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Synchronized Frames for Gödel's Universe

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We exhibit Gödel's geometry in terms of a set of gaussian systems of coordinates, the union of which constitutes a complete cover for the whole manifold. We present a mechanism which induces a particle to follow a closed time-like line (CTL) present in this geometry. We generalize the construction of special class of observers (Generalized Milne Observers) which provides a way to define the largest causal domain allowing a standard field theory to be developed.

1. INTRODUCTION

A simple glance into any book of Relativistic Cosmology displays an interesting common characteristic: all cosmological models are depicted in gaussian systems of coordinates with just one remarkable exception, Gödel's 1949 rotating Universe [1].

This particularity is in general interpreted to be nothing but a consequence of the well-known impossibility of constructing a unique global gaussian system in this geometry. However, such a property does not forbid the use of a *local* gaussian system.

Indeed, the theory of Riemannian differentiable manifolds asserts that it is always possible, at least in a restricted domain, to represent point-events by means of a gaussian coordinate system. The extension of this system beyond a given domain depends on properties of the geometry at large.

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Although there have been some comments in the literature concerning synchronized systems of Gödel's cosmological model, an explicit form has never appeared. We intend to remedy this situation in this article by exhibiting a set of gaussian systems of complementary domains, in such a way that their union constitutes a complete cover for the whole manifold.

The restriction on each synchronized frame can be understood as a consequence of the highly confining property of Gödel's geometry. A question then arises: How can one reconcile such confinement with the homogeneity property of this metric? How could a point (any point) of such homogeneous space-time act as an irresistible attractor? This is precisely the condition to limit the extension of a chosen family of time-like geodesics, inhibiting it from going beyond a certain domain, and so restricting the region covered by the associated chart. To understand this one should look more carefully into the dynamical behaviour of free particle. Since the velocity of photons is the highest allowed one, let us just consider their propagation.

From electrodynamics and gravity standard coupling photons travel along null geodesics. Now, from the behaviour of geodesics in Gödel's geometry [2,3] one obtains that the photons' trajectory, which passes through a point P, can be equivalently described as if the particle feels an attraction to P by a potential $V(r) = V_0 \tanh r$ (in which V_0 is a constant) having an energy $E < V_0$ [2]. This means that the net consequence of such a potential is to forbid the particle to leave the region $\mathcal{D}(P)$ which consists in the points encircling P of a given radius. The actual value of the maximal allowable radius depends on the strength of the vorticity Ω. Thus, any geodesic which passes an (arbitrary) point P remains—for its complete history—confined in a cylinder around P of radius r_0. This has an immediate consequence, which we referred to previously: if one displays a gaussian coordinate system from a point $\mathcal{O}$ (arbitrary) then this system cannot be extended beyond r_0. This is a consequence of the dependence of the gaussian system on a particular choice of time-like geodesics $\Gamma_{(\mathcal{O})}$ which precisely yields the identification of the local (gaussian) time to the proper time of $\Gamma_{(\mathcal{O})}$.

We can build another gaussian system centered on another point $\mathcal{O}'$ distinct from $\mathcal{O}$. This new system can be located either within the domain of the previous Gauss-I system, in the region $0 < r < r_0$ or beyond it. We can then follow the same procedure as in the previous case and define a new gaussian chart (call it Gauss-II) based on point $\mathcal{O}'$. This method can be repeated successively and complete the covering of the whole manifold. We present in Section 2 a short resumé of such peculiar behaviour of time-like geodesics in Gödel's Universe.

The possession of each synchronized system has a direct by-product: it allows one to establish in a standard way the foundations of a field theory in the Gödel background. The ultimate reason for this can be understood in the following way. One of the most fundamental principles of physics is the one associated to the set up of a Cauchy initial-value problem for a given field. In order to follow this program one starts by fixing a given foliation of the background space-time. This is in general provided by the identification of flat surfaces in which the time coordinate is constant. One can, alternatively, deal with other imbeddings, e.g. by considering space-time hyperboloids in which translational time invariance is not explicitly guaranteed. Nevertheless, any of these choices is a good one as long as it provides a set of Cauchy successive surfaces in which the standard methods of causal modelling in physics could be applied.

However, this is not the kind of situation we face in Gödel's Universe. The impossibility of global synchronization inhibits the set up of a standard Cauchy initial-value problem. The true origin for such a difficulty rests in the existence of closed time-like lines (CTL) in this geometry. In order to overcome this situation we will follow a procedure which has its roots in Milne's characterization of fundamental observers in flat space-time. In Section 4 we will review briefly the properties of Milne's frame in the case for which it was originally created, that is, of a Minkowski background.

We shall see that a very similar class of observers can be set up in Gödel's geometry, inducing a framework which allows a description of a restricted causal domain on this Universe. However, there is a crucial difference between Milne's restricted space-time in Minkowski background (which we call the $\mathcal{U}^+$ region) and the analogous construction in Gödel's space-time (which we call the $\mathcal{G}^+$ region). The $\mathcal{U}^+$ region has a fictitious big bang that generates an ever-expanding structure, while the $\mathcal{G}^+$ region has not only a non-homogeneous initial singularity (a false big bang) but also an ending configuration (a false big crunch). Thus a well posed Cauchy problem can be set up in the $\mathcal{G}^+$ domain.

All the above arguments, which led us to the limited Gauss domains, come from classical physics (e.g. the behaviour of null geodesics in Gödel's background). One might wonder if these considerations should be modified if some new effects at the quantum level change the confining property. The proof that this is not the case has been presented some years ago [6]. We review this briefly in the appendix. The main idea runs as follows. From the reduction of the behaviour of null geodesics to the examination of the dynamics of a particle submitted to an effective potential $V(r)$ we can generate the corresponding Schrödinger's equation yielding a one-particle problem in a Posch–Teller potential. The net result of such direct quantization confirms the classical confining behaviour. One could suspect this from the properties of the infinite range Posch–Teller potential.

Finally, it remains to analyse a question concerning the closed timelike lines (CTL). It has been known, since Gödel's original paper, that these curves are not paths of free particles, and a problem then appears: What are the characteristics of the force which induces a particle to follow such a strange path? We present a solution to this question in Section 6 in which we show that a (weak) magnetic field can induce a charged particle (say, an electron) to follow this path.

