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Inhomogeneous Two-Fluid Cosmologies

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A new class of expanding cosmological solutions is derived. The matter content of these models is a mixture of two interacting simple fluids: the first one, homogeneous and isotropic with equation of state $p = (\gamma - 1)\rho$, the dynamics of which is given by the FRW equation and the second one an inhomogenous dust. The limiting case of two dusts corresponds to the Szekeres' universes of class II. A large subclass of the models evolve to a FRW phase for all physically meaningful values of the polytropic index γ and the curvature parameter k. A gauge condition, under which the metric is invariant, is shown to exist for $k \neq 0$. In particular, it explains why the parabolic model is a peculiar solution in the class found by Szekeres.

1. INTRODUCTION

Some time ago, Szekeres [1] derived a remarkable set of inhomogeneous exact solutions of the Einstein field equations (EFE) without cosmological constant. The source of curvature of the models is an expanding, irrotational, and geodesic dust. These solutions are divided in two classes usually denoted by I and II. Here, we are particularly interested in the models of the second class. As shown by Bonnor and Tomimura [2] (hereafter referred to as BT paper), some models of this class evolve to Friedmann dust models with curvature parameter $k = 0, -1$. In fact, as remarked elsewhere [3], a Friedmannian era is established for all values of k. Thus, at least in principle, these solutions may describe an earlier inhomogeneous phase of the present universe [4].

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The Szekeres' spacetimes have been extended introducing pressure terms due to matter [5–8], adding an isotropic radiation (9), including dissipative processes in the cosmic fluid [10–12], and cosmological constant [13]. However, unlike the class found by Szekeres, the matter content of these solutions with pressure does not obey any equation of state. Of course, this is a rather undesirable feature of these models.

In the Szekeres' universes, the scale factor R, of their isotropic bidimensional section, has its dynamics driven by the Friedmann dust equation. In a certain sense, such property explains the evolutive behavior of these models. On the other hand, recently [14] the FRW differential equation was solved in unified form, i.e., for all values of k and the adiabatic index of the "gamma-law" $p=(\gamma-1)\rho$.

By combining these facts we propose, in particular, a possible solution to the well-known equation of state problem in the Szekeres background. In the next section, a unified approach involving FRW- and Szekeres'-type models is developed and a new set of exact inhomogeneous models with pressure is derived. The canonical form of solutions is given in Section 3, and some special solutions are shown in Section 4. Finally, the evolution of a large subclass of models is examined in the Section 5.

2. UNIFIED APPROACH FOR FRW AND SZEKERES' MODELS, CLASS II

In order to make explicit the relation between the Szekeres'-type solutions class II and the FRW ones, they will be derived here in the coordinate system used in paper BT.

2.1. FRW Models

Spatially homogeneous and isotropic cosmological models are locally described by the FRW line element. As we see presently, a convenient, although unusual, expression for it is the following

$$ds^2 = dt^2 - A^2R^2dx^2 - R^2(dy^2 + h^2dz^2) \tag{1}$$

where

$$A = A(x, y, z), \qquad R = R(t) \qquad \text{and} \qquad h = h(y) \tag{2}$$

The functions A and h are given by

$$A = (\sigma \cos z + \nu \sin z)\frac{\sin k^{1/2}y}{k^{1/2}} + \omega \cos k^{1/2}y \tag{3}$$

and

$$h = \frac{\sin k^{1/2} y}{k^{1/2}} = \sin y \quad \text{if} \quad k = 1$$
$$= y \quad \text{if} \quad k = 0 \tag{4}$$
$$= \sinh y \quad \text{if} \quad k = -1$$

In the above expressions, σ, v, and w are arbitrary functions of x, and k is the curvature parameter. Expression (3) for A was chosen as it appears in the Szekeres' models for $k = +1$. Note that unlike the BT paper we are using here the method in which the metrics are analytical continuation of a given one by variation of the parameter k. They used, for $k = -1$, $h = \cosh y$ instead of $h = \sinh y$. Unified solutions given by analytical continuation were obtained, for Gödel's like cosmologies [15].

By using (3) and (4) and comoving frame ($V^\mu = \delta_0^\mu$), the nontrivial EFE for perfect fluid in the metric (1) can be reduced to (Appendix A)

$$\rho = \frac{3}{R^2} (\dot{R}^2 + k) \tag{5}$$

and

$$p = -2 \frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} \tag{6}$$

where ρ and p are the mass energy density and pressure, respectively, and an overdot means time derivative.

From (5) and (6), the scale-factor R obeys the FRW differential equation

$$R\ddot{R} + \left(\frac{3\gamma - 2}{2}\right) \dot{R}^2 + \left(\frac{3\gamma - 2}{2}\right) k = 0 \tag{7}$$

where γ is the adiabatic index of the usual equation of state $p = (\gamma - 1)\rho$.

A first integral of (7) can be written as

$$\dot{R}^2 = (R_0/R)^{3\gamma - 2} - k \tag{8}$$

where $R_0^{3\gamma - 2}$ is a suitable integration constant.

