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Solutions to Einstein's Field Equations with Kantowski–Sachs Symmetry and String Dust Source

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Einstein's field equations are solved with a two-parameter family of classical strings as the source for the gravitational field. The solutions have Kantowski–Sachs symmetry. The singularities of the solutions and the kinematical properties of the string world sheets are discussed.

INTRODUCTION

Among the theories which involve strings, superstring theory and cosmic string theory receive the most attention at present. There is, however, another aspect of string theory that has been given little attention. This is the classical general relativistic theory of strings described by Stachel [1], where strings, surface-forming simple bivector fields, are treated as generalizations of structureless point particles in a strictly classical way.

Stachel assumes an arbitrary underlying spacetime geometry, and he discusses the properties of the energy momentum tensor of a perfect dust of such strings; he derives the conversation laws and the equations of motion of the string dust by analogy with the more common particle description. In this paper the approach is pursued further and the existence of consistent solutions to Einstein's field equations with a string dust source is demonstrated. The solutions are chosen to have Kantowski–Sachs [2] symmetry and the source is a two-parameter family of surface-forming simple bivector fields, also called string dust or a thickened string.

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THE STRING ENERGY MOMENTUM TENSOR

In suitable local coordinates, metrics with Kantowski–Sachs symmetry can be written in the form

$$ds^2 = dt^2 - X^2\,dr^2 - Y^2(d\theta^2 + f^2\,d\phi^2) \tag{1}$$

where $X = X(t)$, $Y = Y(t)$, and $f = \sin\theta$ or $\sinh\theta$ depending on the Lie algebra chosen. To use the index notation the coordinates are labeled $\{x^i\}$; $i, j, k = 0, 1, 2, 3$ with

$$t = x^0, \qquad r = x^1, \qquad \theta = x^2, \qquad \phi = x^3 \tag{2}$$

A simple bivector is an antisymmetric tensor of rank 2 that can be written as the alternating product of two vectors

$$S^{ij} = A^i B^j - A^j B^i \tag{3}$$

For a bivector field to be surface-forming it must satisfy

$$*S_{ij}(\partial_k S^{ij}) = 0 \tag{4}$$

where

$$*S_{ij} = \varepsilon_{ijkl} S^{kl} \tag{5}$$

and ε_{ijkl} is the Levi–Civita symbol in four dimensions. The bivector is called timelike (spacelike, null) if it spans a timelike (spacelike, null) surface element.

The energy momentum tensor associated with a string field is defined [1] to be

$$T_i^j = (-g)^{1/2}\,\mu S_i^{\,k} S_k^{\,j}, \qquad \mu \geqslant 0 \tag{6}$$

where S^{ij} is a timelike surface-forming simple bivector field which is normalized so that

$$S^{ij} S_{ij} = -2 \tag{7}$$

and μ is a scalar field.

The symmetries of the Kantowski–Sachs metric suggest the following vectors to span the bivector field

$$A^i = (\alpha, \beta, 0, 0) \tag{8}$$

$$B^i = (\pi, \tau, 0, 0) \tag{9}$$

where the α, β, π, τ are functions of differentiability class C^2 in the local coordinates, chosen so that the vector fields are timelike and spacelike, respectively. Without loss of generality the vector fields spanning the bivector field may be normalized to be unit vectors. Condition (7) implies that the vectors must be orthogonal. The two conditions, once imposed on the vector fields, reduce the arbitrariness of the components

$$\beta = \pi/X \tag{10}$$

$$\tau = \alpha/X \tag{11}$$

and

$$\pi^2 = \alpha^2 - 1 \tag{12}$$

Substitution into (3) and into (6) yields

$$S^{01} = -S^{10} = 1/X \tag{13}$$

$$T_0{}^0 = (-g)^{1/2}\,\mu, \qquad T_1{}^1 = (-g)^{1/2}\,\mu \tag{14}$$

the other components of both tensor fields are all zero.

The conservation equations which Stachel [1] proved to be equivalent to the condition that a simple bivector field be surface forming, are

$$\partial_i[(-g)^{1/2}\,\mu S^{ij}] = 0 \tag{15}$$

where ∂_i denotes the partial derivative operator with respect to the local coordinates. Integration of these equations yields

$$\mu = F/Y^2 \tag{16}$$

where F is an arbitrary function of integration which can at most be dependent on θ and ϕ. From the field equations which follow shortly it will be seen that μ may only be dependent on t and, hence it follows that F must be a constant.