2. PRELIMINARIES: GÖDEL'S GEOMETRY

In this section we will briefly review some basic properties of Gödel's geometry. The manifold has the structure of $H^3 \otimes \mathbb{R}$ of a 3-dimensional hyperboloid—in which coordinate (t, r, ϕ) are defined with the range $-\infty < t < \infty$, $0 \leq r < \infty$, $0 \leq \phi \leq 2\pi$, respectively—times the infinite linear coordinate (z) defined on the real line $\mathbb{R}$. In this cylindrical coordinate system the fundamental element of length takes the form[2]

$$ds^2 = a^2[dt^2 - dr^2 - dz^2 + 2h(r)dt\, d\phi + g(r)d\phi^2]. \tag{1}$$

The constant a is related to the amplitude of the vorticity Ω of the matter $\Omega^2 = 2/a^2$. The functions $g(r)$ and $h(r)$ are given by

$$\begin{aligned} g(r) &= \sinh^2 r(\sinh^2 r - 1) \\ h(r) &= \sqrt{2}\, \sinh^2 r. \end{aligned} \tag{2}$$

The source of this geometry is a perfect fluid with density of energy ρ and no pressure:

$$T_{\mu\nu}\rho V_\mu V_\nu. \tag{3}$$

[2] Note that under this form of geometry it is not possible to obtain the limit metric for $\Omega \to 0$. This, however, can be achieved if before the limit is taken we make a re-scaling by setting

$$t \to t' = at, \quad r \to r' = ar, \quad \phi \to \phi' = \phi, \quad z \to z' = az.$$

Then the metric takes the form

$$ds^2 = \left[dt' + \frac{2}{\Omega} \sinh^2 \frac{\sqrt{2}}{2}\, \Omega r' d\phi'\right]^2 - \frac{1}{2\Omega^2} \sinh^2 \sqrt{2}\, \Omega r' d\phi'^2 - dr'^2 - dz'^2.$$

In this form the limit $\Omega \to 0$ yields the Minkowskian geometry in the cylindrical system of coordinates.

In the cylindrical coordinate system, $V_\mu = \delta^0_\mu$. The congruence of the fluid has no expansion ($\Theta = 0$), no shear ($\sigma_{\mu\nu} = 0$) but has a non null vorticity

$$\omega_{\mu\nu} = \tfrac{1}{2} h^\alpha_\mu h^\beta_\nu [V_{\alpha;\beta} - V_{\beta;\alpha}] \tag{4}$$

with $h^\nu_\mu = \delta^\nu_\mu - V_\mu V^\nu$. The vorticity vector $\omega^\tau = \frac{1}{2}\eta^{\alpha\beta\rho\tau}\omega_{\alpha\beta}V_\rho$ has components

$$\omega_\alpha = (0, 0, 0, \Omega). \tag{5}$$

Thus at each point of this space-time a privileged direction is defined,. Einstein's equations with a cosmological constant Λ are satisfied if between constants a,Λ and the energy ρ the following relation holds:

$$\rho = 2\Omega^2 = \frac{4}{a^2} = -2\Lambda. \tag{6}$$

Although the above cylindrical system of coordinates can be used throughout the whole manifold it does not allow a bona fide formulation of the Cauchy problem.

The best way to provide the necessary conditions to establish a well defined formulation of the initial value problem in this geometry is to jump into a frame in which a synchronization can be made (at least in some region of Gödel's space-time). We will follow this procedure in this work. The first step towards this is to select a given set of time-like geodesics and solve a corresponding Hamilton–Jacobi equation $g^{\mu\nu}(\partial S/\partial x^\mu)(\partial S/\partial x^\nu) = 1$ for the new time coordinate S. The remaining associated spatial coordinates $\tilde{x}^i$ are obtained from the solution $S(x^\mu, \lambda^i)$ of this Hamilton–Jacobi equation through the derivatives $\tilde{x}^i = \partial S/\partial \lambda^i$. Let us then look into the possible classes of time-like geodesics in order to make a definite choice.

The geodesics in Gödel's Universe were studied by Chandrasekhar and Wright [3] and in an alternative manner by Novello, Soares and Tiomno (NST) [2]. We will follow the NST version in the present paper.

The equations for the geodesics $x^\mu = x^\mu(s)$ with four-velocity $v^\mu = dx^\mu/ds = (\dot{t}, \dot{r}, \dot{\phi}, \dot{z})$ are

$$\begin{aligned}
\dot{z} &= C_0, \\
\dot{\phi} &= \frac{\sqrt{2}\,A_0}{\cosh^2 r} - \frac{B_0}{\sinh^2 r \cosh^2 r}, \\
\dot{t} &= A_0\left[1 - \frac{2\sinh^2 r}{\cosh^2 r}\right] + \frac{\sqrt{2}\,B_0}{\cosh^2 r}, \\
\dot{r} &= A_0^2 - D_0^2 - \left[\frac{\sqrt{2}\,A_0 \sinh r}{\cosh r} - \frac{B_0}{\sinh r \cosh r}\right]^2.
\end{aligned} \tag{7}$$

Instead of going into the immediate integration of this set of equations it is more convenient for our purposes to pause for a while and look into the generic behaviour of the time-like families of geodesics from the examination of their effective potential. In this vein, let us re-write the equation of $\dot{r}$ in the form

$$\dot{r}^2 = A_0^2 - V(r) \tag{8}$$

in which the effective potential $V(r)$ is given by

$$V(r) = D_0^2 + \left[\frac{\sqrt{2}\,A_0 \sinh r}{\cosh r} - \frac{B_0}{\sinh r \cosh r}\right]^2 \tag{9}$$

and

$$D_0^2 = C_0^2 + \frac{1}{a^2}\,. \tag{10}$$

We can thus interpret the constant A_0^2 as the square of the total energy (per unit of mass) and B_0 as the total angular momentum. Indeed, if we define the momenta $P_\mu = g_{\mu\nu}\dot{x}^\nu$ it then follows that

$$P_0 = A_0, \qquad P_r = -\dot{r}, \qquad P_\phi = B_0, \qquad P_z = -C_0. \tag{11}$$

A complete characterization of the main features of the behaviour of the geodesics can be obtained by just examining eq. (9). We distinguish three cases,

$$B_0 > 0, \qquad B_0 = 0, \qquad B_0 < 0, \tag{12}$$

once the associated potentials have distinct features.

It seems worth defining the parameters $\gamma = B_0/A_0$ and $\beta^2 = (D_0/A_0)^2$. Once we are interested only in time-like geodesics we will limit our analysis to the case in which $0 \leq \beta^2 \leq 1$. The forms of the potentials are depicted in Figure 1 for the three cases.

A direct inspection of these graphs gives the information we are looking for. For any geodesic the radial coordinate r oscillates between the values r_1 and r_2 given by

$$\sinh^2 r_i = \frac{1 + 2\sqrt{2}\,\gamma - \beta^2}{2(1+\beta^2)} \pm \frac{\sqrt{1-\beta^2}\,\sqrt{(2\gamma+\sqrt{2}\,)^2 - (1+\beta^2)}}{2(1+\beta^2)}\,.$$

This represents a true confinement in the classical regime once the total energy A_0^2 is a fixed quantity (for each geodesic) and thus the trajectories