Substituting (8) into (5), we obtain for the energy density and pressure

$$\rho = \frac{3}{R_0^2} \left(\frac{R_0}{R}\right)^{3\gamma} \tag{9}$$

$$p = \frac{3(\gamma - 1)}{R_0^2} \left(\frac{R_0}{R}\right)^{3\gamma} \tag{10}$$

This class of spacetimes defined in the comoving frame, are conformally flat, the flow of matter is nonrotating and shear-free, and the expansion parameter is $\theta = 3(\dot{R}/R)$. Thus, at least locally (1) and the standard FRW line element are equivalent.

The general solution of (7), given by Assad et al. [14], can be rewritten as

$$t - t_0 = \frac{2R_0}{3\gamma - 2}(1-k)^{1/2} F_1 - \frac{2R_0}{3\gamma - 2}\left[1 - k\left(\frac{R}{R_0}\right)^{3\gamma - 2}\right]^{1/2}\left(\frac{R}{R_0}\right)^{3\gamma/2} F_2 \quad (11)$$

where t_0 is a new integration constant, F_1 and F_2 are two hypergeometric functions

$$F_1 = F\left[\frac{3\gamma - 1}{3\gamma - 2}, 1; \frac{3}{2}; 1 - k\right] \quad (11a)$$

$$F_2 = F\left[\frac{3\gamma - 1}{3\gamma - 2}, 1; \frac{3}{2}; 1 - k\left(\frac{R}{R_0}\right)^{3\gamma - 2}\right] \quad (11b)$$

The adiabatic index γ in (7)–(11b) is not restricted to any interval. In particular, vacuum solutions derived by using a cosmological constant are recovered taking $\gamma = 0$, i.e., $p = -\rho$. In this case, by (9) and (10) the cosmological constant is $\Lambda = 3/R_0^2$. The constant t_0 in (11) is adjustable for each γ in order to fix the time scales used in the literature.

2. Szekeres'-Type Models

Consider now the line element of Szekeres' cosmological models as given in the BT paper

$$ds^2 = dt^2 - Q^2 dx^2 - R^2(dy^2 + h^2 dz^2) \quad (12)$$

where

$$Q = AR + TR_0 \qquad R = R(t) \qquad T = T(x, t) \qquad \text{and} \qquad A = A(x, y, z) \quad (13)$$

Note that due to the factor constant R_0, the functions A and T are dimensionless. The functions R and T are arbitrary and will be determined by the EFE. The function h is given again by (4), but to the function A a new term is added (cf. Eq. 3)

$$A = 4\alpha\left(\frac{\sin k^{1/2} y/2}{k^{1/2}}\right)^2 + (\sigma \cos z + \nu \sin z)\frac{\sin k^{1/2} y}{k^{1/2}} + \omega \cos k^{1/2} y \quad (14)$$

where α is a new arbitrary function of x. For the sake of brevity we prefer to define the function A but, in fact, it can be obtained integrating some of the field equations (Appendix A).

Taking the limit $k \to 0$, the function A is reduced to

$$A = \alpha y^2 + (\sigma \cos z + v \sin z)\, y + \omega \tag{15}$$

which seems not to coincide with the expressions given in BT for the parabolic case. However, transforming to new variables $y' = y \sin z$ and $z' = y \cos z$, the line element of the section $t = \text{const.}$, $x = \text{const.}$, takes a new form, viz. $dl'^2 = dy'^2 + dz'^2$ and the function A can be rewritten as

$$A = \alpha(y'^2 + z'^2) + vy' + \sigma z' + \omega \tag{16}$$

which is the expression of the BT paper for $k = 0$. Equation (14) for $k = \pm 1$ is the same one given in BT, only if $\alpha = 0$, but, as remarked before, the case $k = -1$ as given there cannot be obtained by analytic continuation as in (14).

The general form of Q function in (13) is invariant under the following gauge transformation

$$A \to A' = A + \delta \tag{17}$$

$$T \to T' = T - \delta(R/R_0) \tag{18}$$

where δ is an arbitrary function of x. In particular, as will be seen later, for $k = \pm 1$ the α function in (14) can always be ruled out through a specific gauge.

In the comoving frame the nontrivial EFE for perfect fluid in the background (12)–(14) can be rewritten as (Appendix A)

$$\rho = \frac{3AR(\dot{R}^2 + k) + 2RR_0\dot{R}\dot{T} + TR_0(\dot{R}^2 + k) - 4\alpha R}{(AR + TR_0)\, R^2} \tag{19}$$

$$p = -2\frac{\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} \tag{20}$$

$$R\ddot{T} + \dot{R}\dot{T} - T\left(\ddot{R} + \frac{\dot{R}^2 + k}{R}\right) = \frac{2\alpha}{R_0} \tag{21}$$

These equations show that, with $T = \lambda(x) R$, where λ is an arbitrary function, (19)–(21) reduce to (5) and (6) and, as expected, locally FRW models are recovered.

As the pressure p in (20) is a function of t alone, the usual equation of state cannot be imposed without loss of generality. In fact, an algorithm involving a definite choice of p has been often used in the literature in order to generate exact inhomogeneous solutions [5–9]. In the majority of cases some functional relationship uniting R and p has been considered, but they do not lead to any equation of state. We propose now an alternative point of view about the matter content that seems to avoid this problem.