2. THE FIELD EQUATIONS AND THEIR SOLUTIONS

The two cases of Kantowski–Sachs metrics can be handled together if a parameter k is introduced, where $k=1$ for $f(\theta)=\sin\theta$ and $k=-1$ for $f(\theta)=\sinh\theta$. If the parameter k is set equal to 0 then the field equations are those of a locally rotationally symmetric Bianchi-type 1 metric, i.e., $f(\theta)=1$ in the metric (1) [3].

The field equations are

$$2(\dot{X}\dot{Y})/(XY) + (k + \dot{Y}^2)/Y^2 = \mu \tag{17}$$

$$2\ddot{Y}/Y + (k + \dot{Y}^2)/Y^2 = \mu \tag{18}$$

$$\ddot{X}/X + \ddot{Y}/Y + (\dot{X}\dot{Y})/(XY) = 0 \tag{19}$$

Since X and Y are functions of t only, it follows that μ is also a function of t only. Subtraction of (18) from (17) and integration of the resulting equation gives

$$a\dot{Y} = X \tag{20}$$

where a is an integration constant. Substitution of this into (19), changing the independent variable to Y, and integration gives

$$\dot{Y}^2 = c - b/Y$$

where c and b are constants of integration. The explicit solutions for Y, X, and μ are given by

$$t = \int \frac{dY}{(c - b/Y)^{1/2}} + t_0 \tag{21}$$

$$X = a(c - b/Y)^{1/2} \tag{22}$$

$$\mu = C/Y^2 \tag{23}$$

where $C = (k + c)$, which confirms the result in (16). The vacuum solutions are given by $C = 0$, i.e., $c = -k$.

The final integration depends on the choice of the constants. If $b = 0$ or $c = 0$, the integration is easy.

Solution ia: $(b = 0)$

$$Y = \Omega(t - t_0) \tag{24}$$

$$X = a\Omega \tag{25}$$

$$\mu = C[\Omega(t - t_0)]^{-2} \tag{26}$$

where

$$\Omega^2 = c \tag{27}$$

Solution ib: $(c = 0,\ b < 0)$

$$Y = [(3/2)(-b)^{1/2}(t - t_0)]^{2/3} \tag{28}$$

$$X = a(-b)^{1/2}\, Y^{-1} \tag{29}$$

$$\mu = kY^{-2} \tag{30}$$

The remaining solutions are clearest if expressed in terms of a parameter $\sigma(t)$. Recall that $\mu = C/Y^2$ and so need not be written out each time.

Solution ii: $(b > 0,\ c > 0)$

$$Y = d^2 \cosh^2 \sigma \tag{31}$$

$$X = a\Omega \tanh \sigma \tag{32}$$

$$t = t_0 + (d^2/\Omega)(\sigma + \tfrac{1}{2}\sinh 2\sigma) \tag{33}$$

where (27) has been used and $d^2 = (b/c)$, $\Omega^2 = c$.

Solution iii: $(b < 0,\ c > 0)$

$$Y = d^2 \sinh^2 \sigma \tag{34}$$

$$X = a\Omega \coth^2 \sigma \tag{35}$$

$$t = t_0 + (d^2/\Omega)(\tfrac{1}{2}\sinh 2\sigma - \sigma) \tag{36}$$

$$d^2 = -(b/c), \qquad \Omega^2 = c$$

Solution iv: $(b < 0,\ c < 0)$

$$Y = d^2 \sin^2 \sigma \tag{37}$$

$$X = a\Omega \cot \sigma \tag{38}$$

$$t = t_0 + (d^2/\Omega)(\sigma - \tfrac{1}{2}\sin 2\sigma) \tag{39}$$

$$d^2 = (b/c), \qquad \Omega^2 = -c$$

3. SOME PROPERTIES OF THE SOLUTIONS

Singularity Structure

Solutions (i), (iii), and (iv) have density singularities at time $t = t_0$, which are cigar-type [4] for solutions (iii) and (iv). Solution (ii) does not have a density singularity anywhere, but it does have a conformal singularity at $t = t_0$ [5].

Kinematical Properties

Instead of considering the bivectors, we follow Stachel [6] and consider the corresponding congruences of two-dimensional subspaces of the

Kantowski–Sachs metrics. The metric tensor breaks into two degenerate orthogonal submetric tensors

$$g_{ij} = a_{ij} + b_{ij} \tag{40}$$

with

$$a_{ij} = \mathrm{diag}(1, \ -X^2, 0, 0)$$

the timelike-subspace metric, corresponding to the string, and

$$b_{ij} = \mathrm{diag}[0, 0, \ -Y^2, \ -Y^2 f^2(\theta)]$$

the spacelike-subspace metric, clearly orthogonal to the timelike one.

