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Geodesics for the NUT Metric and Gravitational Monopoles

Robert L. Zimmerman[1] **and Bijan Y. Shahir**[1]

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In order to provide insight about the physical interpretation of the NUT parameter, we solve the geodesic equations for the NUT metric. We show that the properties of NUT geodesics are similar to the properties of trajectories for charged particles orbiting about a magnetic monopole. In summary, we show that (1) the orbits lie on the surface of a cone, (2) the conserved total angular momentum is the sum of the orbital angular momentum plus the angular momentum due to the "monopole" field, (3) the monopole field angular momentum is independent of the separation between the source of the gravitational field and the test particle, and (4) the geodesics are "almost" spherically symmetric. The strong similarities between the NUT geodesics and the electromagnetic monopole suggest that the NUT metric is an exact solution for a gravitational magnetic monopole. However, the subtle difference of being only almost spherically symmetric implies that the analogy is not perfect. The almost spherically symmetric nature of the NUT geodesics suggest that the energy of the "Dirac string" makes a contribution to the solution. We also construct exact solutions for special orbits, discuss a twin paradox, and speculate about the Dirac quantization condition for a gravitational magnetic monopole.

1. INTRODUCTION

The NUT metric is a vacuum solution of Einstein's equations which has a long and interesting history. Two unrelated physical interpretations for the metric have been stated in the literature. One viewpoint is motivated by the similarity between the group structure for the NUT symmetry and the group structure for spherical symmetry. The identification of these symmetries leads to the interpretation that the NUT metric corresponds to a vacuum cosmological-like solution with periodic time. The peculiar proper-

[1] Institute of Theoretical Science, University of Oregon, Eugene, Oregon 97403.

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ties of the NUT space that follow from this identification are best summarized by a statement made by Misner [2]: "A space which does not admit an interpretation without a periodic time coordinate, a space without reasonable space-like surfaces, and an asymptotically zero curvature space which apparently does not admit asymptotically rectangular coordinates."

The second viewpoint is based upon the observation that the linearized Einstein equations for the NUT metric are analogous to the case in electromagnetism of a semiinfinite magnetic solenoid or a magnetic monopole. That is, the NUT metric is a particle-like solution whose spherically symmetric source has both ordinary mass m and "magnetic-like" mass m^*. We support the latter viewpoint by showing similarity between the NUT geodesics and the orbits of an electric charged test particle orbiting in an electromagnetic field generated by a spherically symmetric source that has both electric and magnetic charge.

The NUT metric, expressed in a Schwarzschild-like coordinate system, is [1]

$$ds^2 = U[dt + 4l\sin^2(\theta/2)\, d\phi]^2 - U^{-1}\, dr^2 - (r^2 + l^2)[d\theta^2 + \sin^2(\theta)\, d\phi^2] \tag{1}$$

where

$$U = l - 2\frac{mr^2 + l}{r^2 + l^2}$$

The constant m is the "ordinary mass" of the source and l is the NUT parameter, soon to be identified with the gravitational "magnetic" mass m^*. In the limit that $l = 0$ the NUT metric reduces to the Schwarzschild metric.

For a nonzero NUT parameter the vanishing of the metric determinant in Eq. (1) identifies the singularities at $\theta = 0$ and $\theta = \pi$. It is this axial singularity which is responsible for the different physical interpretations of this metric. Misner [2] considered the singularities at $\theta = 0$ and $\theta = \pi$ to be the degeneracies associated with spherical coordinates on a 3-sphere. In order to impose this interpretation Misner had to make the time coordinate periodic; consequently this makes the NUT metric an uninteresting particle-like solution.

Seeking to avoid a periodic time coordinate, Bonnor [3] imposed only part of Misner's identification on the NUT metric. With Bonnor's identification the singularity at $\theta = 0$ was removed but not the one at $\theta = \pi$. Bonnor related the singularity at $\theta = \pi$ singularity to a semiinfinite massless source of angular momentum directed along the symmetry axis. Dowker [4] pointed out that this was analogous to representing the magnetic monopole in electromagnetic theory by a semiinfinite solenoid. The singularity along the z axis is analogous to the Dirac string. Earlier, Demianski and Newman [5] had suggested that the NUT parameter corresponded to a "magnetic" gravitational monopole. Numerous authors [6–9] have subsequently discussed this gravitational magnetic monopole interpretation for the NUT parameter.

The NUT metric has properties that are similar to both the Kerr and the Schwarzschild metrics. Like the Kerr and Schwarzschild, the NUT space is Petrov type D and has a Killing horizon that surrounds the origin at a distance of $r_0 = r + (r^2 + l^2)^{1/2}$. Like the Schwarzschild metric, the single nonvanishing Riemann curvature scalar is a function of only r, i.e., the curvature scalar is spherically symmetric. Also like the Schwarzschild space, the NUT space contains a four-parameter group of motion whose three space-like generators have the same commutator algebra as do the generators for angular momentum.

The NUT metric in Eq. (1) is Kerr-like in the sense that it has a crossed space-time metric component $g_{o\phi}$ which generates gravimagnetic effects. The cross term in the Kerr metric not only breaks spherical symmetry but also generates an ergosphere and produces frame dragging. The cross term in Eq. (1) does not produce an ergosphere but it does produce an effect similar to the dragging of inertial frames. In addition, even though the cross term in Eq. (1) singles out the z axis and appears to break spherical symmetry, the space components of the geodesics as a function of proper time are spherically symmetric. However, the geodesic coordinate time component is not spherically symmetric. Because the time component is dependent on the orientation of the "Dirac string," we say that the geodesics are only "almost" spherically symmetric.

The NUT metric appears to be axially symmetric yet the scalar curvature invariant is spherical symmetric. How do we reconcile this apparent difference? The situation is precisely like that encountered in the theory of electromagnetic monopoles where the magnetic field $\vec{H}$ is spherically symmetric but the vector potential $\vec{A}$ is not. The vector potential has a Dirac string which breaks the spherical symmetry. In classical electrodynamics the vector potential is not physically measurable and the observables depend only on the magnetic field $\vec{H}$.

In quantum mechanics it is necessary to use the vector potential to formulate the equations and the vector potential introduces a singularity. The singularity runs from infinity along some curve, usually taken to be a line along the $\theta = \pi$ direction, ending on the monopole. In order to make the physical observables in quantum theory independent of the singularity, i.e., the Dirac string, one is forced to quantize the electric and magnetic charges. A similar result follows from the linearized version of Einstein's equations when they are allowed to have a gravitational monopole mass source. However, based on arguments that follow from the NUT geodesics that are calculated in this article and contain all nonlinear effects, we argue that the results based on linear theory might not be correct. When the nonlinear terms are accounted for, the mass that appears in the Dirac quantization condition becomes the energy of the system. Using the energy of the system instead of the mass in the Dirac quantization condition leads to a rather ambiguous result. This argument is discussed in more detail in Section 9.

The similarities between the NUT parameter and the charge of a magnetic monopole are immediately evident when components for the vector potential $\vec{A}$ in electromagnetism are compared with the space-time metric components g_{oi} which generate gravimagnetic effects. The only nonvanishing vector component for the magnetic monopole in spherical coordinates is $A_\phi = 2q \sin^2(\theta/2)$, where q is the magnetic monopole charge. From Eq. (1) it follows that the only nonvanishing space-time component for the NUT metric is $g_{o\phi} = 4l \sin^2(\theta/2)$. The formal analogy between $g_{o\phi}$ and A_ϕ suggests that the NUT parameter can be identified with a magnetic mass. In the following we define $l \equiv m^*$ in order for the notation to be more symbolic of a magnetic mass.

The magnetic mass m^* (the NUT parameter) is also the source term that generates the components of the dual Riemann curvature tensor, just as the magnetic monopole charge generates the components for the dual of the electromagnetic tensor. The close analogy that is established in this article between the NUT geodesics and the magnetic monopole trajectories strengthens the identification of the NUT parameter with a gravitational magnetic mass. However, we also show that there are subtle differences between the NUT geodesics and the electromagnetic trajectories. For example, the NUT geodesics are only almost independent of the direction of the Dirac string, unlike the electromagnetic analogy.