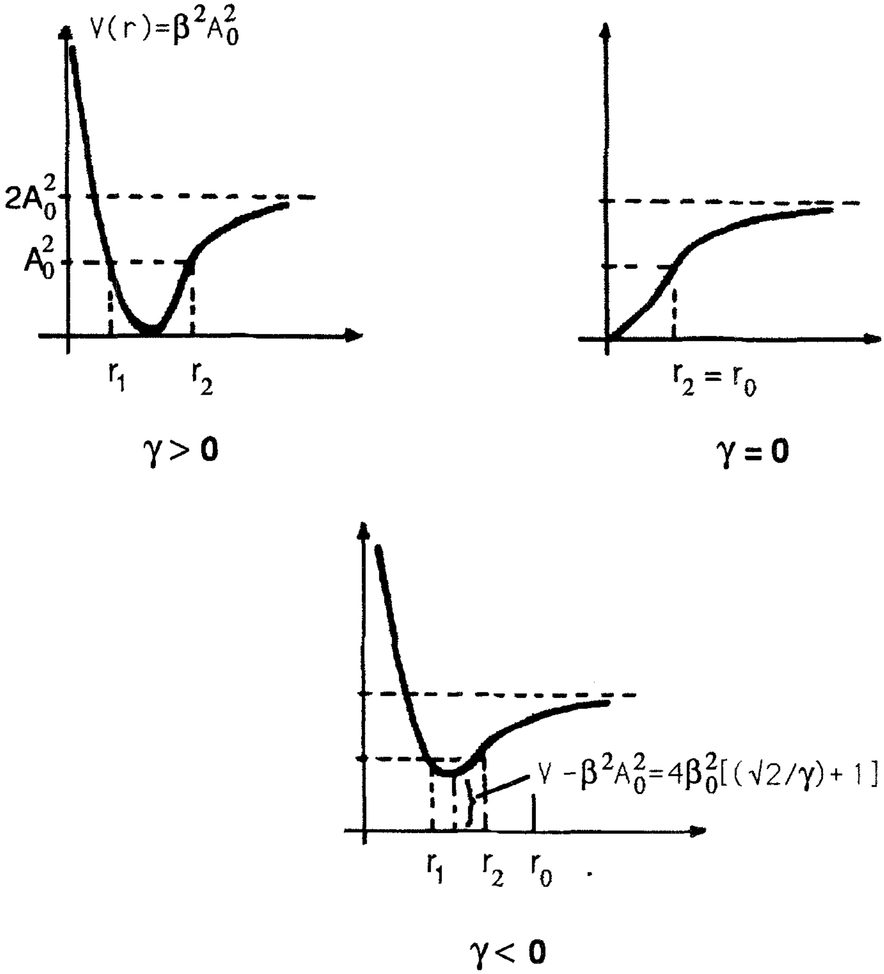

Figure 1. Graphs of the effective potential for the free particles in Gödel's geometry.

are kept within the cylindrical shell $r_1 \leq r \leq r_2$. For $\gamma \leq 0$ the potential $V(r)$ produces the phenomenon of confinement of all trajectories within the cylinder $r \leq r_c$ with $\sinh r_c = 1$.[3] Such powerful attraction of gravity is

[3] Note that in case $\gamma > 0$ the behaviour of the particles depends on the momentum along the axis z; the width of the cylindrical shell diminishes and can attain the value zero for $\beta^2 = 1$.

the reason that forbids the extension of any *local* gaussian system beyond a certain region.

3. FROM TIME-LIKE GEODESICS TO THE GAUSSIAN SYSTEM OF COORDINATES

We are now prepared to undertake the first step toward a synchronized frame. In the geometry (1), the equation satisfied by the associated Hamilton–Jacobi equation S takes the form

$$\left[\frac{\sinh^2 r - 1}{\cosh^2 r}\right]\left(\frac{\partial S}{\partial t}\right)^2 - \frac{2\sqrt{2}}{\cosh^2 r}\frac{\partial S}{\partial t}\frac{\partial S}{\partial \phi} + \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{\sinh^2 r \cosh^2 r}\left(\frac{\partial S}{\partial \phi}\right)^2 + \left(\frac{\partial S}{\partial z}\right)^2 + a^2 = 0. \quad (13)$$

We can make the ansatz

$$S(t, r, \phi, z) = \lambda_1 t + \lambda_2 \phi + \lambda_3 z + F(r) \quad (14)$$

where the λ_i are constants.

The problem is then reduced to the integration of the equation for $F(r)$. A straightforward calculation yields

$$F(r) = \frac{\sqrt{P}}{2}\arcsin\frac{(-2Px + Q)}{\sqrt{Q^2 - 4P\lambda_2^2}} - \lambda_2^2 \arcsin\frac{(Qx - 2\lambda_2^2)}{x\sqrt{Q^2 - 4P\lambda_2^2}} + \frac{\sqrt{|Q + P + \lambda_2^2|}}{2}\arcsin\frac{(Q + 2P)x + Q - 2\lambda_2^2}{(x+1)\sqrt{Q^2 - 4P\lambda_2^2}}$$

in which

$$x \equiv \sinh^2 r, \quad P \equiv \lambda_1^2 + \lambda_3^2 + a^2, \quad Q \equiv 2\sqrt{2}\,\lambda_1\lambda_2 + \lambda_1^2 - \lambda_3^2 - a^2.$$

Inserting $F(r)$ into the expression (14) completes the definition of the new time. From it we can obtain the remaining spatial components $\tilde{x}^i$ by taking the derivatives of S with respect to the parameters λ_i, where $i = 1, 2, 3$. This procedure then provides a local synchronization through the construction of a set of hypersurfaces which are space-like and such that the family of geodesics chosen intersects each surface just once. It only remains to identify among the time-like geodesics which satisfy (7) those characterized by the above choice (14) of $S(\lambda_i, t, r, \phi, z)$. This can be made easily once in NST an explicit integrated form for these geodesics is given. The result of this identification can be summarized as follows.

From the previous qualitative analysis of the behaviour of the geodesics (see Fig. 1) one can infer that in order to set up a gaussian system the origin of which starts at (an arbitrary) point $\mathcal{O}$ we must select the value of the constant B_0 as being null.

From now on we will call such a system a Gauss-I system (centered at $\mathcal{O}$). To complete the system we make the following choice for the values of constant A_0, B_0, C_0 and D_0 which appeared in expression (7):

$$A_0 = \frac{\mu}{a}\,, \qquad B_0 = C_0 = 0, \qquad D_0 = \frac{1}{a}\,, \tag{15}$$

in which we have defined $\mu \equiv \lambda_1/a$ and set $\lambda_2 = \lambda_3 = 0$, in order to conform with the solution (14) and $F(r)$.

Let us display here for future reference the explicit formula of passage from the cylindrical coordinate system (t, r, ϕ, z) to the Gauss-I system $(\tilde{t}, \tilde{\xi}, \tilde{\eta}, \tilde{z})$:

$$\begin{aligned} \tilde{t} &= \mu a t + \frac{a}{2}\sqrt{\mu^2+1}\arcsin\Psi + \frac{\mu a}{\sqrt{2}}\arcsin\Delta \\ \tilde{\xi} &= t + \frac{\mu}{2\sqrt{\mu^2+1}}\arcsin\Psi + \frac{1}{\sqrt{2}}\arcsin\Delta \\ \tilde{\eta} &= \left(\phi - \frac{\pi}{4}\right) + \frac{1}{2}\arcsin\Delta \\ \tilde{z} &= z \end{aligned} \tag{16}$$

in which

$$\begin{aligned} \Psi &= 1 - 2\frac{\mu^2+1}{\mu^2-1}\sinh^2 r \\ \Delta &= \frac{3\mu^2+1}{\mu^2-1}\,\frac{\sinh^2 r}{\sinh^2 r+1} - \frac{1}{\sinh^2 r+1}\,. \end{aligned} \tag{17}$$