For notational convenience Stachel [6] defines the Christoffel derivatives

$$a_{\{ij;k\}} = \tfrac{1}{2}(a_{kj;i} + a_{ik;j} + a_{ij;k}) \tag{41}$$

where the semicolon on the right-hand side denotes the covariant derivative in the spacetime. From these the following kinematical quantities are defined [6]

$$\text{Acceleration} \qquad A_{k(ij)} = a^p{}_i a^q{}_j b^r{}_k a_{\{pq;r\}} \tag{42}$$

$$\text{Rotation} \qquad R_{i[jk]} = a^p{}_i b^p{}_{[j} b^r{}_{k]} a_{\{pq;r\}} \tag{43}$$

$$\text{Rate of deformation} \qquad H_{k(ij)} = a^p{}_k b^q{}_i b^r{}_j a_{\{pq;r\}} \tag{44}$$

where the brackets and parentheses applied to indices refer to the anti-symmetric and symmetric parts, respectively.

Rotation of the spacelike congruence of subspaces is

$$R'_{i[jk]} = -b_i{}^p a_{[j}{}^q a_{k]}{}^r a_{\{pq;r\}} \tag{45}$$

Note that because $g_{ij;k} = 0$ it follows that $a_{\{ij;k\}} = -b_{\{ij;k\}}$. The Stachel quantities can be used to define an expansion and shear by analogy with the definitions in the particle case (Stachel does not do it) i.e., by separating out trace and trace-free parts of the rate of deformation tensor. The expansion vector is defined to be

$$E_k = b^{ij} H_{k(ij)} \tag{46}$$

and the shear tensor by

$$\sigma_{k(ij)} = H_{k(ij)} - \tfrac{1}{3} E_k b_{ij} \tag{47}$$

The Kantowski–Sachs timelike subspaces are acceleration- and rotation-free; also $R'_{i[jk]}=0$, which is to be expected because both the subspaces are surface-forming. The rate of deformation has two nonzero components

$$H_{0(22)} = -Y\dot{Y} \tag{48}$$

$$H_{0(33)} = -Y\dot{Y}f^2(\theta) \tag{49}$$

The expansion vector has one nonzero component, $E_0 = 2\dot{Y}/Y$, and the nonzero components of the shear tensor are $\sigma_{0(22)}f^2(\theta) = \sigma_{0(33)} = \frac{1}{3}Y\dot{Y}f^2(\theta)$. The shear is defined by $s^2 = \frac{1}{2}\sigma^{i(jk)}\sigma_{i(jk)}$ which gives

$$s^2 = \tfrac{1}{3}\dot{Y}/Y \tag{50}$$

For the solutions with a string dust source

$$\dot{Y}/Y = (1/Y)(c - b/Y)^{1/2} \tag{51}$$

so as time passes the expansion vector and shear decrease for solutions (i), (iii), and (iv). For solution (ii) the behavior is less obvious. When $\sigma = 0$ the shear and expansion are zero, and their values increase with Y until $Y = \frac{3}{2}(b/c)$ after which they decrease again tending to zero as Y tends to infinity. The behavior of all these quantities is consistent with the kinematical interpretations that Stachel has assigned to them.

4. WORLD TUBES OF STRING DUST

Stachel [1] gives the boundary conditions for a string dust world tube, with timelike boundary $F(x^i) = \text{constant}$, as

$$T^i{}_j F_{,i} = 0 \tag{52}$$

and he points out that these conditions are equivalent to one of the following

$$\text{I} \qquad S_j{}^i F_{,i} = 0 \tag{53}$$

$$\text{II} \qquad S_j{}^i F_i = V_j \qquad \text{and} \qquad S_k{}^j V_j = 0 \tag{54}$$

From (13) it follows that

$$S_0{}^1 = X^{-1}, \qquad S_1{}^0 = X \tag{55}$$

and the remaining components are zero.

Consider first a boundary $F(r, t) = \text{constant}$. In this case

$$F_{,i} = (\partial F/\partial t)\,\delta_i^0 + (\partial F/\partial r)\,\delta_i^1 \tag{56}$$

It follows that

$$V_j = S_j^i F_{,i} = X\delta_j^1(\partial f/\partial t) + X^{-1}\delta_j^0(\partial f/\partial r) \tag{57}$$

For condition I to hold, V_j must be the zero vector, and for condition II

$$S_k^{\ 1}X(\partial F/\partial t) + S_k^{\ 0}X^{-1}(\partial F/\partial r) = 0 \tag{58}$$

so no solutions exist for function F.