In Section 2, we review the relevant properties of the NUT metric and derive the geodesic equations. The relation between the angular variables and constants that follow from the Killing symmetries are discussed in Section 3. We also show in Section 3 that the geodesics lie on the surface of a cone whose axis is defined by the Killing constants. The almost spherically symmetric property of the geodesics is established in Section 4. A potential analysis of the radial coordinate is given in Section 5 and explicit solutions for circular orbits are given in Section 6. With the exact circular solutions we also illustrate a twin paradox in which the proper periods for counterrevolving orbits are the same but their coordinate periods are different.

Solutions for bound orbits and null geodesics are discussed in Sections 7 and 8, respectively. In Section 9 we speculate about the analogous Dirac quantization condition that follows from the NUT geodesics.

2. THE NUT METRIC AND GEODESIC EQUATIONS

If m^* corresponds to a gravitational "magnetic" monopole mass, then there should be a close analogy between the NUT geodesics and the orbits for an electric charged particle moving in a potential generated by a spherically symmetric source with both electrical and magnetic charge. The properties for these charged particle orbits are well known and can be summarized as:

(i) The field created by an electric charge and a magnetic monopole charge has an angular momentum $\vec{S}$ directed along the line joining the two charges. The magnitude of this angular momentum is independent of their separation and has a value proportion to eq, where e is the electric charge and q is the magnetic charge.

(ii) For an alectric charged particle orbiting in the potential of a magnetic monopole, the conserved total angular momentum is $\vec{J}=\vec{L}+\vec{S}$, where $\vec{L}$ is the orbital angular momentum and $\vec{S}$ is the field angular momentum. The magnitudes of the orbital and field angular momenta, $L\equiv|\vec{L}|$ and $S\equiv|\vec{S}|$, are separately conserved, however, their directions are not.

(iii) The trajectories for an electric charge in the potential of a monopole lie on a surface of a cone where the angle α is given by $\cos(\alpha)=|\vec{S}|/|\vec{J}|$.

(iv) The vector potential of a monopole is singular along the "Dirac string"; however, the direction of the singularity does not affect the trajectories so the orbits exhibit spherical symmetry.

To show that most of these magnetic monopole properties are shared by the NUT geodesics, let us consider the geodesic equations

$$\frac{d}{ds}(g_{\mu\nu}\dot{x}^{\nu})-\frac{1}{2}\frac{d}{dx^{\mu}}(g_{\tau\omega})\,\dot{x}^{\tau}\dot{x}^{\omega}=0 \tag{2}$$

The dots above the variables denote derivatives with respect to proper time, i.e. $\dot{A}=dA/ds$. The Greek letters run from 0 to 3 and correspond to the (t, r, θ, ϕ) coordinates, respectively.

Writing out the components of Eq. (2) for the NUT metric in (1), we get the following equations:

$$\nu=0, \qquad \frac{d}{ds}\{U[\dot{t}+2m^*(1-\cos\theta)\,\dot{\phi}]\}=0 \tag{3a}$$

$$\nu=1, \qquad \ddot{r}-\frac{1}{2}\frac{dU}{dr}U^{-1}\dot{r}^2-U\left[r\sin^2\theta-2m^{*2}(1-\cos\theta)^2\frac{dU}{dr}\right]\dot{\phi}^2$$
$$-Ur\dot{\theta}^2+2m^*U(1-\cos\theta)\frac{dU}{dr}\dot{t}\dot{\phi}+\frac{1}{2}U\frac{dU}{dr}\dot{t}^2=0 \tag{3b}$$

$$\nu=2, \qquad \frac{d}{ds}[(r^2+m^{*2})\,\dot{\theta}]+[-(r^2+m^{*2})\sin\theta\cos\theta$$
$$+4m^{*2}U\sin\theta(1-\cos\theta)]\,\dot{\phi}^2+2m^*U\sin\theta\dot{\phi}\dot{t}=0 \tag{3c}$$

$$\nu=3, \qquad \frac{d}{ds}\{[4m^{*2}U(1-\cos\theta)^2-(r^2+m^{*2})\sin^2\theta]\,\dot{\phi}$$
$$+2m^*U(1-\cos\theta)\,\dot{t}\}=0 \tag{3d}$$

An additional equation follows from $u^\nu u_\nu=\varepsilon$ ($\varepsilon=1, 0$):

$$\varepsilon=U\dot{t}^2-U^{-1}\dot{r}^2-(r^2+m^{*2})\,\dot{\theta}^2+[-(r^2+m^{*2})\sin^2\theta$$
$$+4m^{*2}U(1-\cos\theta)^2]\,\dot{\phi}^2+4m^*U(1-\cos\theta)\,\dot{t}\dot{\phi} \tag{3e}$$

where u^ν is the tangent to the geodesic path.

Two constants of motion follow immediately from integrating Eqs. (3a) and (3d); they are

$$E=U\{\dot{t}+2m^*(1-\cos\theta)\,\dot{\phi}\} \tag{4a}$$

$$-J_z+2m^*E=[4m^{*2}U(1-\cos\theta)^2-(r^2+m^{*2})\sin^2\theta]\,\dot{\phi}$$
$$+2m^*U(1-\cos\theta)\,\dot{t} \tag{4b}$$

The constant E is defined as the energy of the orbit and the constant J_z is chosen to agree with the z component of the conserved total angular momentum. The conserved angular moment J is discussed in Section 3.

We can eliminate $\dot{t}$ from Eqs. (3) by substituting Eq. (4a) into Eqs. (3b)–(3e); it follows that

$$\ddot{r}-\frac{1}{2}\dot{r}^2\frac{dU}{dr}U^{-1}+\frac{1}{2}\frac{dU}{dr}E^2U^{-1}-rU(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2)=0 \tag{5a}$$

$$\frac{d}{ds}\left[(r^2+m^{*2})\,\dot{\theta}\right]+2Em^*\sin\theta\dot{\phi}-(r^2+m^{*2})\sin\theta\cos\theta\dot{\phi}^2=0 \quad (5b)$$

$$\dot{\phi}=\frac{J_z-2mE\cos\theta}{(r^2+m^{*2})\sin^2\theta} \quad (5c)$$

$$\varepsilon=E^2U^{-1}-\dot{r}^2U^{-1}-(r^2+m^{*2})(\dot{\theta}^2+\sin^2\theta\dot{\phi}^2) \quad (5d)$$

Using (5c) to eliminate the $\dot{\phi}$ variable in (4a), we get the following equation for $\dot{t}$:

$$\dot{t}=\frac{E}{U}-\frac{2m^*(J_z-2m^*E\cos\theta)}{(r^2+m^{*2})(1+\cos\theta)} \quad (6)$$

It is now an easy task to eliminate the angular variables in Eqs. (5) and get an equation which is a function of only r and its derivative. The general properties for the radial variable then follow from a potential analysis. Before we continue with the detailed discussion of the orbital coordinates we examine the angular momentum properties and constants of motion that follow from the Killing symmetries.

3. KILLING VECTORS AND ANGULAR VARIABLES

Like the Schwarzschild metric, the NUT metric has sufficient Killing symmetries to establisch an exact relation between the ϕ and the θ coordinates. The four Killing vectors [2–3] that characterize the NUT symmetries are

$$\kappa_t^{\nu}=(1, 0, 0, 0) \quad (7a)$$

$$\kappa_z^{\nu}=(-2m^*, 0, 0, 1) \quad (7b)$$

$$\kappa_x^{\nu}=[-2m^*\cos\phi\tan(\theta/2), 0, -\sin\phi, -\cos\phi\cot\theta] \quad (7c)$$

$$\kappa_y^{\nu}=[-2m^*\sin\phi\tan(\theta/2), 0, \cos\phi, -\sin\phi\cot\theta] \quad (7d)$$

The subscripts, t, x, y, and z, denote the particular vector and the subscript, ν, denotes the tensor indice. The time-like Killing vector in (7a) describes the symmetry associated with the usual time independent solutions. The three space-like Killing vectors in (7b)–(7d) characterize the orbit's angular dependence and they generate a conserved total angular momentum vector.