Gödel's geometry in the Gauss-I system takes the from

$$ds^2 = d\tilde{t}^2 - a^2(\mu^2-1)d\tilde{\xi}^2 + a^2 g(\tilde{t},\tilde{\xi})d\tilde{\eta}^2 + 2ha^2(\tilde{t},\tilde{\xi})d\tilde{\xi}\,d\tilde{\eta} - a^2 d\tilde{z}^2 \tag{18}$$

in which the functions g and h are the same as those given in (2) with the substitution of the radial coordinate r in terms of the new ones. From the transformations (16) it follows that

$$1 - \sin M = 2\frac{\sinh^2 r}{\sinh^2 r_c} \tag{19}$$

with

$$\sinh^2 r_c = \frac{\mu^2 - 1}{\mu^2 + 1} \tag{20}$$

and

$$M = \frac{2}{a}\sqrt{\mu^2 + 1}\,(\tilde{t} - \mu a \tilde{\xi}). \tag{21}$$

Thus, making use of these relations, we can write

$$\begin{aligned} g(\tilde{t}, \tilde{\xi}) &= -\frac{1}{4}\frac{\mu^2 - 1}{(\mu^2 + 1)^2}(1 - \sin M)\,[\mu^2 + 3 + (\mu^2 - 1)\sin M] \\ h(\tilde{t}, \tilde{\xi}) &= \frac{\sqrt{2}}{2}\frac{\mu^2 - 1}{\mu^2 + 1}(1 - \sin M). \end{aligned} \tag{22}$$

Let us point out that such a synchronization procedure generated by the new system $(\tilde{t}, \tilde{\xi}, \tilde{\eta}, \tilde{z})$ is valid only in a restricted domain. In terms of the r coordinate it is given by $0 \leq r \leq r_c$ for r_c defined by (20). Let us make a final comment on this.

Distinct values of $\mu = \lambda_1/a$ yield (within the same class of geodesics, e.g. $B_0 = C_0 = 0$ and $D_0 = 1/a$) different types of curves and consequently distinct, although equivalent, coordinate systems. Now, for each fixed family (e.g. fixed values of λ_1) and by noting that $\mu = \lambda_1/a = (\sqrt{2}/2)\lambda_1\Omega$, it follows that when the vorticity increases $(\omega \to \infty)$ we achieve the maximum possible value for the gaussian domain: $\sinh^2 r_c = 1$. On the other hand, by means of a simple re-scaling of the cylindrical coordinates—as we did earlier—we can show that if the vorticity vanishes the domain of validity of the above synchronization can be extended through the whole space-time manifold (i.e. it reduces to the empty Minkowski geometry).

4. THE FUNDAMENTAL OBSERVERS OF GAUSS-I (GENERALIZED MILNE FRAMES)

The complete characterization of the gaussian observers can be achieved through a direct integration of the geodesic equations. In the cylindrical coordinate system this is accomplished by setting (NST)

$$\begin{aligned}
z &= z_0 \\
\frac{\sinh^2 r}{\sinh^2 r_c} &= \frac{1}{2}\left[1 + \cos\frac{2}{a}\sqrt{\mu^2+1}\,(s-s_0)\right] \\
\cos(\phi - \phi_0) &= \frac{\sqrt{2}\,\mu}{\sqrt{\mu^2-1}}\,\frac{1}{\sinh^2 r + 1} \\
\tan\frac{\sqrt{2}}{2}\left[t + \frac{\mu}{a}(s-s_0)\right] &= \frac{\sqrt{\mu^2+1}}{\mu\sqrt{2}}\,\tan\frac{\sqrt{\mu^2+1}}{a}(s-s_0).
\end{aligned} \tag{23}$$

We can thus read from these formulae the value of the components of the four-vector l^μ of this observer. We obtain

$$\begin{aligned}
l^0 &= -\frac{\mu}{a}\,\frac{\sinh^2 r - 1}{\cosh^2 r} \\
l^1 &= \frac{\sqrt{\mu^2 - 1 - (\mu^2+1)\sinh^2 r}}{a\cosh r} \\
l^2 &= \frac{\sqrt{2}\,\mu}{a}\,\frac{1}{\cosh^2 r} \\
l^3 &= 0.
\end{aligned} \tag{24}$$

This vector l^μ in the Gauss-I system of coordinates takes the value $\tilde{l}^\mu = \delta^\mu_0$, just by construction. We have seen in Section 2 that the matter flow has components $V^\mu = (1/a)\delta^\mu_0$, which in the gaussian system take the form $\tilde{V}^\mu = (\mu, 1/a, 0, 0)$. This can be used to give a simple geometrical interpretation for the parameter μ which we have used to distinguish among the infinite set of equivalent systems of transformation (16): it measures the angle between the fluid four-velocity V^μ and the geodesic l^μ. Indeed, from the above expressions it follows that

$$\mu = V_\alpha l^\alpha. \tag{25}$$

Just for completeness let us make one more remark concerning such kinematical properties. Although the matter content of Gödel's Universe is conveniently represented by the form (3), when represented in terms of the gaussian observers it appears as a more complicated fluid with non-vanishing pressure and heat flow. This reflects the fact that a tensor

$t_{\mu\nu}$ can be represented by projecting into non-equivalent frames. This is explicitly realized by the equality

$$\rho V_\alpha V_\beta = \tilde{\rho} l_\alpha l_\beta - \tilde{p}(g_{\alpha\beta} - l_\alpha l_\beta) + \tilde{q}_{(\alpha} l_{\beta)} + \tilde{\pi}_{\alpha\beta} \tag{26}$$

in which

$$\tilde{\rho} = \rho\mu^2, \qquad \tilde{p} = \tfrac{1}{3}\rho(\mu^2 - 1), \qquad \tilde{q}_\lambda = \rho\mu(V_\lambda - \mu l_\lambda). \tag{27}$$

Let us now turn our attention to the behaviour of the congruences generated by the l^μ-geodesics. From the definition of the congruence $\Theta = l^\mu{}_{;\nu}$ it follows, using (24) and (18), that

$$\Theta = -2\frac{\sqrt{\mu^2+1}}{a} \tan M \tag{28}$$

in which M is given by (21). Thus, Θ diverges at the boundaries of validity of the gaussian system (see Fig. 2).

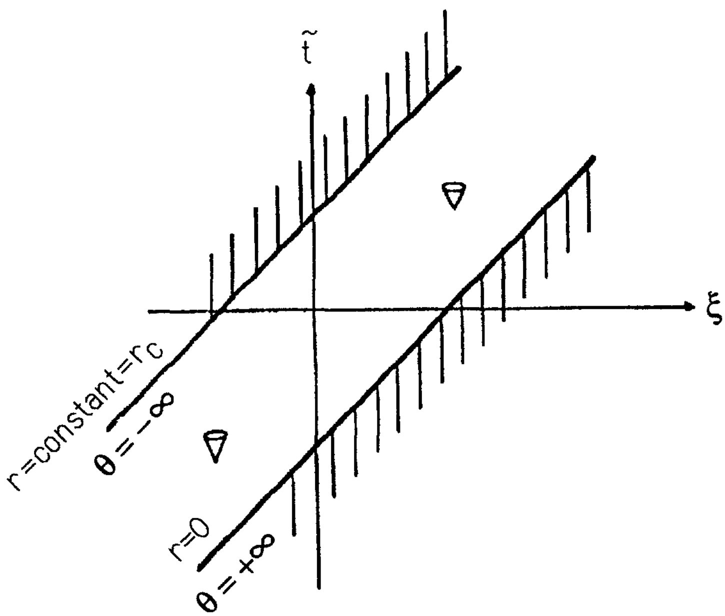

Figure 2. The domain of Gauss-I system of coordinates constitutes the region inside the lines $r = 0$ and $r = r_c$, which we will call the G^+ Universe. The null cones are provided by $d\tilde{t} = a\sqrt{\mu^2 - 1}d\xi$.