A second possibility is to choose $\theta = \theta_1$, a constant ($\phi = \text{constant}$ and $F(\theta) = \text{constant}$ lead to a similar analysis). In this case

$$F_{,i} = \delta_i^2 \tag{59}$$

so

$$S_j^i F_{,i} = S_j^2 = 0 \tag{60}$$

and condition I is satisfied, which suggests that a timelike boundary can be found. To complete the analysis it is interesting to examine the matching conditions across the boundary in more detail using the Darmois conditions [7]. The conditions require that the first and second fundamental forms, calculated as functions of the common coordinates on the matching hypersurface Σ, are identical. Further, Σ must be covered by the same domain of the common coordinates in both representations.

Consider a region U in which the energy momentum tensor is that of string dust. The metric, for $k = 1$, is given by

$$g_{ij} = \text{diag}(1, -X^2, -Y^2, -Y^2 \sin^2\theta) \tag{61}$$

where Y and X are given by (21) and (22) with

$$0 \leqslant \theta < \pi \qquad \text{and} \qquad 0 \leqslant \phi < 2\pi \tag{62}$$

and recall that a, b, c are integration constants.

On the 3-surface choose

$$u^1 = t, \qquad u^2 = r, \qquad u^3 = \phi \tag{63}$$

Then the first fundamental form is given by

$$g_{\alpha\beta} = \text{diag}(1, -X^2, -Y^2 \sin^2\theta_1) \tag{64}$$

with

$$u^1 - u_0{}^1 = \int (c - b/Y)^{-1/2}\, dY \tag{65}$$

$$X = a(c - b/Y)^{1/2} \tag{66}$$

Now consider the vacuum region $\bar{U}$. The metric is again given by (21) and (22) except that $c = -k$ so there is one fewer parameter. Rescale the coordinate ϕ by

$$\bar{\phi} = h\phi \tag{67}$$

where h is a constant and leaves the other coordinates unchanged. The metric becomes

$$\bar{g}_{ij} = \operatorname{diag}(1, -\bar{X}^2, -\bar{Y}^2, -\bar{Y}^2 \sin^2\theta) \tag{68}$$

where the overbar is used to indicate that $c = -k$. Redefine

$$\tilde{Y} = h^{-1}\bar{Y} \tag{69}$$

then (21) with $c = -k$ becomes

$$t - t_0 = \int (-kh^{-2} - h^{-3}\bar{b}/\tilde{Y})^{1/2}\, d\tilde{Y} \tag{70}$$

where a bar over the parameter is used to denote that the metric refers to the vacuum region. If the parameters in the vacuum region, h, a, b, are chosen to be related to those in the non-vacuum region as follows

$$h = (-k/c)^{1/2}, \qquad \bar{a} = ah^{-1}, \qquad \bar{b} = h^3 b \tag{71}$$

and the coordinates u^1, u^2, and u^3 are related to the coordinates t, r, ϕ in the region U by

$$u^1 = t, \qquad u^2 = r, \qquad u^3 = \bar{\phi} \tag{72}$$

then the first fundamental form on $F(\theta) = \theta_1$ is given by (64–66). It is easily shown that the second fundamental forms match. However, as a result of the transformation (67) the domains of the two sets of coordinates on the common hypersurface are not the same, and so the Darmois conditions are not satisfied. The problem arises from the global character of the matching conditions which is not addressed by conditions (52). While this does not prove that the Kantowski–Sachs string dust solutions cannot be matched to a vacuum solution, the physically interesting case $F(r, t) = \text{constant}$ is eliminated, as are the obvious boundaries—θ or ϕ constant.

This result can be strengthened. Stachel [1] shows that open strings can only end on a boundary if the end points move at the speed of light, i.e., if the nonnormalized form of the string bivector becomes null. From (8) and (9) the condition for this to happen is that

$$(\alpha\tau - \pi\beta)^2 = 0 \tag{73}$$

which can only be satisfied if the bivector becomes degenerate, in the sense that the vectors A^i and B^i are parallel. Consequently the strings are unbounded.

In this paper it has been shown that the Stachel [1] string dust (thickened string) theory is self-consistent in so far as there exist solutions to Einstein's field equations with an unbounded string dust source, which can play the role of the metric for this discussion. This does not complete the picture because the existence of solutions with a bounded string dust source enclosed in a timelike world tube needs to be established. An approach to this further problem, using a source suggested by Stachel's rotating string source in a cylindrically symmetric spacetime, is being investigated.

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