The three space-like vectors form a subgroup with the same structure constants that are obeyed by spherically symmetric solutions. However, the

NUT killing vectors have nonzero time components and act on a three-dimensional hypersurface, which is contrary to the spherically symmetric case, which acts on only a two-dimensional hypersurface.

Contracting (7b)–(7d) with the four-velocity vector and using Eq. (4a) to eliminate the $\dot{t}$ terms, we get the following three components for the conserved total angular momentum (per unit mass):

$$J_x = -\kappa_x^\nu u_\nu = -(r^2 + m^{*2}) \sin \phi \dot{\theta} - (r^2 + m^{*2}) \times \cos \theta \sin \theta \cos \phi \dot{\phi} + 2m^* E \sin \theta \cos \phi \tag{8a}$$

$$J_y = -\kappa_y^\nu u_\nu = (r^2 + m^{*2}) \cos \phi \dot{\theta} - (r^2 + m^{*2}) \times \cos \theta \sin \theta \sin \phi \dot{\phi} + 2m^* E \sin \theta \sin \phi \tag{8b}$$

$$J_z = -\kappa_z^\nu u_\nu = (r^2 + m^{*2}) \sin^2 \theta \dot{\phi} + 2m^* E \cos \theta \tag{8c}$$

It is obvious from Eqs. (8) that the conserved total angular momentum $\vec{J}$ can be expressed as the sum of two vectors, the field angular momentum $\vec{S}$ and the orbital angular momentum $\vec{L}$,

$$\vec{J} = \vec{L} + \vec{S} \tag{9}$$

The orbital angular momentum (per unit mass) is defined by the usual expression but with r^2 replaced by $r^2 + m^{*2}$,

$$L_x \equiv -(r^2 + m^{*2}) \sin \phi \dot{\theta} - (r^2 + m^{*2}) \times \cos \theta \sin \theta \cos \phi \dot{\phi} \tag{10a}$$

$$L_y \equiv (r^2 + m^{*2}) \cos \phi \dot{\theta} - (r^2 + m^{*2}) \times \cos \theta \sin \theta \sin \phi \dot{\phi} \tag{10b}$$

$$L_z \equiv (r^2 + m^{*2}) \sin^2 \theta \dot{\phi} \tag{10c}$$

The components of the field angular momentum generated by the magnetic mass are defined by

$$s_x = 2m^* E \sin \theta \cos \phi \tag{11a}$$

$$s_y = 2m^* E \sin \theta \sin \phi \tag{11b}$$

$$s_z = 2m^* E \cos \theta \tag{11c}$$

The field angular momentum $\vec{S} = 2m^* E \hat{e}_r$ is independent of the coordinate r and is directed along a line joining the test particle and the source. This field angular momentum has a constant magnitude and is perpendicular to $\vec{L}$, precisely the same as the angular momentum generated by the field between an electric charge and a magnetic monopole. It is the time component in Eqs. (7b)–(7d), that is responsible for generating the field angular momentum.

Squaring the total angular momentum, we get

$$J^2 = J_x^2 + J_y^2 + J_z^2 = 4m^{*2}E^2 + L^2 \tag{12}$$

Since J^2 and S^2 are constants it follows that the magnitude of the orbital angular momentum L is also constant, but not its vector components.

Let us now separate $\dot{\theta}$ and $\dot{\phi}$ in Eqs. (8) and obtain relations for the ϕ and θ coordinates. Multiplying Eqs. (8a) by $-\sin\phi$ and Eq. (8b) by $\cos\phi$, and then adding, we get

$$\dot{\theta} = \frac{-A\sin(\phi - \eta)}{(r^2 + m^{*2})} \tag{13}$$

where we have defined the constants A and η by

$$J_x \equiv A\cos(\eta) \qquad \text{and} \qquad J_y \equiv A\sin(\eta) \tag{14}$$

In addition, multiplying Eq. (8a) by $\cos\phi$ and Eq. (8b) by $\sin\phi$ and adding, we get

$$A\cos(\phi - \eta)\sin\theta = 2m^*E - J_z\cos\theta \tag{15}$$

The relation between the angular variables given by (15) is the relation that restricts the motion of the test particle to the surface of a cone whose axis is along $\vec{J}$.

An expression for the cone angle can be obtained by dotting the normalized position vector $\hat{r}$ with the conserved angular momentum vector $\vec{J}$. It then follows from (9) that the cone axis along $\vec{J}$ makes an interior cone angle α given by

$$\cos\alpha = \frac{2m^*E}{|\vec{J}|} \tag{16}$$

Let us now use the relations that we have established for the angular variables to obtain an equation for the radial coordinate. The behavior of the angular variables allows us to derive a potential equation for the radial coordinate which is independent of the angular variables.

Squaring and adding Eqs. (10), it follows that L^2 is related to the angular coordinates by

$$(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) = \frac{L^2}{(r^2 + m^{*2})^2} \tag{17}$$

In order for θ and ϕ to be real it follows that the allowable range of J^2 and L^2 are $L^2 \geqslant 0$ and $J^2 \geqslant 4m^{*2}E^2$.

Using (17) to eliminate the angular variables in (5d), we get

$$\dot{r}^2 = (E^2 - 1) - V(r) \tag{18a}$$

where

$$V(r) = U - 1 + \frac{L^2 U}{(r^2 + m^{*2})} \tag{18b}$$

Equation (18b) defines an effective potential for the radial coordinate. Only the magnitude of $\vec{L}$ enters in (18a), therefore, the radial equation is obviously spherically symmetric.

Let us now consider the behavior of the angular coordinates, ϕ and θ, for the special orientation of the orbit in which $J_x = J_y = 0$ $(A = 0)$. For $A = 0$, the conserved momentum is along the z axis. Setting $A = 0$ it follows from (15) that

$$\cos\theta = \frac{2m^*E}{|\vec{J}|} \tag{19}$$

where $|\vec{J}| = J_z$. In the Schwarzschild limit, $m^* = 0$, Eq. (19) reduces to $\theta = \pi/2$. That is, the origin of the coordinates lies in the plane of the orbit and the plane of the orbit is orthogonal to the z axis. With a nonzero m^*, the orbit is lifted in the direction of the z axis by the gravimagnetic effects generated by the magnetic monopole mass.

From (19) it follows that $\sin^2\theta = (J^2 - 4m^{2*}E^2)/J^2$ so Eq. (5c) reduces to

$$\dot{\phi} = \frac{|\vec{J}|}{(r^2 + m^{*2})} \tag{20}$$

Equations (19) and (20) describe the behavior for the angular variables in the special case that $A = 0$.

The angular properties that follow from the group symmetries are identical to the analogous angular properties for an electric charge spiraling along the magnetic field lines generated by a magnetic monopole. Let us now show that the spatial coordinates, r, θ, and ϕ, for the orbits are spherically symmetric, i.e., the spatial coordinates are independent of the orientation of the orbit.

4. ROTATION OF THE COORDINATES TO CANONICAL FORM

For the case of the spherical symmetry Schwarzschild metric one can always pick a coordinate system, without the loss of generality, so that $J_x = J_y = 0$. This result can be verified by rotating the coordinates to a system with the z axis pointing along the orbital angular momentum vector. The resultant equations will be identical to the equations one gets by setting J_x and J_y equal to zero. A similar result is also valid in the classical electromagnetic for the motion of a charged particle in the field of a magnetic monopole, even though the Dirac string breaks spherical symmetry in the vector potential.