Initially we remark that (20) for pressure p is the same one of FRW models (cf. Eq. 6). Moreover, the energy density ρ given in (19) can be rewritten as

$$\rho = \rho_{\text{FRW}} + \Delta\rho \tag{22}$$

where ρ_{FRW} is given by (5) and

$$\Delta\rho = \frac{2RR_0\dot{R}\dot{T} - 2TR_0(\dot{R}^2 + k) - 4\alpha R}{(AR + TR_0)\,R^2} \tag{23}$$

Therefore, the EFE imply that the matter content of these models can be seen as a mixture of two interacting simple fluids: the first one homogeneous and isotropic and the second one, an inhomogeneous dust, the energy density of which is given by (23). Now, it seems natural to impose for the isotropic component the usual equation of state $p = (\gamma - 1)\rho_{\text{FRW}}$. Of course, as for dust $p = 0$, the Szekeres' universes are a limiting case in which the mixture is reduced to two dusts.

As in the FRW models, the function R also obeys (7) and, substituting it into (21), we find the final form of the differential equation of T

$$R\ddot{T} + \dot{R}\dot{T} + \left(\frac{4 - 3\gamma}{3\gamma - 2}\right)\ddot{R}T = \frac{2\alpha}{R_0} \tag{24}$$

the solution of which, as shown in Appendix B, is given by

$$T = \beta\left(\frac{R}{R_0}\right)F_3 + \mu\left(\frac{R}{R_0}\right)^{(3\gamma - 4)/2} F_4 + \frac{2\alpha}{k}\left(\frac{R}{R_0}\right)(F_3 - 1) \tag{25}$$

where β and μ are two new arbitrary functions of x, and F_3, F_4 are two hypergeometric functions

$$F_3 = F\left[\frac{1}{3\gamma - 2}, \frac{1}{3\gamma - 2}; \frac{3\gamma + 2}{2(3\gamma - 2)}; k\left(\frac{R}{R_0}\right)^{3\gamma - 2}\right] \tag{25a}$$

$$F_4 = F\left[\frac{3\gamma - 4}{2(3\gamma - 2)}, \frac{3\gamma - 4}{2(3\gamma - 2)}; \frac{9\gamma - 10}{2(3\gamma - 2)}; k\left(\frac{R}{R_0}\right)^{3\gamma - 2}\right] \tag{25b}$$

The inhomogeneous solutions are completely specified by (14) for the function A, (25) for T, and by the solution of R given in (11a, b). Of course, ρ_{FRW} and p are defined in (9) and (10) and the density of the inhomogeneous dust is established substituting T, R, and A in (23).

If $k \neq 0$, the functions A and T can be rewritten as

$$A = (\sigma \cos z + v \sin z)\frac{\sin k^{1/2}y}{k^{1/2}} + \bar{\omega} \cos k^{1/2}y + \frac{2\alpha}{k} \tag{26}$$

and

$$T = \bar{\beta}\left(\frac{R}{R_0}\right) F_3 + \mu \left(\frac{R}{R_0}\right)^{(3\gamma-4)/2} F_4 - \frac{2\alpha}{k}\left(\frac{R}{R_0}\right) \tag{27}$$

where

$$\bar{\omega} = \omega - (2\alpha/k) \qquad \text{and} \qquad \bar{\beta} = \beta + (2\alpha/k) \tag{28}$$

By comparing (26) and (27) with (17) and (18), we can see the existence of a specific gauge in which the function δ is given by $\delta = 2\alpha/k$. Thus, if $k \neq 0$, the arbitrary function α can be eliminated of the expression (14) and (25) without loss of generality. This means that if $\alpha \neq 0$, the parabolic models are a special class of solutions, and as the gauge is γ-independent, this is valid for any value of γ. In particular, this explains why the Szekeres' parabolic model ($k = 0$, $\gamma = 1$) has, for instance, an anomalous behavior if $\alpha \neq 0$, but not if $\alpha = 0$ (see BT).

3. THE CANONICAL FORM OF THE SOLUTIONS

By using the BT notation we exhibit a canonical form for all models presented in the preceding section. The parabolic case is determined taking the limit $k \to 0$ in all expressions with the term $\beta(R/R_0)$ of the T function absorbed in AR. For $k \pm 1$, the gauge freedom has been used in order to eliminate the α function.

3.1. Parabolic Models ($k = 0$)

$$A = \alpha y^2 + (\sigma \cos z + v \sin z)\, y + \omega \tag{29}$$

$$T = \mu \left(\frac{R}{R_0}\right)^{(3\gamma-4)/2} + \frac{4\alpha}{(3\gamma-2)(3\gamma+2)}\left(\frac{R}{R_0}\right)^{3\gamma-1} \tag{30}$$

$$R(t) = R_0\left[1 + \frac{3\gamma}{2}\left(\frac{t-t_0}{R_0}\right)\right]^{2/3\gamma} \qquad Q = AR + T \tag{31}$$

$$\rho = \rho_{\text{FRW}} + \Delta\rho \qquad \rho_{\text{FRW}} = \frac{3}{R_0^2}(R_0/R)^{3\gamma} \tag{32}$$

$$\Delta\rho = \frac{(3\gamma-6)\,\mu(R/R_0)^{-3\gamma/2} - 12\alpha(3\gamma+2)^{-1}(R/R_0)}{R_0^2\{A(R/R_0)^3 + \mu(R/R_0)^{3\gamma/2} + [4\alpha(R/R_0)^{3\gamma+1}/(3\gamma-2)(3\gamma+2)]\}} \tag{33}$$