The region covered by this system, the $\mathcal{G}^+$ Universe, can be described in an alternative way as the evolution of a solidary, unique, compactified region which we will now try to describe.

The behaviour of the expansion Θ suggests the interpretation of the Gauss-I system as the establishment of a frame generated by a fictitious class of observers in Minkowski background as proposed many years ago by Milne [4].

In order to understand this let us briefly review the properties of this frame in the case for which it was first created, i.e. that of a Minkowski background.

According to Milne's idea, from an arbitrary point $\mathcal{O}$ of Minkowski space-time an infinite number of idealized particles (without any material properties, i.e. no mass, no volume—almost ghosts) is shot out in all directions in a completely random way, with all possible velocities. Thus at $\mathcal{O}$ there exists a sort of space-time creation mechanism, a false big bang, which is nothing but the reduction of the whole Minkowski space-time to a small portion of it, the region which we denote by $\mathcal{U}_{(+)}$. Such $\mathcal{U}_{(+)}$ consists of the region inside the light cone $\mathcal{H}$ generated from $\mathcal{O}$. This $\mathcal{U}_{(+)}$ region is called the Milne Universe.

The geometry at $\mathcal{U}_{(+)}$ takes the form

$$ds^2 = dt^2 - t^2 d\sigma^2. \tag{29}$$

Milne's fundamental observers are comoving, i.e. $V^\mu = \delta^\mu_0$. The expansion factor of the congruence Θ takes the calue $\Theta = 1/t$; it diverges at $\mathcal{H}$, in which $t = 0$.

Thus, Milne's frame contains a trivial Cauchy horizon once it consists of a chosen limited gaussian system of coordinates, which however can be extended beyond $\mathcal{H}$ by another choice of coordinates.

In this sense it has been argued—not completely without foundation—that Milne's Universe is nothing but a self-limited artificial construction of a handicapped frame used to describe Minkowski space-time.

Nevertheless, in some other class of space-times there are global properties such that the accomplishment of a class of observers similar to Milne's does not suffer from the above criticism; instead this becomes precisely the most adaptable frame in which a causal history of events could be displayed.

We will provide an example of this assertion by looking at the properties of Gödel's Universe.

From the previous analysis of the behaviour of geodesics in Gödel's geometry we concluded that any material particle (or photon) which passes through an arbitrary point, say A, is confined into a cylinder of radius r_c encircling A. Once this geometry is completely homogeneous, such confinement is guaranteed for any of its points. Such a curious confinement aspect has been analysed extensively [2,3].

Thus if one intends to obtain a gaussian system of coordinates for this geometry by means of timelike geodesics one faces the above limitation, which is nothing but the counterpart of the occurrence of closed time-like curves (non-geodesic) in this geometry.

We are thus prepared to undertake the construction of Milne-type observers in Gödel's background.

In every plane $z =$ constant we consider an infinite number of test particles shot out, in this plane, in all directions in a chaotic way. The infinite "source" of these observers is a string which we may locate arbitrarily at the origin of the r-coordinate. In the Minkowski case, the origin of the bang is a point. Here, in Gödel's, it is a string. This means that these observers are devised in such a way as to exhibit the background symmetry. Thus, for $r = 0$ this part of Gödel's Universe, expressed through Milne's coordinates, evolves as a closed Universe, once Θ diverges both at $r = 0$ and at $r = r_c$, where the critical radius r_c is given by (23) as $\sinh^2 r_c = (\mu^2 - 1)/(\mu^2 + 1)$. There is a false big bang at $r = 0$ and a false big crunch at $r = r_c$.

Let us note that inspection of the above dependence of the original coordinate r on the gaussian coordinates $(\tilde{t}, \tilde{\xi})$ shows that the bang is not homogeneous: it can be depicted as a configuration similar to a lagging core typical of a white hole (see Fig. 4). That is, for the gaussian system there is no unique moment of creation; separate parts enter the gaussian domain at distinct moments, viewed in terms of the cosmic time $\tilde{t}$.

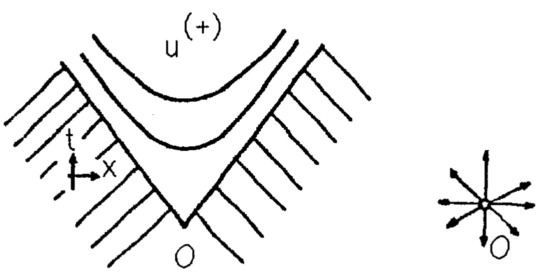

Figure 3. Milne's ghost observers in Minkowski space-time. The false big bang occurs at (an arbitrary) point $\mathcal{O}$.

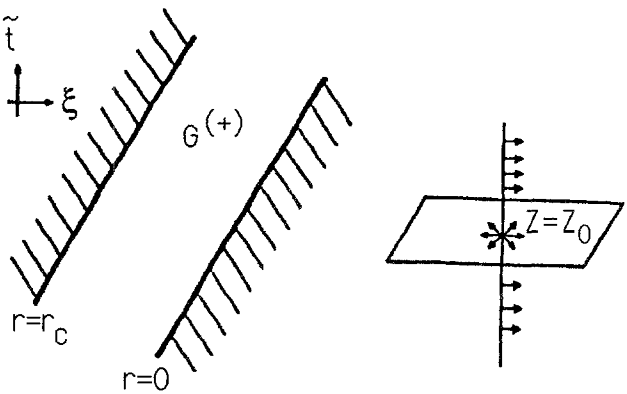

Figure 4. Milne's ghost observers in Gödel space-time. The false big bang occurs at $r = 0$. The false big crunch occurs at $r = r_c$.

From the theory of geodesics in Gödel's geometry [2,3] it follows that the region covered by such Milne observers—which we call the $\mathcal{G}^+$ Universe—is the largest causal domain of Gödel's model (see Fig. 4). Thus the hypersurfaces $\tilde{t} =$ constant provide Cauchy surfaces for all points in the $\mathcal{G}^+$ Universe.

This in turn can be used to examine the evolution properties of any field theory described by initial data on surfaces $\tilde{t} =$ constant in $\mathcal{G}^+$. Equivalently, one can define standard commutation relations at $\tilde{t} =$ constant and propagate them throughout $\mathcal{G}^+$ in order to elaborate a canonical quantum version of a field theory.

Although in these considerations we have been concerned with Gödel's Universe, the construction of similar Milne-type observers for different classes of geometries can be achieved through the generalization of the procedure shown in the above example. Some of those (e.g. Minkowski) are of limited importance (generating trivial Cauchy horizon), but others can be of crucial value (e.g. the $\mathcal{G}^+$ Universe) in the set up of a field theory.