In this section, we show that a similar result follows for the NUT geodesics. That is, if one rotates to a coordinate system with the z axis pointing along $\vec{J}$, the angular equations reduce to the canonical form given by (19) and (20). It is already obvious from Eq. (18) that the radial coordinate is independent of the orientation of the coordinate system. However, a similar result does not follow for the time coordinate. The time coordinate is a function of the direction of the singularity (Dirac string) along the z axis.

Let us now construct a transformation that rotates the coordinates to a system where the z' axis lies along the direction of the conserved angular momentum vector. In the rotated coordinates it will follow that Eqs. (8) reduce to the form generated by the special case where we have set $J_x = J_y = 0$.

The rotation matrix that generates a coordinate system (x', y', z') such that the z' axis lies along $\vec{J}$ is obtained by rotation first about the z axis by an angle η and then about the y axis by an angle ω. Taking the product of these two rotations we get the following rotation matrix that connects the coordinates (x', y', z') and (x, y, z):

$$R(\eta, \omega) = \begin{Bmatrix} \cos\eta\cos\omega & \sin\eta\cos\omega & -\sin\omega \\ -\sin\eta & \cos\eta & 0 \\ \cos\eta\sin\omega & \sin\eta\sin\omega & \cos\omega \end{Bmatrix} \tag{21a}$$

$$\eta = \tan^{-1}(J_y/J_x), \qquad \omega = \operatorname{ctn}^{-1}(J_z/A) \tag{21b}$$

Expressing the Cartesian coordinates in terms of angular variables, it follows from (21a) that we get the following relations between the rotated angular variables (θ', ϕ') and the unrotated angular variables (θ, ϕ):

$$\sin\theta' \cos\phi' = \cos\omega[\sin\theta\cos(\phi - \eta) - \tan\omega\cos\theta] \tag{22a}$$

$$\sin\theta'\sin\phi' = \sin\theta\sin(\phi-\eta) \tag{22b}$$

$$\cos\theta' = \sin\omega[\sin\theta\cos(\phi-\eta) + \operatorname{ctn}\omega\cos\theta] \tag{22c}$$

and

$$\cos\theta = -\sin\omega\sin\theta'\cos\phi' + \cos\omega\cos\theta' \tag{22d}$$

Let us now use the relations in (22) to express Eqs. (8) in terms of the rotated coordinates (θ', ϕ'). It follows that

$$\cos(\theta') = \frac{2m^*E}{|\vec{J}|} \tag{23a}$$

An expression for the angular variable ϕ' follows from taking a derivative of (22d) and using (15), (21b), and (22), we get

$$\dot{\phi}' = \frac{|\vec{J}|}{(r^2 + m^{*2})} \tag{23b}$$

Equations (23) are identical to Eqs. (19) and (20) where $A = 0$ and they are the expected results that follow for a spherically symmetric field.

The equations that govern the radial and angular variables as a function of proper time are spherically symmetric—the direction of the singularity along the negative z axis does not affect their behavior. An equivalent statement cannot be made about the time coordinate. To show this we express the time coordinate in terms of the rotated coordinates and compare the result with the equations that follow from setting $A = 0$ in (6).

Substituting (22d) and (23a) into (6), we get

$$\dot{t} = \frac{E}{U} + \frac{-2m^*\{J_zJ^2 - 2m^*E[-AL\cos\phi' + 2m^*EJ_z]\}}{[JJ_z - AL\cos\phi' + 2m^*EJ_z](r^2 + m^{*2})} \tag{24a}$$

This is not the same equation that follows from setting $A = 0$ in (6) which is simply

$$\dot{t} = \frac{E}{U} - \frac{2m^*[J - 2m^*E]}{(r^2 + m^{*2})} \tag{24b}$$

In general, the time coordinate does depend on the orientation of the orbit relative to the direction of the line singularity. Only for the special case of radial infall, i.e., $\vec{L} = 0$, does Eq. (24a) reduce to (24b) and, therefore, reflect spherical symmetry. In general only the spatial components of the geodesics are independent of the orientation of the orbit and reflect spherical symmetry.

5. POTENTIAL ANALYSIS FOR THE RADIAL VARIABLE

In this section we study the properties of the NUT geodesic that follow from the radial equation. Properties for the radial coordinate follow immediately from a potential analysis of Eqs. (18). Expressing the effective potential in (18b) as the sum of four terms, we have

$$V(r) = -\frac{2mr^3}{(r^2+m^{*2})^2} + \frac{(L^2-2m^{*2})\,r^2}{(r^2+m^{*2})^2} - \frac{2mr(L^2+m^{*2})}{(r^2+m^{*2})^2} - \frac{m^{*2}(L^2+2m^{*2})}{(r^2+m^{*2})^2} \tag{25}$$

The characteristics of this potential are similar to the Schwarzschild potential. The first three terms on the right-hand side of Eq. (25) are just the usual Schwarzschild terms with only a slight modification from m^*. The first term is just the classical Newtonian potential, the second term corresponds to the centrifugal force barrier, and the third term is analogous to the usual general relativistic correction that occurs in the Schwarzschild potential. The main effect of m^* on these three terms is to modify the angular momentum barrier. The potential barrier appears in the second term of (25) and is proportional to (L^2-2m^{*2}). The fourth terms is unique to the NUT metric and makes only a short-range contribution. In addition, unlike the Schwarzschild potential, the NUT potential is nonsingular at $r=0$, which follows from the fact that r^2 appears only in the form r^2+m^{*2}.

The qualitative features of the potential for $L \neq 0$ in (25) are shown in Fig. 1. The shape of the potential is determined by the number of extremum points. Setting the derivative of the potential in (25) equal to zero, it follows that the extremum points are located at the roots of the quartic equation,

$$mr^4 + (2m^{*2}-L^2)\,r^3 + 3L^2mr^2 + (2m^{*4}+3L^2m^{*2})\,r - m^{*2}m(m^{*2}+L^2) = 0 \tag{26}$$

Equation (26) has four roots, one of which is always negative. In order of decreasing distance from the origin, the four roots are denoted by r_1, r_2, r_3, and r_4. An approximate solution of (26) is

$$r_1 \approx \frac{L^2}{2m}\left[1-\frac{2m^{*2}}{L^2}\right]\left\{\left[1+\frac{8m^{*2}}{9L^2}\right]+\left[1-\frac{12m^2}{L^2}-\frac{16m^{*2}}{9L^2}\right]^{1/2}\right\} \tag{27a}$$

$$r_2 \approx \frac{L^2}{2m}\left[1-\frac{2m^{*2}}{L^2}\right]\left\{\left[1+\frac{8m^{*2}}{9L^2}\right]-\left[1-\frac{12m^2}{L^2}-\frac{16m^{*2}}{9L^2}\right]^{1/2}\right\} \tag{27b}$$

$$r_3 \approx (1/3)^{1/2}\, m^* - (4/9m)\, m^{*2} \tag{27c}$$

$$r_4 \approx -(1/3)^{1/2}\, m^* - (4/9m)\, m^{*2} \tag{27d}$$

Equation (26) or Eqs. (27) reduce to the standard Schwarzschild results when $m^* = 0$. Setting $m^* = 0$ in Eqs. (27), we have

$$r_1 = \frac{L^2}{2m}\left\{1+\left[1-\frac{12m^2}{L^2}\right]^{1/2}\right\} \tag{28a}$$

$$r_2 = \frac{L^2}{2m}\left\{1-\left[1-\frac{12m^2}{L^2}\right]^{1/2}\right\} \tag{28b}$$

and $r_3 = r_4 = 0$.