$$p = (\gamma-1)\rho_{\text{FRW}} \tag{34}$$

3.2. Elliptic and Hyperbolic Models ($k = \pm 1$)

$$A = (\sigma\cos z + \nu\sin z)(k^{-1/2}\sin k^{1/2}y) + \omega\cos k^{1/2}y \tag{35}$$

$$T = \beta\left(\frac{R}{R_0}\right) F\left[\frac{1}{3\gamma-2} + \frac{1}{3\gamma-2}; \frac{3\gamma+2}{2(3\gamma-2)}; k\left(\frac{R}{R_0}\right)^{3\gamma-2}\right] + \mu\left(\frac{R}{R_0}\right)^{(3\gamma-4)/2} F\left[\frac{3\gamma-4}{2(3\gamma-2)}, \frac{3\gamma-4}{2(3\gamma-2)}; \frac{9\gamma-10}{2(3\gamma-2)}; k\left(\frac{R}{R_0}\right)^{3\gamma-2}\right] \tag{36}$$

$$\rho = \rho_{\text{FRW}} + \Delta\rho \qquad p = (\gamma-1)\,\rho_{\text{FRW}} \tag{37}$$

$$\Delta\rho = \frac{2R_0R\dot{R}\dot{T} - 2R_0T(R/R_0)^{3\gamma-2}}{(AR + TR_0)\,R^2} \tag{38}$$

where the function R is given in (11).

All solutions can be put in parametric form defining the conformal time by $dt = Rd\tau$. In this case, the scale-factor $R(\tau)$ takes the form [14]

$$R(\tau) = R_0\left(k^{-1/2}\sin k^{1/2}\left|\frac{(3\gamma-2)}{2}\right|\tau\right)^{2/3\gamma-2} \tag{39}$$

where the k-dependent range of τ is given by $0 \leqslant \tau \leqslant 2\pi/|3\gamma-2|$ if $k=1$ and $0 \leqslant \tau < \infty$ if $k = 0, -1$. The functions $t(\tau)$ and $T(x, \tau)$ are obtained, in general, substituting (39) into (11) and (25), respectively.

For any value of k, by a transformation in x, one arbitrary function can be made constant, and as t_0 can be adjusted freely, the models depend on four arbitrary functions and one positive constant R_0. Note also that only two arbitrary functions, β and μ if $k = \pm 1$, α and μ if $k = 0$, are related with these inhomogeneous models. In fact, if $k = \pm 1$ and β, μ are constants, the solutions (35)–(38) generalize the Kantowski–Sachs models and Bianchi VI-type ones, respectively [16]. If $k = 0$ and α, μ are constants, Bianchi I-type models have been extended.

4. SPECIAL SOLUTIONS

The existence of the FRW-type component implies that from a cosmological point of view, the most interesting cases of the models presented in the latter section are just $\gamma = 0$ (vacuum plus dust), $\gamma = 1$ (two dusts), and $\gamma = 4/3$ (radiation plus dust).

4.1. Parabolic Models ($k = 0$)

In this case, the solutions with $\gamma = 0$, 1 and 4/3 are trivially obtained by using (30)–(34). We observe that, considering the usual one fluid description, the Szekeres' parabolic model is reobtained taking $\gamma = 1$.

4.2. Elliptic and Hyperbolic Models ($k = \pm 1$)

In general, the hypergeometric functions are not reducible to elementary functions. However, this occurs if $\gamma = 0$, 1, and 4/3 (Appendix C).

(i) $\gamma = 0$ (vacuum plus dust)

$$T = \frac{3\mu}{k}\left\{1 - \left[\left(\frac{R}{R_0}\right)^2 - k\right]^{1/2} \frac{\arcsin k^{1/2}(R/R_0)}{k^{1/2}}\right\} + \beta\left[\left(\frac{R}{R_0}\right)^2 - k\right]^{1/2} \tag{40}$$

$$\rho = \Lambda + \Delta\rho \qquad \Lambda = 3/R_0^2 = -p \tag{41}$$

$$\Delta\rho = -6\mu R_0/QR^2 \tag{42}$$

$$\begin{aligned} R = R_0 \cosh(t/R_0) \qquad & \text{if} \qquad k = 1 \\ R_0 \sinh(t/R_0) \qquad & \text{if} \qquad k = -1 \end{aligned} \tag{43}$$

(ii) $\gamma = 1$ (two dusts)

In parametric form we have [see (39)]

$$T = \mu k^{1/2} \cot k^{1/2} \frac{\tau}{2} + \frac{3\beta}{k}\left(1 - \frac{\tau}{2} k^{1/2} \cot k^{1/2} \frac{\tau}{2}\right) \tag{44}$$

$$R = R_0\left(k^{-1/2} \sin k^{1/2} \frac{\tau}{2}\right)^2 \qquad t = \frac{R_0}{2k}\left[\tau - (\sin k^{1/2}\tau/k^{1/2})\right] \tag{45}$$

$$\rho = \frac{3}{R_0^2}\left(\frac{R_0}{R}\right)^3 + \Delta\rho \qquad p = 0 \tag{46}$$

$$\Delta\rho = 3R_0(\beta R - TR_0)/QR^3 \tag{47}$$

where $0 \leqslant \tau \leqslant 2\pi$ if $k = +1$ and $0 \leqslant \tau < \infty$ if $k = -1$.

