{1 + A'\xi^2 + B'\xi^4}{1 + C'\xi^2 + D'\xi^4} \quad (7)$$

where A', B', C', and D' are constants. Equalizing (6) and (7), we have

$$A' = \frac{\sigma(12208n^2 + 80346n - 208600) - 1246n^2 + 6657n - 8750}{252\{\sigma(240n + 3000) - 51n + 150\}}$$

$$B' = \frac{\sigma(18416n^3 + 44964n^2 - 581480n + 672000) - 1290n^3 + 10849n^2 - 29100n + 24500}{15120\{\sigma(240n + 3000) - 51n + 150\}}$$

$$C' = \frac{\sigma(12208n^2 + 81858n - 57400) - 1246n^2 + 4515n - 2450}{252\{\sigma(240n + 3000) - 51n + 150\}} \quad (8)$$

$$D' = \frac{\sigma(18416n^3 + 108384n^2 - 23300n) - 1290n^3 + 4815n^2 - 2850n}{15120\{\sigma(240n + 3000) - 51n + 150\}}$$

In view of the boundary conditions (3), solution of the quadratic equation [numerator of (7)]

$$B'\eta^2 + A'\eta + 1 = 0 \qquad (\eta = \xi^2) \quad (9)$$

would define the boundary value ξ_1 of ξ which are given in tabular form (Table I) for $n = 0$, 1.0 (0.5) 3.0 and $\sigma = 0.00\,(0.002)\,0.01, 0.04$. Specific solution results for $n = 0$ are

$$\theta = 1 - \frac{1 + 4\sigma}{6}\xi^2, \qquad \xi_1 = \left(\frac{6}{1 + 4\sigma}\right)^{1/2} \quad (10)$$

Table I. Boundary Values ξ_1 of the Physical Parameter

$\sigma \backslash n$	0	1.0	1.5	2.0	2.5	3.0
0.00	2.44949	3.14572	3.68684	4.40921	5.44374	6.9211
	2.44949*	3.14159*	3.65375*	4.35287	5.35528	6.89685*
	0.00000	1.31×10^{-3}	9.05×10^{-3}	1.29×10^{-3}	1.652×10^{-2}	3.52×10^{-3}
0.002	2.43975	3.14283	3.68566	4.41182	5.50207	7.22884
0.004	2.43012	3.16202	3.72320	4.48823	5.87617	7.16213
0.006	2.42061	3.20456	3.80088	4.64509	8.72166	7.13619
0.008	2.41121	3.25414	3.92448	4.91605	–	–
0.010	2.40192	3.37635	4.10686	5.39870	–	7.11364
0.040	2.27429	2.67646	–	–	4.28534	7.08592

which coincides with Tooper's value, if we introduce post-Newtonian approximation in his expression (3.4).

The upper entry in ξ_1 of each of the columns of Table I refers to the value (for $\sigma = 0$ and $n = 0$) obtained by our analytical method; the middle entry presents Chandrasekhar's values [3], as marked by an asterisk (*); and the lower entry gives the relative errors $(\xi_{\text{ana}} - \xi_{\text{num}}/\xi_{\text{num}})_{\xi=\xi_1}$ (suffixes "ana" and "num" denote the value obtained by analytical and numerical methods, respectively) between these. Likewise for θ, we can express the mass function $U(\xi)$ [equation (5)] as a Padé (2, 2) approximant:

$$U_{22}(\xi) = \frac{\xi^3}{3} \frac{1 + E\eta + F\eta^2}{1 + G\eta + H\eta^2} \tag{11}$$

and determine the coefficients E, F, G, and H with the help of (5); for the case $n = 1$, the Padé (2, 2) approximant in final form is given by

$$U_{22}(\xi) = \frac{\xi^3}{3} \cdot \frac{166320(488\sigma + 13) - 252(22330\sigma + 633)\xi^2 + (65180\sigma + 2647)\xi^4}{166320(488\sigma + 13) + 1260(2662\sigma + 45)\xi^2 + 35(3076\sigma + 17)\xi^4} \tag{12}$$

For $n = 0$ (homogeneous liquid case), we have from (5),

$$U(\xi) = \xi^3/3 \tag{13}$$

the same expression as obtained by Tooper. Numerical values of $\rho_c/\bar{\rho}$, $2GM/c^2R$, and $U(\xi_1)$ for $n = 0$ and 1, $\sigma = 0.00(0.002)\,0.01, 0.04$ as calculated with the help of formulas

$$\frac{\rho_c}{\bar{\rho}} = \frac{1}{3} \frac{\xi_1^3}{U(\xi_1)} \tag{14}$$

$$\frac{2GM}{c^2R} = \frac{2\sigma(n + 1)\, U(\xi_1)}{\xi_1} \tag{15}$$

Table II. Characteristics of Physical Parameters $U(\xi)$, $\rho_c/\bar{\rho}$, and $2GM/c^2R$

$\sigma \backslash n$	0			1		
	$U(\xi)$	$\rho_c/\bar{\rho}$	$2GM/c^2R$	$U(\xi)$	$\rho_c/\bar{\rho}$	$2GM/c^2R$
0.000	4.89898	1.00000	0.00000	3.14572	3.300	0.00000
	4.89898*	1.00000*	0.00000*	3.14159*	3.290*	0.00000*
	0.00000	0.00000	0.00000	1.31×10^{-3}	3.31×10^{-3}	0.00000
0.002	4.84077	1.00000	0.00794	3.10339	3.334	0.00790
0.004	4.78368	1.00000	0.01574	3.06678	3.436	0.01552
0.006	4.72774	1.00000	0.02344	3.02658	3.624	0.02267
0.008	4.67287	1.00000	0.03100	2.97890	3.856	0.02929
0.010	4.61907	1.00000	0.03846	2.86581	4.477	0.03395
0.040	3.92118	1.00000	0.13794	2.65818	2.404	0.15891

and (12) (using ξ_1 from Table I), respectively, appear in Table II. The suffix "1" means the boundary value. In this table, the upper and middle entries in $U(\xi_1)$, $\rho_c/\bar{\rho}$, and $2GM/c^2R$ (for $n = 0$ and 1; $\sigma = 0.00$), respectively, refer to the author's and Tooper's values [marked by an asterisk (*)]. The relative error between these two is shown in the lower entry. Calculations for other values of n can be similarly done.

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