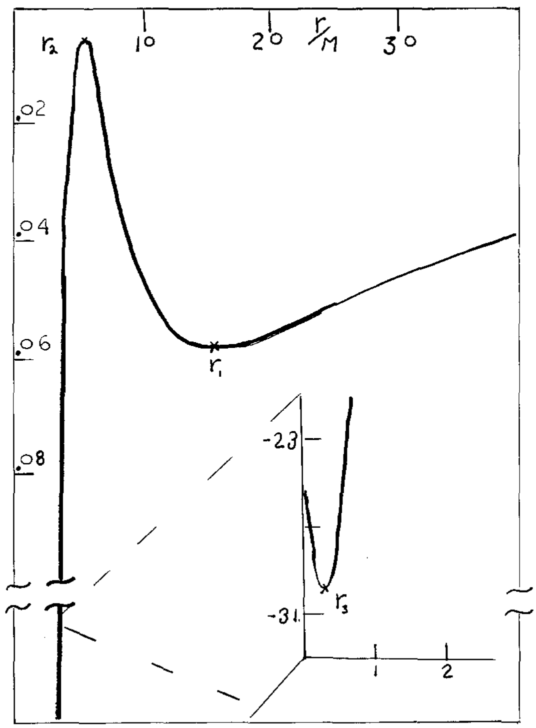

Fig. 1. The qualitative features of the effective potential V in terms of r/m, where $m = 1$ and $L^2 = 22$ for nonzero m^*.

It follows from (26) that r_4 is always negative (unphysical) so there are only three possible circular orbits, which are located at r_1, r_2, and r_3. The orbits located at r_1 and r_3 are stable, while the orbit at r_2 is unstable. The two circular orbits at r_1 and r_2 lie outside the horizon and are analogous to the stable and unstable orbits in the Schwarzschild potential—the circular orbit at r_3 is unique to the NUT potential and is due to the short-ranged fourth term in (25). The circular orbit at r_3 is always located inside the horizon and lies in the allowable positive range of r only for appropriate values of L and m^*.

The turning points of the orbit are obtained by setting $\dot{r}=0$ in Eq. (18) and solving for r. They are located at the roots of the equation

$$E^2(r^2+m^{*2})-U(r^2+m^{*2}+L^2)=0 \tag{29}$$

Let us now solve the orbit for the explicit case of radial motion. From Eq. (17) it follows that the radial orbits have $L=0$. Setting $L=0$ in (18), we have

$$\dot{r}^2=(E^2-U) \tag{30}$$

Taking the square root and integrating, we get

$$S=\int \frac{[r^2+m^{*2}]^{1/2}\,dr}{[(E^2-1)\,r^2+2mr+(E^2+1)\,m^{*2}]^{1/2}} \tag{31}$$

Equation (31) is an expression for the radial coordinate as a function of the proper time. In a similar manner, we get the radial position as a function of the coordinate time. Setting $\dot{\phi}=0$ in (4a), and using the resulting expression between the coordinate and the proper times to eliminate the proper time in (30), it follows that

$$T=\int \frac{E[r^2+m^{*2}]^{3/2}\,dr}{(r^2-2mr-m^{*2})[(E^2-1)\,r^2+2mr+(E^2+1)\,m^{*2}]^{1/2}} \tag{32}$$

The results in Eqs. (31) and (32) are similar to the Schwarzschild result. From (31) it follows that only a finite amount of proper time is needed to fall to the origin, $r=0$. On the other hand, it follows from (32) that the coordinate time goes to infinity as the test particle approaches the horizon. These results are illustrated in Fig. 2.

Notice that the solution is independent of the direction of infall, i.e., the radial orbits are independent of the singularity along the $\theta=\pi$ direction. This is consistent with what one would expect from a magnetic monopole field—the motion along the magnetic field lines does not affect

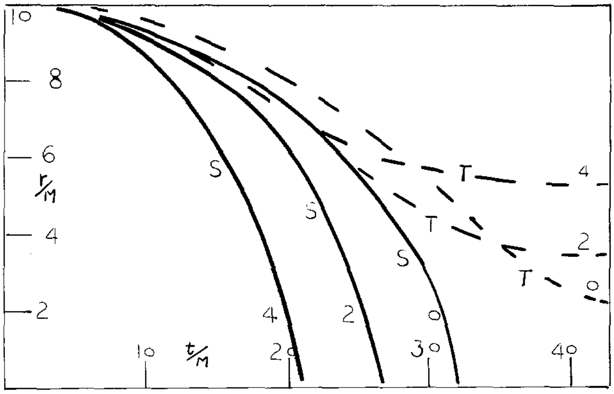

Fig. 2. The solid (proper times) and the dashed (coordinate time T) curves are plotted by Eqs. (29) and (30) for $M^* = 0, 2, 4$.

the motion of the particle. The NUT radial trajectories are not twisted by the gravimagnetic term, whereas the similar radial trajectories in the Kerr metric are given an angular component by the dragging of the inertial frame produced by the gravimagnetic term.

6. CIRCULAR ORBITS AND THE TWIN PARADOX

In this section we construct the exact solution for circular orbits. Circular orbits are complex enough to show the basic effects of the magnetic mass yet simple enough to be transparent. The exact solution for circular orbits illustrates the gravimagnetic effects that are produced by the $g_{\phi o}$ space-time term. In particular, there is an effect which is analogous to the dragging of the inertial frame in the Kerr metric. The basic difference between the Kerr and the NUT term is that the Kerr cross term, $g_{\phi o}$, produces dipole-like field, whereas the NUT term is a monopole-like field.

The effects of frame dragging are most apparent in the Kerr solution when one compares the periods of counterrevolving orbits. The dragging of the inertial frame produces age differences for twins in similar counter-revolving orbits. We illustrate the dragging of inertial frames in the NUT metric by considering a similar twin paradox.

To illustrate frame dragging in the NUT space, we compare the periods of counterrevolving orbits. We show that twins in counterrevolving orbits with the same radius do not have the same coordinate period even though they have the same proper period. The difference in coordinate periods is attributed to frame dragging.

Properties for circular orbits are determined from the location of the extremum points of the potential and the expression for the turning points of the orbit. The position for the extremum is given by (26). Solving this equation for L^2 and denoting the radial coordinate by $r=a$, we have

$$L^2=\frac{(a^2+m^{*2})^2\,(dU/dr)}{[2Ua-(a^2+m^{*2})(dU/dr)]} \tag{33}$$

The turning points are given by (29). Using (33) to eliminate L^2 in (29) and solving for E, we have

$$E^2=\frac{2U^2a}{[2Ua-(a^2+m^{*2})(dU/dr)} \tag{34}$$

The expression for the conserved angular momentum, $J^2=4m^{*2}E^2+L^2$, follows from Eq. (33) and (34); we have

$$J^2=\frac{[(a^2+m^{*2})^2\,(dU/dr)+8aU^2m^{*2}]}{[2Ua-(a^2+m^{*2})(dU/dr)]} \tag{35}$$

Solutions for the circular orbits are well defined only if the denominators in equations (33)–(35) are positive. The vanishing of the denominators determines the position of the innermost circular orbit. As m^* goes to zero the value of the innermost circular orbit approaches the Schwarschild value of $3m$. From Eq. (16) it follows that circular orbits lie on the surface of a cone with cone angle α given by

$$\cos\alpha^2=\frac{4m^{*2}E^2}{J^2}\equiv\frac{8m^{*2}aU^2}{[(a^2+m^{*2})^2\,(dU/dr)+8aU^2m^{*2}]} \tag{36}$$

As m^* goes to zero, α goes to zero and the origin of the coordinates lies in the plane of the orbit as it should.

The a appearing in the orbital equations is *not* the radius of the circular orbit; rather, it is the distance from the origin of the coordinates along the cone to the orbit (Fig. 3). The radius of the circle R is related to the radial distance from the origin to the orbit by

$$R=\frac{La}{J} \tag{37}$$

Substituting Eqs. (33) and (35) into (37), we have

$$R^2=\frac{a^2(dU/dr)[a^2+m^{*2}]^2}{[(a^2+m^{*2})^2\,(dU/dr)+8aU^2m^{*2}]} \tag{38}$$

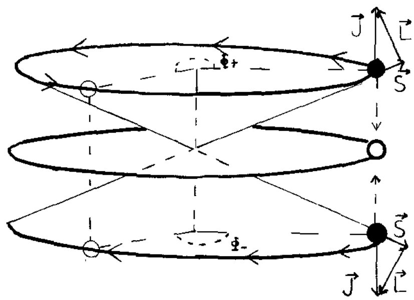

Fig. 3. The proper time for the co- and counterrevolving twins where they are displaced in the opposite direction along the total angular momentum by the distance $2D$.