As in the case $k=0$, the Szekeres' models can be recovered if we adopt the one fluid description in which the energy density (46) takes the form

$$\rho = 3R_0(A+\beta)/QR^2 \tag{48}$$

Equations (44) and (48) may be compared to the respective results of BT paper. There, the numerical factor 3 in (44) was absorbed into the β function and, for $k=+1$, the same occurred with a negative sign explaining, in the latter case, the positive sign in (48).

(iii) $\gamma = 4/3$ (radiation plus dust)

$$T = \beta\tau + \mu \tag{49}$$

$$R = R_0 \sin k^{1/2}\tau/k^{1/2} \qquad t = R_0[(1-\cos k^{1/2}\tau)/k^{1/2}]) \tag{50}$$

$$\rho = (3/R_0^2)(R_0/R)^4 + \Delta\rho \qquad p = R_0^2/R^4 \tag{51}$$

$$\Delta\rho = 2R_0[\beta k^{1/2} \cot k^{1/2}\tau - (\beta\tau+\mu)\, k \csc^2 k^{1/2}\tau]/QR^2 \tag{52}$$

where $0 \leqslant \tau \leqslant \pi$ if $k=1$, and $0 \leqslant \tau < \infty$ if $k=-1$.

5. KINEMATICAL QUANTITIES AND EVOLUTION

As in the Szekeres' universes, our models have no killing vectors, are type D in the Petrov classification, the 3-spaces are conformally flat, and the flow of matter is irrotational and geodetic. The expansion and shear parameters are

$$\theta = 2\frac{\dot{R}}{R} + \frac{A\dot{R} + \dot{T}R_0}{AR + TR_0} \tag{53}$$

and

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{3R_0^2}{R^2}\left(\frac{R\dot{T} - T\dot{R}}{AR + TR_0}\right)^2 \tag{54}$$

In the framework of the two-fluid interpretation, (53) can be rewritten as

$$\theta = \theta_{\rm FRW} + \Delta\theta \tag{55}$$

where $\theta_{\rm FRW} = 3(\dot{R}/R)$ and

$$\Delta\theta = \frac{R_0}{R}\left(\frac{R\dot{T} - T\dot{R}}{AR + TR_0}\right) \tag{56}$$

Now, by using (54) and (56)

$$\sigma^2 = \tfrac{1}{3}(\Delta\theta)^2 \tag{57}$$

is easily obtained. Thus, the shear tensor and the "anomalous" part of expansion $\Delta\theta$ are closely related and depend strongly on the inhomogeneous dust, since T proportional to R implies $\Delta\theta = \sigma^{\mu\nu} = 0$.

The asymptotic behavior (in time) of the models can be studied using the canonical form of solutions and taking into account (55)–(57).

If the isotropic component obeys the "strong energy condition" ($\gamma > 2/3$), the models are always singular in the early times. In this case, as in the FRW models, the solutions are essentially parabolic near the singularity ($R \ll R_0$). In the course of time, if $k = 0, -1$, the scale factor expands indefinitely, thus the asymptotic behavior must be studied for large values of the cosmological time ($R \gg R_0$). However, if $k = 1$ and $\gamma > 2/3$, R_0 is a maximum value of R. Then, if a FRW phase is expected, the correct limit to consider is $R \to R_0$. In what follows, the parameter γ is restricted to the physical interval ($1 \leqslant \gamma \leqslant 2$). All limits were computed retaining the leading terms in the respective expressions.

5.1. Approach to Singular Point

By using (30) we find that for $R \ll R_0$, $AR + TR_0 \sim \mu R_0 (R/R_0)^{(3\gamma-4)/2}$. Therefore, after a trivial variable change, the (12) takes, in this limit, the following form

$$ds^2 \sim dt^2 - R_0^2 (R/R_0)^{3\gamma-4}\, dx'^2 - R^2(dy'^2 + dz'^2) \tag{58}$$

which is homogeneous and anisotropic. In fact, from (55) and (56), a suitable anisotropy scale is measured by $\Delta\theta/\theta_{\mathrm{FRW}} \sim (\gamma - 2)/2$ in this limit. The anisotropy strength diminishes as γ increases, in particular, if $\gamma = 2$ the model is isotropic in the early times.

By using (32) and (33) we can readily obtain, with the same degree of accuracy

$$\lim_{R \ll R_0} \Delta\rho \sim \frac{3\gamma - 6}{R_0^2} \left(\frac{R_0}{R}\right)^{3\gamma} \tag{59}$$

and

$$\lim_{R \ll R_0} \rho \sim \frac{3(\gamma - 1)}{R_0^2} \left(\frac{R_0}{R}\right)^{3\gamma} \tag{60}$$

From (60), (32), and (34), we find $p \sim \rho$ regardless of the value of γ. Then, near the singularity, the mixture behaves as a simple fluid obeying the stiff equation of state. Note that in this limit, the density of the inhomogeneous dust given in (59) is negative. However, the net energy density $\rho = \rho_{\text{FRW}} + \Delta\rho$ is always positive in accordance with the weak energy condition. In fact, since near the singularity the dust concept is meaningless, the mixture is to be regarded, for all values of γ, as a macroscopic representation of stiff matter in this limit. This interpretation was suggested in [9] for a mixture of isotropic radiation ($\gamma = 4/3$) and dust with negative density.