Therefore, from what we have learned we conjecture that Milne's observers provide the most natural frame in which a Quantum Field Theory may be established in space-times which contain closed time-like curves.

5. BEYOND THE GAUSS-I SYSTEM

We have seen the reasons which forbid the analytical continuation of the Gauss-I system beyond a certain finite radius r_c. However, we can find another gaussian system for $r > r_c$ just by choosing another set of fundamental observers.

This is equivalent to defining a new local time $\tilde{t}_2$ through the choice of another class of time-like geodesics. Thus, in the new system (call it Gauss-II), instead of the values (15) for the parameters which characterize the congruence in Gauss-I, we set

$$A_0 = \frac{\mu}{a}, \quad B_0 = \frac{\nu}{a}, \quad C_0 = 0, \quad D_0 = \frac{1}{a}. \tag{30}$$

Then the constants γ and β^2 take the values

$$\gamma = \frac{\nu}{\mu}, \qquad \beta^2 = \frac{1}{\mu^2}. \tag{31}$$

From the previous analysis it follows that we must select $\mu^2 > 1$. Then $0 < \beta^2 < 1$ guarantees the existence of the Gauss-II system of coordinates. We note that the parameter ν is the true factor which defines the domain of validity of Gauss-II. The explicit transformation formula which relates the cylinder system of coordinates (1) to the Gauss-II system is given by

$$\begin{aligned} \tilde{t}_2 &= \mu a t + \nu a \phi + \frac{a}{2}\sqrt{\mu^2+1}\arcsin\Psi_2 \\ &\quad - \frac{\nu a}{2}\arcsin\chi_2 + \frac{a}{2}(\nu + \sqrt{2}\,\mu)\arcsin\Delta_2 \\ \tilde{\xi}_2 &= t + \frac{\mu}{2\sqrt{\mu^2+1}}\arcsin\Psi_2 + \frac{1}{\sqrt{2}}\arcsin\Delta_2 \\ \tilde{\eta}_2 &= \phi + \frac{1}{2}\arcsin\Delta_2 - \frac{1}{2}\arcsin\chi_2 \\ \tilde{z}_2 &= z \end{aligned} \tag{32}$$

in which

$$\begin{aligned} \Psi_2 &= \frac{1}{\sqrt{q}}\left[-2(\mu^2+1) - 2(\mu^2+1)\sinh^2 r + 2\sqrt{2}\,\mu\nu + \mu^2 - 1\right] \\ \Delta_2 &= \frac{1}{\sqrt{q}}\left[\frac{(2\sqrt{2}\,\mu\nu + 3\mu^2 + 1)\sinh^2 r - (\mu^2 + \sqrt{2}\,\nu)^2 + 1}{\sinh^2 r + 1}\right] \\ \chi_2 &= \frac{1}{\sqrt{q}}\left[\frac{-2\nu^2 + (2\sqrt{2}\,\mu\nu + \mu^2)\sinh^2 r}{\sinh^2 r}\right] \end{aligned} \tag{33}$$

in which

$$q \equiv (2\sqrt{2}\,\mu\nu + \mu^2 - 1)^2 - 4\nu^2(\mu^2 + 1). \tag{34}$$

Just for completeness, we can exhibit the components of the four-velocity p^μ of the Gauss-II observers which in the cylindrical coordinate system are

$$\begin{aligned}
p^0 &= \frac{1}{a^2\cosh^2 r}\,[\mu(1 - \sinh^2 r) + \sqrt{2}\,\nu] \\
p^1 &= \frac{1}{a\cosh r\sinh r}\,[\,(\mu^2 - 1)\cosh^2 r\sinh^2 r - (\sqrt{2}\,\mu\sinh^2 r - \nu)^2] \\
p^2 &= \frac{1}{a\cosh^2 r\sinh^2 r}\,[\sqrt{2}\,\mu\sinh^2 r - \nu] \\
p^3 &= 0.
\end{aligned} \tag{35}$$

We can use this to give the direct proof that the angle between Gauss-I observers four-velocity p^μ and the four-velocity V^μ of the matter content of Gödel's Universe is precisely the same as the Gauss-I observers,

$$p^\alpha V_\alpha = l^\alpha V_\alpha = \mu. \tag{36}$$

This reflects the continuity property which yields a coincidence of the two classes of Gauss observers in the limit $\nu \longrightarrow 0$. Combining the values of the parameters A_0,B_0,C_0 and D_0 from (30) with the form (9) of the effective potential we conclude that the domain of validity of the Gauss-II system is bounded by the range $r_1 \leq r \leq r_2$ as given previously by Fig. 1 which is the classically accessible region for a free particle.

One can give an overview of those coordinate systems in the following terms. The system Gauss-I covers a part of Gödel's geometry that consists in those points limited from an arbitrary origin (say point $\mathcal{O}$) which we gauge $r = 0$ to $r = r_c$ with $\sinh^2 r_c = (\mu^2 - 1)/(\mu^2 + 1)$. Thus the extension of such a domain depends on the value of the parameter μ.

The range of the second (Gauss-II) system depends not only on μ but also on the value of the parameter ν, and it is bounded by r_1 and r_2. We can choose these parameters in such a way that we have $r_1 \leq r_c$, with a non-vanishing intersection of both systems. This allows for the continuity of the covering of the Gödel geometry from $r = 0$ to $r = r_2$.

We can proceed further and define a third gaussian system (call it Gauss-III) which will display the same feature as Gauss-II. This procedure can be repeated. The net result of this is nothing but a piece-wise synchronization of the whole Gödel Universe.

Let us pause for a while and consider what we have achieved. From what we have learned above it follows that the impossibility of defining a unique global time can be interpreted as the requirement of the uses of non-gravitational forces to accelerate an observer in order to provoke its passage from a given geodesic congruence (which provides a definite local synchronization) to another one (correspondingly, to another gaussian time).

The set of gaussian observers which we use to define the Gauss system encounters some unusual properties due to the existence of closed time-like lines (CTL) in this geometry. It seems worthwhile then not only to look into the properties of these CTL but also to try to answer the question of how a material particle could be accelerated in order to follow a CTL. Besides this, one could contemplate that the difficulty of extending the path of a real particle beyond a certain region in Gödel's Universe might be a classical hindrance. Is it possible that some sort of quantum tunneling effect might lead a particle to escape from the confinement which we have described earlier? In the next section we will elaborate an answer to these two questions.

6. BACKWARDS TIME TRAVEL

Gödel's geometry has five Killing symmetries. One of them, which in a given basis can be characterized by $\partial/\partial\phi$, is of particular interest. This is because of its unusual property which guarantees that the vector $\partial/\partial\phi$ is space-like for $r < r_c$ and becomes time-like for $r > r_c$. This property has been employed, since Gödel's original discovery, to adapt an observer to a closed time-like line (CTL). Indeed, the line defined by[4] $r = r_0 = \text{constant}$, $z = z_0 = \text{constant}$, $t = t_0 = \text{constant}$ and $0 \leq \phi \leq 2\pi$ is a CTL for $r_0 > r_c$. Once this curve cannot be a geodesic (cf. our previous remarks), the question then arises of what the required force is which constrains a particle to follow such an unusual path.