Solving Eq. (38) for a as a function of R and keeping only the lowest-order terms in m and m^*, we have

$$a \approx R\left[1+\frac{2m^{*2}}{mR}\left(1-\frac{4m}{R}\right)\right] \tag{39}$$

Counterclockwise and clockwise revolving orbits do not lie in the same plane. The orbits are displaced in the direction of the orbital angular momentum vector; therefore, orbits revolving in the clockwise and counterclockwise directions are displaced in opposite directions by a distance $2D$. D is given by

$$D=\frac{2m^*ER}{L} \tag{40}$$

The relation for the proper orbital angular frequency follows from substituting Eq. (35) into (20); we have

$$\dot{\phi}^2=\frac{[(a^2+m^{*2})^2\,(dU/dr)+8aU^2m^{*2}]}{[2Ua-(a^2+m^{*2})(dU/dr)][a^2+m^{*2}]^2} \tag{41}$$

The proper period S for the orbit follows from inverting (41), taking the square root, and integrating, i.e., multiplying by 2π. It follows that the square of the proper period is

$$S^2=\frac{4\pi^2[2Ua-(a^2+m^{*2})(dU/dr)][a^2+m^{*2}]^2}{[(a^2+m^{*2})^2\,(dU/dr)+8aU^2m^{*2}]} \tag{42}$$

The proper period goes to zero as the radius of the orbit approaches the position of the inner most circular orbit. Substituting Eq. (39) into

Eq. (42), it follows that the proper periods can be written in terms of R. To lowest order in m^*, we have

$$S^2(R) \approx 4\pi^2 \Omega_s^{-2}(1 - 3\Omega_s^2 R^2)\left(1 - \frac{15m^{*2}}{R^2}\right) \tag{43}$$

where $\Omega_s = (m/R^3)^{1/2}$ is the Schwarzschild angular velocity.

It follows from Eq. (42) or Eq. (43) that m^* shortens the proper period in the NUT solution relative to the classical Schwarschild proper period. In addition, the proper period is independent of the direction of revolution, which is not the case for the coordinate period.

An expression for the coordinate period t follows from (4a). For counterclockwise revolutions we have $\cos\theta = \cos\alpha$ and $\dot{\phi} = |\dot{\phi}|$, while for clockwise revolutions we have $\cos\theta = \cos(\pi - \alpha) = -\cos\alpha$ and $\dot{\phi} = -|\dot{\phi}|$. Substituting these results in (4a), we have

$$\dot{t}_+ = E/U - 2m^*(1 - \cos\alpha)\,|\dot{\phi}| \tag{44a}$$

and

$$\dot{t}_- = E/U + 2m^*(1 + \cos\alpha)\,|\dot{\phi}| \tag{44b}$$

where + is for the counterclockwise revolution and − for the clockwise revolution. Integrating (44) gives the coordinate period for the two counterrevolving orbits. The difference between the coordinate periods is attributed to the gravimagnetic effect generated by the cross term g_{tr} in the metric.

An expression for the angular velocity in terms of the coordinate time can be written

$$\left|\frac{d\phi}{dt}\right|_\pm = |\dot{\phi}|\left(\frac{1}{\dot{t}_\pm}\right) \tag{45}$$

This difference in coordinate periods leads to a twin paradox for observers that are revolving in opposite directions.

Let us now illustrate the gravimagnetic effect of frame dragging with a calculation for the ages of two twins revolving with the same radius but in different directions. Consider two orbits located in the $x-y$ $(A=0)$ plane with the same radial coordinate $r = a$ but revolving in opposite directions. If the two orbits start from $\phi = 0$ at coordinate time $t = 0$, they next pass each other at coordinate time $t = T$ (Fig. 3). The co- and counterrevolving twins will have angular coordinates denoted by ϕ_+ and ϕ_-, respectively. When the twins meet, i.e., pass over and under one another, their coordinate positions will be given by

$$\phi_{\pm} = |\dot{\phi}| \left(\frac{1}{\dot{t}_{\pm}}\right) T \tag{46a}$$

which satisfy the relation

$$\phi_{+} + \phi_{-} = 2\pi \tag{46b}$$

From (45) and (46) it follows that the time of passage is given by

$$T = 2\pi \frac{(\dot{t}_{+})(\dot{t}_{-})}{|\dot{\phi}|(\dot{t}_{+} + \dot{t}_{-})} \tag{47}$$

Substituting (47) into (46) we get the following expressions for their angular coordinates at the time of passage:

$$\phi_{+} = 2\pi(\dot{t}_{-})(\dot{t}_{+} + \dot{t}_{-})^{-1} \tag{48a}$$

and

$$\phi_{-} = 2\pi(\dot{t}_{+})(\dot{t}_{+} + \dot{t}_{-})^{-1} \tag{48b}$$

The proper time that each twin measures from $T = 0$ to the next time of passage, T, is obtained by taking the square root of (41) and integrating from 0 to $\phi_{\pm}$, i.e.,

$$S_{\pm} = \int^{\phi_{\pm}} \frac{d\phi}{|\dot{\phi}|} \tag{49}$$

The fractional difference in proper time becomes

$$\frac{S_{+} - S_{-}}{1/2(S_{+} + S_{-})} = \frac{4m^{*}J}{(r^{2} + m^{*2})} \tag{50}$$

To lowest order in m^{*} it follows from (34) and (35) that $E = U = 1$ and $J \approx (mr)^{1/2}$ so Eq. (50) becomes

$$\frac{S_{+} - S_{-}}{1/2(S_{+} + S_{-})} \approx 4m^{*}\Omega s \tag{51}$$

7. BOUND QUASI-ELLIPTICAL ORBITS

In this section we show how the magnetic mass affects bound orbits. The general solutions for bound "quasi-elliptical" orbits have all the characteristics of circular orbits and, in addition, are nonplaner and exhibit a perihelion precession. In the following section we consider the properties of elliptical orbits for the case that $A = 0$.

For $A = 0$ the θ dependence is given by Eq. (19) and the radial dependence as a function of ϕ follows from Eqs. (18) and (20):

$$\frac{J^2}{(r^2+m^{*2})^2}\left[\frac{dr}{d\phi}\right]^2 = (E^2-1) - V(r) \tag{52}$$

where

$$V(r) = U - 1 + \frac{L^2 U}{(r^2+m^{*2})}$$

If we let $v = 1/r$, then Eq. (52) reduces to

$$J^2\left[\frac{dv}{d\phi}\right]^2 = a_1 v^4 + a_2 v^3 + a_3 v^2 + 2mv + (E^2-1) \tag{53a}$$

where

$$a_1 = m^{*2}(L^2 + m^{*2} + m^{*2}E^2),\ a_2 = 2m(L^2 + m^{*2}),\ a_3 = m^{*2}E^2 - L^2 \tag{53b}$$

Factoring the right-hand side of Eq. (53a), we get

$$a^2(1-e^2)^2 J^2\left[\frac{dx}{d\phi}\right]^2 = a_1(x-1+e)(x-1-e)(x-x_3)(x-x_4) \tag{54a}$$

where

$$v \equiv \frac{x}{(1-e^2)\,a} \tag{54b}$$

The two turning points of the orbit are at $a(1-e)$ and $a(1+e)$.