5.2. Behavior at large values of R

As in the course of time the contributions of the curvature terms are not negligible, the models are separately examined.

5.2.1. *Parabolic Models* (k = 0)

If $\alpha = 0$, from (30) and (13) it is easily obtained that, for $R \gg R_0$, $Q = AR + TR_0 \sim AR$. Then, taking into account the results of Section 2.1. about the FRW models, it follows that the homogeneous and isotropic phase is reached. In fact, by using (32) and (33), it can be computed that in this limit $\Delta\theta/\theta_{\text{FRW}} \sim 0$, $\rho \sim \rho_{\text{FRW}}$, and $p \sim (\gamma - 1)\,\rho$. For $\alpha \neq 0$, similar computations show that the models are homogeneous but anisotropic for $R \gg R_0$. However, as $\Delta\rho$ is negative, an unreasonable result in this limit (33), these solutions with $\alpha \neq 0$ can be ruled out in the framework of the two-fluid interpretation.

5.2.2. *Hyperbolic Models* (k = −1)

In this case, the hypergeometric functions present in (36) are given in terms of oscillating power series; thus, a direct analysis from these equations about the limit $R \gg R_0$ cannot be made by this method. However, this problem can be circumvented through a linear transformation formula of the hypergeometric functions. By using the identity [17 p. 559, Eq. 5.3.4] $F(a, b, c, z) = (1 - z)^{-a} F(a, c - b; c; z/z - 1)$ and taking the limit $R \gg R_0$, it is easy to see that, for $k = 1$

$$F_3 \sim c_1 \left(\frac{R_0}{R}\right)\left[1 + 0\left(\frac{R_0}{R}\right) + \cdots\right]$$

and

$$F_4 \sim c_2 \left(\frac{R_0}{R}\right)^{(3\gamma - 4)/2}\left[1 + 0\left(\frac{R_0}{R}\right) + \cdots\right]$$

where c_1 and c_2 are two γ-dependent constants.

Substituting these results into (36) it follows that for $R \gg R_0$, $T \sim c_1 \beta + c_2 \mu$; in consequence, $AR + TR_0 \sim AR$. Thus, the FRW phase for large values of the cosmological time is independent of the choice of the arbitrary functions.

5.2.3. Ellyptic models (k = +1)

For this case, as remarked before, a FRW phase can be expected to occur when the "radius" R is near its maximum value R_0. The analysis is simplified by observing that in the neighborhood of R_0 we have $\dot{R} \sim 0$, $\dot{T} = (\partial T/\partial R)\, \dot{R} \sim 0$. In fact, from (56) we find that in this limit $\Delta\theta \sim 0$, and, thus, $\theta \sim \theta_{\mathrm{FRW}}$. Moreover, from (36) and (13), it is easy to show, absorbing the functions β and μ into the function A, that for $R \to R_0$, $Q \sim AR$. Then, as in the hyperbolic case, analogous results can be derived from the hypergeometric functions, computing the appropriate limits.

FINAL REMARKS

We have examined the existence of inhomogeneous cosmological models with Szekeres'-type metric class II and a different two-fluid, mater content. These fluids are explicitely taken as an inhomogeneous dust and a FRW polytropic fluid. A unified approach analysis revealed several aspects concerning the relation between the FRW and Szekeres'-type cosmological models.

In the two-fluid solutions, the energy-momentum tensor of each component is not separately conserved. Thus, there is interaction between them. However, the evolution of the models is fully adiabatic, i.e., only entropy exchanges between the components are performed.

Another feature, worth mentioning, closely related with the two-fluid interpretation is the simplicity of the solutions. It was possible to obtain exact solutions for all values of k and γ. These solutions are, in general, expressed in terms of hypergeometric functions. For $k \neq 0$ they assume elementary form for certain values of γ, among them the Szekeres' solutions. In the case $k = 0$, the geometrical and physical quantities are given for all values of γ as power functions of t (compare the results of [9] with ours for $\gamma = 4/3$; in fact, they do not have the same dynamics).

Finally, we observe that the Szekeres' parabolic model with $\alpha \neq 0$ (β in the notation of the BT paper) is an "anomalous" but physical solution; this fact remains unaltered if we adopt the one-fluid description for the solutions with $\alpha \neq 0$ presented here. However, in the two-fluid interpretation, as was shown in Section 5.2.1., they become unphysical solutions. Thus, the unified solutions presented in (35)–(38) are the most comprehensive set of cosmological solutions generated by this mixture of two-fluids.