The most natural candidate for this enterprise should be a rocket which carries enough fuel to undertake such an incredible journey. However, it has been shown in a very direct way [5] that the fuel consumption on such a path would be so high (almost 100% of the initial mass contained in the rocket) that it would preclude such backwards time travel.

Instead of considering some sort of technological argument which might produce more economic utilisation of the fuel, we will turn to a

[4] We are using here the cylindrical coordiante system.

more fundamental question: How should a machine operate in order to send a real particle of well-known properties travelling backwards in time?

We will discuss the case of setting an electron to undertake such travel. We decide to proceed in this way because it seems to us that, if we answer this question for the electron, then we would quite naturally gain some insight into the corresponding question of travelling backwards in time for macroscopic bodies, at least as far as theoretical arguments are concerned. The problem can thus be clearly stated as follows.

Consider an electron which follows the trajectory Γ characterized (in the cylindrical coordinate system) by

$$t = t_0 = \text{const.}, \quad r = r_0 = \text{const.}, \quad z = z_0 = \text{const.}, \quad 0 \leq \phi \leq 2\pi. \tag{37}$$

The normalized four-velocity b^μ of the electron is given by

$$b^\mu = \left(0, 0, \frac{1}{a \sinh r \sqrt{\sinh^2 r - 1}}, 0\right) \tag{38}$$

The corresponding four-vector acceleration $\dot{b}^\mu$ defined by $\dot{b}^\mu = b^\mu_{;\nu} b^\nu$ is directed along $\partial/\partial r$, that is

$$\dot{b}^\mu = \left(0, \frac{\cosh r(\sinh^2 r - 1)}{a^2 \sinh r(\sinh^2 r - 1)}, 0, 0\right). \tag{39}$$

As we have stated above, the gravity field alone cannot provide for such travel. The question we face is then the following: What force maintains the electron in such a strange orbit? What is responsible for such motion?

Our strategy for solving this problem is simply to look for a combined effect of gravity and electromagnetic forces. We decided to turn our attention to these two forces because they are the only long-range forces known. This seems a good criterion, once we know that the breakdown of synchronization is a global effect (see the analysis in the previous section of the limitation of gaussian domain for a coordinate system).

We will examine the orbit (37) in Gödel's Universe. Thus, any electromagnetic field present on it must be treated just as a small perturbation. In order to simplify our model we restrict the analysis to a sourceless pure magnetic field, as viewed in the frame comoving with the rotating matter, which is the source of the geometry. In the cylindrical coordinate system, the unique non-vanishing component of the electromagnetic field is set to be

$$F^{12} = H. \tag{40}$$

For the comoving matter with four-velocity V^μ [cf. eq. (3)] this is indeed a pure magnetic field directed in the z-direction. However, in the electron's frame which has four-velocity b^μ [cf. eq. (38)] this is *not* a pure magnetic field. Indeed, for the electron we have a non-null electric tensor given by

$$\hat{E}_\mu = F_{\mu\nu} b^\mu = F_{\mu 2} b^2 \neq 0. \tag{41}$$

The Lorentz force provides the necessary acceleration to keep the electron in the orbit Γ, for $r = r_0$, if the strength of the field is given by

$$H(r) = \frac{m}{ea^2} \frac{\cosh r (2\sinh^2 r - 1)}{\sinh r}. \tag{42}$$

Solving Maxwell's equations as a test field in the background of Gödel's geometry yields the dependence of H,

$$H = \frac{H_0}{\sinh 2r}. \tag{43}$$

Combining eqs. (42) and (43) for a given value $r = r_0$ yields the value of the intensity H_0, that is,

$$H_0 = \frac{2m}{ea^3} \frac{\cosh^2 r_0 (2\sinh^2 r_0 - 1)}{\sinh r_0 (\sinh^2 r_0 - 1)^{3/2}}. \tag{44}$$

This is the strength of the magnetic field which maintains the electron in the Γ orbit. It remains to be proved that the energy of such field is small enough compared to the matter density ρ as given by (6) in order not to disturb significantly the geometry of the background.

Let us look for the physical components of the stress energy tensor in a local inertial frame. We choose to work in a tetrad e^A_α such that

$$e^A_{(\alpha)} e^B_{(\beta)} \eta_{AB} = g_{\alpha\beta} \tag{45}$$

in which indices $A, B, \ldots$ are tetrad indices, η_{AB} is the Minkowski metric, diag. $(+ - - -)$. Using the cylindrical coordinate system we have

$$\begin{aligned} e^0_{(0)} &= e^1_{(1)} = e^3_{(3)} = a \\ e^0_{(2)} &= \sqrt{2}\, a \sinh^2 r \\ e^2_{(2)} &= a \sinh r \cosh r \end{aligned} \tag{46}$$

and its inverse

$$
\begin{aligned}
e_0^{(0)} &= e_1^{(1)} = e_3^{(3)} = \frac{1}{a} \\
e_2^{(0)} &= -\frac{\sqrt{2}}{a}\frac{\sinh r}{\cosh r} \\
e_2^{(2)} &= \frac{1}{a}\frac{1}{\sinh r \cosh r}\,.
\end{aligned}
\tag{47}
$$

In the tetrad frame the energy of the matter is a constant given by [cf. eq. (6)]

$$T_{00} = \rho = \frac{4}{a^2}\,. \tag{48}$$

The energy of the magnetic field is also given by a constant

$$T_{00}^{(\text{mag.})} = \frac{a^4}{4}\,H_0^2. \tag{49}$$

We should then require

$$H_0^2 \ll \frac{16}{a^6}\,. \tag{50}$$

Substituting the previous results in this inequality yields

$$\frac{\cosh^2 r_0(2\sinh^2 r_0 - 1)}{\sinh r_0(\sinh^2 r_0 - 1)^{3/2}} \ll 2\frac{e}{m}\,. \tag{51}$$

Thus, given the value of the ratio e/m, (51) yields a limit value for r_0 in order not to violate the above requirement. Using the values of the properties of the electron we conclude that the minimal value of the radius r_0 of the orbit Γ is near the critical value r_c. Thus, it suffices to choose a massive charged particle to allow for travel backwards in time, circumventing thus the difficulties pointed out by Malament [5].