Using Eq. (54b) in (53a) and comparing the results with Eq. (54a), we get

$$a_1 x_3 x_4 = a^4(1-e^2)^3\,(E^2-1) \tag{55a}$$

$$a_1(x_3+x_4) = -\,[2a_1 + a(1-e^2)\,a_2] \tag{55b}$$

$$a_3 a^2(1-e^2) + 4ma^3(1-e^2) - a_1 = -(3+e^2)(1-e^2)(E^2-1)\,a^4 \tag{55c}$$

$$a_2 a(1-e^2) + 2a_1 - 2ma^3(1-e^2)^2 = 2a^4(1-e^2)^2\,(E^2-1) \tag{55d}$$

Using Eq. (53b) in Eqs. (55a) – (55d) and keeping only the lowest-order term in m^*, we have

$$L^2 \cong L_s^2\left[1+\frac{2m^{*2}}{ma(1-e^2)}+\frac{m^{*2}(3+5e^2)}{a^2(1-e^2)^2}\right] \tag{55e}$$

$$E^2 \cong E_s^2\left[1+\frac{4mm^{*2}}{(1-e)^2 a^3}\right] \tag{55f}$$

$$J^2 \cong L_s^2\left[1+\frac{6m^{*2}}{ma(1-e^2)}+\frac{(-13+5e^2)}{(1-e^2)}\frac{m^{*2}}{a^2}\right] \tag{55g}$$

where L_s^2 and E_s^2 are the Schwarzschild results,

$$L_s^2=\frac{ma(1-e^2)}{\{1-[m(3+e^2)/a(1-e^2)]\}} \tag{55h}$$

$$E_s^2=\frac{1-\{4m/[a(1-e^2)]\}+\{4m^2/[a(1-e^2)]\}}{1-[(3+e^2)/(1-e^2)](m/a)} \tag{55i}$$

Defining the new variable

$$x=(1+\mathrm{eco}\,s\psi) \tag{56}$$

Eq. (54a) can be simplified

$$-[a(1-e^2)J]^2\left[\frac{d\psi}{d\phi}\right]^2 = a_1[(1+e\cos\psi)^2-(x_3+x_4)(1+e\cos\psi)+x_3x_4] \tag{57}$$

Substituting Eqs. (55a) and (55b) and (53b) into Eq. (57), we have

$$\left(\frac{d\psi}{d\phi}\right)^2 \approx -\frac{1}{J^2}\left[\frac{3m^{*2}L_s^2}{a^2}+\frac{2m}{a}(L_s^2+m^{*2})+a^2(E^2-1)\right] \tag{58a}$$

We assume that $m^* < m < a$ and keep only linear terms in the eccentricity and terms to order $(m^*/a)^2$. Substituting Eqs. (55a)–(55i) in Eq. (58a), taking the inverse square root, and integrating the resulting formula, we have

$$\phi \approx \left(1+\frac{3m}{a}+\frac{3m^{*2}}{ma}+\frac{27m^2}{2a^2}+\frac{9m^{*2}}{a^2}\right)\psi \tag{58b}$$

As ψ increases from 0 to 2π, i.e., from one perihelion to the next, ϕ increases by $2\pi + \Delta\phi$, where

$$\Delta\phi \approx \frac{6\pi m}{a}\left[1 + \frac{m^{*2}}{m^2} + \frac{9m}{8a} + \frac{3m^{*2}}{ma}\right] \tag{59}$$

where we used Eqs. (55e) and (55f) in Eq. (58b).

8. NULL GEODESICS

In this section we consider the effect of the magnetic mass on null geodesics. We first construct the potential equation for null geodesics and then calculate the bending of light due to m^*. Again, without the loss of generality, we assume $A = 0$. The angular dependence for the null geodesics is the same as for the time-like geodesics. The null orbits lie on the surface of a cone with angle given by (16). The characteristics for the radial variable follow from an analysis of the impact potential that is deduced from Eq. (5d) with $\varepsilon = 0$. Eliminating the angular variable in Eq. (5d) by means of Eq. (17) and using (20) to express the radial equation in terms of ϕ, we get the following equation for the radial variable:

$$\frac{1}{(r^2 + m^{*2})^2}\left[\frac{dr}{d\phi}\right]^2 = \frac{(E^2 - 1)}{J^2} - \frac{[B^{-2}(r) - 1]}{J^2} \tag{60}$$

$B^{-2}(r) - 1$ is the impact potential and B^{-2} is given by

$$L^{-2}B^{-2}(r) = \left[\frac{r^2}{(r^2 + m^{*2})^2} - \frac{2mr}{(r^2 + m^{*2})^2} - \frac{m^{*2}}{(r^2 + m^{*2})^2}\right] \tag{61}$$

The constants E and L enter Eq. (61) only in the form of the ratio L/E; therefore, we define a new constant $L' = L/E$.

The impact potential has the same qualitative behavior as the Schwarzschild geodesics. The third term in (61) is a consequence of only the NUT parameter and it makes only a small contribution to the path of a photon.

We now seek a solution for (60) where only the leading order terms are retained. We assume that $r > m > m^*$ and keep only the lowest-order terms. The constant L' can be expressed in terms of the distance of closest approach r_0. Setting $dr/d\phi = 0$ in Eq. (60) and using Eq. (61), we get

$$L'^2 \approx L'^2_s(1 + 3m^{*2}/r_0^2) \tag{62a}$$

where the Schwarzschild value of L'_s is

$$L'^2_s = \frac{r_0^2}{(1 + 2m/r_0)} \tag{62b}$$

We have retained terms only to order $0(m^2, m^{*2})$ in Eqs. (62).

Let $v = 1/r$ and using Eqs. (61) and (62a) in (60) and taking the derivative, with respect to ϕ we have

$$\frac{d^2 v}{d\phi^2} \cong -\left(1 - \frac{6m^{*2}}{L'^2_s}\right) + 3mv^2 + 2m^{*2}v^3 \tag{63}$$

Expanding the solution about the Schwarzschild result, we let

$$v = v_s + v_n \tag{64}$$

where v_s is a solution of

$$\frac{d^2 v_s}{d\phi^2} = -v_s + 3mv_s^2 \tag{65}$$

Substituting (64) into (63) and using (65), we get the following equation for v_n:

$$\frac{d^2 v_n}{d\phi^2} \approx -v_n + 2m^{*2}v_s\left(\frac{3}{r_0^2} + v_s^2\right) \tag{66}$$

Solving Eqs. (65) and (66) for v_s and v_n and substituting the resulting solutions into Eq. (64), we have

$$v \approx A\cos\phi + \frac{3mA^2}{2}\left(1 - \tfrac{1}{3}\cos 2\phi\right) + \frac{3m^2A^3}{4}\left(5\phi\sin\phi + \tfrac{1}{4}\cos 3\phi\right)$$
$$+ \frac{3m^{*2}A}{4}\left[\left(A^2 + \frac{4}{r_0^2}\right)\phi\sin\phi - \left(\frac{A^2}{12}\cos 3\phi\right)\right] \tag{67a}$$

To find the constant A we set $\phi = 0$ at the distance of closest approach $(r = r_0)$ so

$$A \approx \frac{1}{r_0}\left[1 - \frac{m}{r_0}\left(1 - \frac{29m}{16r_0}\right) + \left(\frac{m^{*2}}{16r_0^2}\right)\right] \tag{67b}$$

Therefore, the bending of light is described by (67a), where A is given by (67b).

To find the deflection of the path we let $\phi = \frac{1}{2}\pi + \delta$ where v becomes infinity; we get for the total deflection ($\Delta = 2\delta$)

$$\Delta = \frac{4m}{r_0} - \frac{4m^2}{r_0^2} + \frac{30\pi m^2}{8r_0^2} + \frac{30\pi m^{*2}}{8r_0^2} \tag{68}$$

or

$$\Delta = \frac{4m}{r_0}\left(1 - \frac{m}{r_0} + \frac{15\pi m}{16r_0} + \frac{15\pi m^{*2}}{16mr_0}\right) \tag{69}$$

9. CONCLUSION

We conclude that the properties of the NUT geodesics are remarkably similar to the properties of trajectories for a charged particle orbiting about a magnetic monopole. It follows from this analogy that the source for the NUT metric may be related to a combination of ordinary mass plus a component of "magnetic" mass. This interpretation is also supported by the trajectories calculated from Newtonian gravity that was generalized to include a magnetic mass term. However, the one property where the NUT geodesics differs from a magnetic monopole is in the spherical symmetry of the coordinate time. In electrodynamics the direction of the Dirac string does not affect the classical trajectories. This is not exactly true for the NUT metric. The spatial coordinates are independent of the direction of the line singularity when they are parameterized in terms of the proper time; however, the time coordinate depends on the direction of the line singularity.