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APPENDIX A

In the comoving frame, the EFE $G^{\mu\nu}=T^{\mu\nu}$ for the Szekeres' line element (12) with $T_{\mu\nu}=(\rho+p)\,v_\mu v_\nu-pg_{\mu\nu}$ are (in our units $8\pi G=c=1$)

$$QR^2\rho = Q\dot{R}^2+2R\dot{Q}\dot{R}-Q_{22}-h^{-2}(Q_{33}+hh_2Q_2+hh_{22}Q) \quad \text{(A1)}$$

$$R^2p = -2\ddot{R}R-\dot{R}^2+h^{-1}h_{22} \quad \text{(A2)}$$

$$QRp = -Q\ddot{R}-\dot{Q}\dot{R}-\ddot{Q}R+h^{-2}R^{-1}(Q_{33}+hh_2Q_2) \quad \text{(A3)}$$

$$QRp = -Q\ddot{R}-\dot{Q}\dot{R}-\ddot{Q}R+R^{-1}Q_{22} \quad \text{(A4)}$$

$$0=Q_{23}-h^{-1}h_2Q_3 \quad \text{(A5)}$$

$$0=\dot{Q}_2-Q_2R^{-1}\dot{R} \quad \text{(A6)}$$

$$0=\dot{Q}_3-Q_3R^{-1}\dot{R} \quad \text{(A7)}$$

where an overdot means time partial derivative and $Q_i \equiv \partial Q/\partial x^i$ $(i=2, 3\equiv y, z)$.

APPENDIX B

Here we establish the solution of the differential equation (24) to T function

$$R\ddot{T}+\dot{R}\dot{T}+\left(\frac{4-3\gamma}{3\gamma-2}\right)\ddot{R}T=\frac{2\alpha}{R_0} \quad \text{(B1)}$$

without loss of generality we take

$$T=n^{1/(3\gamma-2)}f(n,x) \qquad n=(R/R_0)^{3\gamma-2} \quad \text{(B2)}$$

Substituting (B2) into (B1) and using (7) and (8), we find that f satisfies the inhomogeneous equation

$$n(1-kn)\frac{\partial^2 f}{\partial n^2}+\left[\frac{3\gamma+2}{2(3\gamma-2)}-\left(\frac{3\gamma}{3\gamma-2}\right)kn\right]\frac{\partial f}{\partial n}-\frac{kf}{(3\gamma-2)^2}=\frac{2\alpha}{(3\gamma-2)^2} \quad \text{(B3)}$$

If $\alpha=0$ and $k=+1$, the above equation is in the canonical form of a hypergeometric differential equation [17, p. 562, Eq. 15.5.1] whose parameters are $a=b=(3\gamma-2)^{-1}$ and $c=(3\gamma+2)/2(3\gamma-2)$. If $k=-1$, transforming $n\to -n$, the same equation is obtained. Then, in the variable kn, the homogeneous solution of (B3) is given by [17, p. 563, Eqs. 15.5.3 and 15.5.4]

$$f=\beta F\left[\frac{1}{3\gamma-2},\frac{1}{3\gamma-2};\frac{3\gamma+2}{2(3\gamma-2)};kn\right]$$
$$+\mu n^{(3\gamma-6)/2(3\gamma-2)}F\left[\frac{3\gamma-4}{2(3\gamma-2)},\frac{3\gamma-4}{2(3\gamma-2)};\frac{9\gamma-10}{2(3\gamma-2)},kn\right] \quad \text{(B4)}$$

where β and μ are two arbitrary functions of x. Note that since $F(a,b,c,o)=1$, a solution to the flat case is readily obtained in the limit $k\to 0$. Finally, instead of taking the particular solution of (B4) $f_p=-2\alpha/k$, which is valid for $k\neq 0$, we take the following unified expression

$$f_p^{(k)}=\frac{2\alpha}{k}\left[F\left(\frac{1}{3\gamma-2},\frac{1}{3\gamma-2};\frac{3\gamma+2}{2(3\gamma-2)};kn\right)-1\right] \quad \text{(B5)}$$

which in the limit $k\to 0$ furnishes

$$f_p^{(0)}=\lim_{k\to 0}f_p^{(k)}=\frac{4\alpha n}{(3\gamma-2)(3\gamma+2)} \quad \text{(B6)}$$

then, by the (B2), (B4), and (B5), the unified solution of T, as a function of R, is

$$T=\beta\left(\frac{R}{R_0}\right)F_3+\mu\left(\frac{R}{R_0}\right)^{(3\gamma-4)/2}F_4+\frac{2\alpha}{k}\left(\frac{R}{R_0}\right)[F_3-1] \quad \text{(B7)}$$

where

$$F_3=F\left[\frac{1}{3\gamma-2},\frac{1}{3\gamma-2};\frac{3\gamma+2}{2(3\gamma-2)};k\left(\frac{R}{R_0}\right)^{3\gamma-2}\right] \quad \text{(B8)}$$

$$F_4=F\left[\frac{3\gamma-4}{2(3\gamma-2)'}\frac{3\gamma-4}{2(3\gamma-2)};\frac{9\gamma-10}{2(3\gamma-2)},k\left(\frac{R}{R_0}\right)^{3\gamma-2}\right] \quad \text{(B9)}$$

Let us observe that the hypergeometric functions in (B4) are linearly independent only if the parameter $c=(3\gamma+2)/2(3\gamma-2)$ is nonintegral. If $\gamma=(4p+2)/(6p-3)$, where p is an integer, it is necessary to obtain the so-called logarithmic solutions, since one of the hypergeometric functions in (A4) becomes meaningless or both become identical. However, the cases $\gamma=0, 1$, and 4/3 are all contained in (B7) (Appendix C). As the most interesting cases can be derived from (B7), we do not consider in this paper the logarithmic case.