Spherical symmetry is also violated in the quantum mechanical treatment of the magnetic monopole since the quantum equations depend on the vector potential. Forcing the quantum equations to be independent of the Dirac string leads to the Dirac quantum condition for the charge. Can one find a similar relation in general relativity that will make the time coordinate in (24a) independent of the axial singularity? It is not clear what the analogous Dirac condition must be to make the time coordinate independent of the axial singularity because the problem is more complex in general relativity.

In general relativity the line singularity creates an effective energy which, in turn, affects the rates of clocks. Hence, the line singularity manifests itself in the relation for the time coordinate. It is not clear whether the NUT solution has to be modified at the beginning in order to distinguish between a solenoid-like solution and a magnetic-like particle solution. Or is it possible to eliminate the effect of the Dirac string the way Dirac did for the quantum mechanical formulation of the magnetic monopole, i.e., by quantizing the charge and not modifying the field.

There are several ways that lead to the Dirac quantization condition. The easiest, and the one most relevant to this article, is the semi-classical approach [15]. The Dirac quantization condition follows equating the magnitude of the field angular momentum equal to $n\hbar$, i.e., $eg = n\hbar$. If we do the same thing for the NUT metric, it follows from Eqs. (11) that $Emm^* = \frac{1}{2}n\hbar$. To lowest order in m^*, $E \approx 1$ and the equation reduces to $mm^* = \frac{1}{2}n\hbar$. However, if one retains all orders in m^*, then the quantization condition is not just a restriction on m and m^* but also involves the total energy of the system. It is not clear what this means or even whether it makes sense to suggest such a restriction. In addition, this quantization does not make the NUT solution spherically symmetric so the monopole string is still manifest in the geodesic time coordinate.

Although a conclusive interpretation for the NUT metric would have been preferable, we must conclude this article the same way we started, i.e., with an indecisive conclusion. Even though the NUT metric has properties that are tantalizingly close to those that one would expect to follow for a gravitation magnetic monopole, the subtle nonspherical behavior of the time coordinate casts some doubt about this interpretation. Can the non-spherical behavior be eliminated by a quantization condition, as is done for quantum monopoles, or must the NUT metric be modified in order to eliminate its solenoidal-like behavior?

APPENDIX: NEWTONIAN THEORY FOR GRAVITY MONOPOLE

To verify further the magnetic behavior of the NUT parameter, we construct the solution for the equations of motion that follow from a theory of Newtonian gravity that has been generalized to include "magnetic" mass. This example further supports the claim that the NUT parameter, m^*, behaves like magnetic mass.

Consider a spherical source that has both ordinary mass m and magnetic mass m^*; m generates the ordinary gravitational field F_e, and m^* generates a "magnetic-type" field F_h. For a spherically symmetric source F_h and F_e are of the form

$$\vec{F}_e = \frac{m}{r^3}\vec{r}, \qquad \vec{F}_h = \frac{m^*}{r^3}\vec{r} \tag{A1}$$

The magnetic mass m^* modifies the usual equation of motion in the same way that a magnetic charge modifies the Lorentz force in electrodynamics. The generalized equations of motion for a test body with inertial mass m_0 orbiting a source with both ordinary mass and magnetic mass are

$$\vec{a}=\left[\frac{-m\vec{r}}{r^3}+\frac{2\vec{v}\times(-m^*\hat{r})}{r^3}\right] \tag{A2}$$

We have assumed that the gravitational mass of the test particle is equal to its inertial mass, i.e., the principle of equivalence.

The particles angular momentum (per mass), $\vec{L}=\vec{r}\times\vec{v}$ does not yield a conserved quantity because the magnetic matter generates a torque on the test particle. Forming the cross product of the vector $\vec{r}$ with Eq. (A2), we get the following expression for the time derivative of the angular momentum:

$$\frac{d\vec{L}}{dt}=-2m^*\frac{d\hat{r}}{dt} \tag{A3}$$

Integrating (A3), we get the conserved angular momentum

$$\vec{J}=\vec{L}+\vec{S} \tag{A4a}$$

where

$$\vec{S}=2m^*\hat{r} \qquad \text{and} \qquad \vec{L}=\vec{r}\times\vec{v} \tag{A4b, c}$$

The conserved angular momentum $\vec{J}$ is the sum of the particles orbital angular momentum $\vec{L}$ and the field angular momentum for the source $\vec{S}$. $\vec{S}$ is directed along the line joining the orbiting particle and the source and is normal to $\vec{L}$. The magnitudes of both $\vec{L}$ and $\vec{S}$ are conserved. Squaring Eqs. (A4), it follows that

$$\vec{S}\cdot\vec{S}=4m^{*2}, \qquad \vec{L}\cdot\vec{L}=\vec{J}\cdot\vec{J}-4m^{*2} \tag{A5}$$

From these results it follows that the orbits lie on the surface of a cone. Dotting the unit position vector $\hat{e}_r$ into $\vec{J}$, we get the constant $\hat{e}_r\cdot\vec{J}=2m^*$. This means that the motion is along a cone with cone angle $\cos\alpha=|\vec{S}|/|\vec{J}|$. Equations (A4) and (A5) are the limiting case of the NUT equations (9)–(12) to lowest order in m^*.

The radial behavior for the orbit follows from the conserved energy. Dotting $\vec{v}$ into (A2), the conserved energy (per mass) is given by

$$E^2=\tfrac{1}{2}v^2-m/r \tag{A6}$$

Expressing (A6) in terms of angular variables, we get

$$E^2 = \tfrac{1}{2}\dot{r}^2 + V(r) \tag{A7a}$$

where

$$V(r) = -\frac{m}{r} + \frac{L^2}{2r^2} \tag{A7b}$$

The potential in (A7b) is identical to the form of a particle in an ordinary gravitational field. The magnetic part of the source does not appear in the energy equations because the force acts in a direction normal to the displacement of the trajectory.

Notice the remarkable similarity between these generalized Newtonian equations and the NUT geodesic equations. Keeping only the terms linear in m and m^* and comparing the results for the Newtonian and NUT geodesic, we get identical results. This close similarity is further support for the identification of the NUT parameter with a gravitational magnetic monopole component for the source.

REFERENCES

1. Newman, E., Tamburino, L., and Unti, T. (1963). *J. Math. Phys.*, **4**. 915.
2. Misner, C. W. (1963). *J. Math. Phys.*, **4**, 924.
3. Bonnor, W. B. (1969). *Proc. Camb. Phil. Soc.*, **66**, 145.
4. Dowker, J. S. (1974). *Gen. Rel. Grav.*, **5**, 603.
5. Demianski, M., and Newman, E. T. (1966). *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.*, **14**, 653.
6. Jackson, J. C. (1970). *J. Math. Phys.*, **11**, 924.
7. Miller, J. G. (1973). *J. Math. Phys.*, **14**, 486.
8. Sackfield, A. (1971). *Proc. Camb. Phil. Soc.*, **70**, 89.
9. Taub, A. (1951). *Ann. Math.*, **53**, 472.
10. Markley, F. L. (1973). *AJP*, **41**, 1246.
11. Dowker, J. S., and Roche, J. A. (1967). *Proc. Phys. Soc.*, **92**, 1.
12. Kobiske, R. A., and Parker, L. (1977). *Phys. Rev.*, **D16**, 3355.
13. Ramaswamy, S., and Sen, A. (1981). *J. Math. Phys.*, **22**, 2612.
14. Barut, A. C. (1971). *Phys. Rev.*, **30**, 1747.
15. Jackson, J. D. (1974). *Classical Electrodynamics*, 2nd ed.