APPENDIX C

The function T for models with $\gamma=0, 1$, and 4/3 are considered. In what follows, the identities below are useful [17, p. 556, Eqs. 15.1.6, 15.1.8; p. 558, Eq. 15.2.26]

$$F(a, b; b; z)=(1-z)^{-a} \tag{C1}$$

$$F(1/2, 1/2; 3/2; z^2)=(1-z^2)^{1/2}\, F(1, 1; 3/2; z^2)=z^{-1} \arcsin z \tag{C2}$$

$$[b-1-(c-a-1)\, z]\, F(a, b; c; z)+(c-b)\, F(a, b-1; c; z) - (c-1)(1-z)\, F(a, b; c-1; z)=0 \tag{C3}$$

Consider now the cases

(i) $\gamma=0$
Equation (36) reduces to

$$T=\beta \frac{R}{R_0} F[-1/2, -1/2; -1/2; k(R/R_0)^{-2}] + \mu(R/R_0)^{-2}\, F[(1, 1; 5/2; k(R/R_0)^{-2}] \tag{C4}$$

Considering the identity (C1), it is sufficient to compute $F(1, 1; 5/2; z^2)$, where $z^2=k(R/R_0)^{-2}$. By using (C2) and (C3), we find $F(1, 1; 5/2; z^2)= 3z^{-2}[1-(1-z^2)^{1/2}\, z^{-1} \arcsin z]$. Substituting into (C4), after some manipulations, it follows that

$$T=\frac{3\mu}{k}\left\{1-\left[\left(\frac{R}{R_0}\right)^2-k\right]^{1/2} \frac{\arcsin k^{1/2}(R/R_0)}{k^{1/2}}\right\}+\beta\left[\left(\frac{R}{R_0}\right)^2-k\right]^{1/2} \tag{C5}$$

(ii) $\gamma = 1$
Now, (36) reads

$$T = \beta\left(\frac{R}{R_0}\right) F\left[1, 1; \frac{5}{2}; k\left(\frac{R}{R_0}\right)\right] + \mu\left(\frac{R}{R_0}\right)^{-1/2} F\left[-\frac{1}{2}, -\frac{1}{2}; -\frac{1}{2}; k\left(\frac{R}{R_0}\right)\right] \tag{C6}$$

Note that the above hypergeometric functions have the same parameters of the latter case. Only the argument has been modified. Defining $z = k(R/R_0)$ and repeating the steps given in case (i), it is readily obtained

$$T = \frac{3\beta}{k}\left[1 - \left(\frac{R_0}{R} - k\right)^{1/2} \frac{\arcsin k^{1/2}(R/R_0)^{1/2}}{k^{1/2}}\right] + \mu\left(\frac{R_0}{R} - k\right)^{1/2} \tag{C7}$$

(iii) $\gamma = 4/3$

$$T = \beta\left(\frac{R}{R_0}\right) F\left[1/2, 1/2; 3/2; k\left(\frac{R}{R_0}\right)^2\right] + \mu \tag{C8}$$

By using (C2) and taking $z = k(R/R_0)^2$ in (C8), we find

$$T = \beta \frac{\arcsin k^{1/2}(R/R_0)}{k^{1/2}} + \mu \tag{C9}$$

REFERENCES

1. Szekeres, P. (1975). *Comm. Math. Phys.*, **41**, 55.
2. Bonnor, W. B., and Tomimura N. (1976). *Mont. Not.*, **175**, 85.
3. Lima, J. A. S. (1986). *Phys. Lett. A*, **116**, 210.
4. Goode, S. W., and Wainwright, J. (1982). *Phys. Rev. D*, **26**, 3315.
5. Tomimura, N. (1976). *Nuovo Cimento B*, **42**, 1.
6. Szafron, D. A., and Wainwright, J. (1977). *J. Math. Phys.*, **18**, 1668.
7. Szafron, D. A. (1977). *J. Math. Phys.*, **18**, 1673.
8. Lima, J. A. S. (1982). MSC thesis, CBPF, Rio de Janeiro.
9. Pollock, M. D., and Caderni, N. (1980). *Mont. Not.*, **190**, 509.
10. Roy, S. R., and Singh, J. P. (1982). *Ind. J. P. Appl. Math.*, **13**, 1285.
11. Lima, J. A. S., and Maia, M. G. (1985). *Phys. Lett. A*, **110**, 366.
12. Goode, S. (1986). *Class. Quant. Grav.*, **3**, 1247.
13. Barrow, J. D., and Stein-Chabes, J. (1984). *Phys. Lett. A*, **103**, 315.
14. Assad, M. J. D., and Lima, J. A. S. (1988). *Gen. Rel. Grav.*, **20**, 527.
15. Rebouças, M. J., and Tiomno, J., (1983). *Phys. Rev. D*, **28**, 1251.
16. Spero, A., and Szafron, D. A. (1978). *J. Math. Phys.*, **19**, 1536.
17. Abramowitz, N. E., and Stegun, I. A. (1965). *Handbook of Mathematical Functions* (Dover, New York).