7. FIELD THEORY IN CAUSAL DOMAINS: A GENERAL SCHEME

From what we have learned in the preceding sections we conclude that the gaussian coordinates $(\tilde{t}, \tilde{\xi}, \tilde{\eta}, \tilde{z})$ consitute a natural framework in which the evolutionary equations for an arbitrary test field $\Phi(x^\mu)$ should be examined. In this section we present an overview of the standard scheme which allows for such analysis. We consider a real scalar field $\Phi(x^\mu)$ which satisfies the wave equation

$$\Box\Phi = 0 \tag{52}$$

where $\Box$ is the d'Alembertian operator in the metric

$$ds^2 = d\tilde{t}^2 - a^2(\mu^2 - 1)d\tilde{\xi}^2 + g(\tilde{t},\tilde{\xi})d\tilde{\eta}^2 + 2h(\tilde{t},\tilde{\xi})d\tilde{\xi}\,d\tilde{\eta} - d\tilde{z}^2 \tag{53}$$

in which the functions g and h are given by

$$\begin{aligned} g &= -\frac{1}{4}\,\frac{\mu^2 - 1}{(\mu^2+1)^2}\,(1 - \sin M)\,[\mu^2 + 3 + (\mu^2 - 1)\sin M] \\ h &= \frac{\sqrt{2}}{2}\,\frac{\mu^2 - 1}{\mu^2 + 1}\,(1 - \sin M) \end{aligned} \tag{54}$$

and

$$M \equiv \frac{2}{a}\sqrt{\mu^2 + 1}\,(\tilde{t} - \mu a\tilde{\xi}). \tag{55}$$

Using the definition of the d'Alembertian in the metric (53) it follows that

$$\begin{aligned} \Box\Phi = {} & \frac{\partial^2\Phi}{\partial\tilde{t}^2} - 2\frac{\sqrt{\mu^2+1}}{a}\,\tan M\frac{\partial\Phi}{\partial\tilde{t}} \\ & - \frac{1}{a^2(\mu^2-1)(\mu^2+1)}\,\frac{1}{1+\sin M}\,[\mu^2 + 3 + (\mu^2 - 1)\sin M]\frac{\partial^2\Phi}{\partial\tilde{\xi}^2} \\ & - \frac{4\sqrt{2}}{a^2(\mu^2-1)}\,\frac{1}{1+\sin M}\,\frac{\partial^2\Phi}{\partial\tilde{\xi}\partial\tilde{\eta}} \\ & - \frac{2\mu}{\sqrt{\mu^2+1}\,a^2(\mu^2-1)}\,\frac{1}{\cos M(1+\sin M)} \times \\ & \qquad \times [\mu^2 + 3 + (\mu^2-1)(\sin M - \cos^2 M)]\frac{\partial\Phi}{\partial\tilde{\xi}} \\ & - \frac{4\sqrt{2}\,\mu\sqrt{\mu^2+1}}{a^2(\mu^2-1)}\,\frac{1}{\cos M(1+\sin M)}\,\frac{\partial\Phi}{\partial\tilde{\eta}} \\ & - \frac{4}{a}\,\frac{\mu^2+1}{\mu^2-1}\,\frac{1}{\cos M}\,\frac{\partial^2\Phi}{\partial\tilde{\eta}^2} - \frac{1}{a^2}\,\frac{\partial^2\Phi}{\partial\tilde{z}^2} = 0. \end{aligned}$$

In order to solve this equation let

$$\Phi_{nk}(x) = e^{n\tilde{\eta}}e^{-ik\tilde{z}}F(M) \tag{56}$$

in which $F(M)$ is a solution of the equation

$$\alpha(M)F'' + \beta(M)F' + \gamma(M)F = 0 \tag{57}$$

where $F' \equiv dF/dM$ and

$$\alpha(M) \equiv \frac{(\mu^2-1)\sin M - (3\mu^2+1)}{(\mu^2-1)(\mu^2+1)(1+\sin M)}$$

$$\beta(M) \equiv \frac{n2\sqrt{2}\,\mu}{(\mu^2-1)\sqrt{\mu^2+1}\,(1-\sin M)} - \tan M + \frac{\mu^2[(\mu^2+3)+(\mu^2-1)(\sin^2 M + \sin M - 1)]}{(\mu^2-1)(\mu^2+1)\cos M(1+\sin M)}$$

$$\gamma(M) \equiv -\frac{n\sqrt{2}\,\mu}{(\mu^2-1)\sqrt{\mu^2+1}\,\cos M(1+\sin M)} - \frac{n^2}{(\mu^2-1)\cos^2 M} + \frac{k^2}{4(\mu^2+1)}\,. \tag{58}$$

We define the scalar product of functions Φ_1 and Φ_2 in the standard way:

$$(\Phi_1, \Phi_2) = i\int_{\partial\Omega} \Phi_1 \overleftrightarrow{\partial}_\mu \Phi_2 d\Psi^\mu. \tag{59}$$

We know that a foliation of space-like surfaces in $\mathcal{G}_+$ is provided by the hypersurfaces $\Psi \equiv \tilde{t} =$ constant. The time-like vector N_μ normal to these hypersurfaces is given by $N_\mu = \partial_\mu \Psi = (1,0,0,0)$ and thus it yields $d\Psi^\mu = N^\mu \sqrt{|\det \tilde{g}_{\mu\nu}|}\, d\tilde{\xi} d\tilde{\eta} d\tilde{z}$.

Let us point out that the time-like Killing vector which allows the above definition of the scalar product is provided by $\tilde{K}^\alpha = (\mu, 1/a, 0, 0)$ which in the cylinder coordinate system (t, r, ϕ, z) reduces to the form $K^\alpha = (1/a, 0, 0, 0)$ tangent to the velocity flow of the matter. Note that $\sqrt{|\det g_{\mu\nu}|}$ is equal to

$$\sqrt{|\det g_{\mu\nu}|} = \frac{a^3}{2}\,\frac{\mu^2-1}{\mu^2+1}\,[(\mu^2-1)(1-\sin^2 M)]^{1/2}.$$

We are thus prepared to pass to the quantization in $\mathcal{G}_+$ by setting the canonical commutator relation, for instance

$$[\Phi(\tilde{t}_1, \tilde{\xi}_1), \Phi(\tilde{t}_2, \tilde{\xi}_2)]_\Sigma = 0. \tag{60}$$

From the Lagrangian

$$\mathcal{L} = \sqrt{-g}\,\partial_\mu\Phi\partial_\nu\Phi g^{\mu\nu} = \frac{a^2(\mu^2-1)}{2\sqrt{\mu^2+1}}\cos M\left[\left(\frac{\partial\Phi}{\partial\tilde{t}}\right)^2 + \left(\frac{\partial\Phi}{\partial\tilde{\xi}}\right)^2 g^{11} + 2\frac{\partial\Phi}{\partial\tilde{\eta}}\,\frac{\partial\Phi}{\partial\tilde{\xi}}\,g^{12} + \left(\frac{\partial\Phi}{\partial\tilde{\eta}}\right)^2 g^{22} - \frac{1}{a^2}\left(\frac{\partial\Phi}{\partial\tilde{z}}\right)^2\right].$$

We can proceed to obtain the associated moment

$$\Pi = \frac{\delta \mathcal{L}}{\delta(\partial \Phi / \partial \tilde{t})} \tag{61}$$

and the remaining conventional procedure of field theory.

It seems worth pointing out that some authors [7] argue that the standard quantum field theory may not make much sense in a space-time which contains closed time-like lines (CTL) as in Gödel's Universe. Note however that by using the generalized Milne coordinates, it is possible to construct a frame in which the local causal structure of the space-time is explicitly guaranteed. One could envisage applying this method to any space-time which contains CTL. We postpone the complete analysis of this question to a forthcoming paper.

8. CONCLUSION

In this paper we have analysed the main properties of the synchronization mechanism in Gödel's Universe.

The impossibility of constructing a unique global gaussian system of coordinates is related to the confinement properties of the geodesics in this geometry. This unusual behaviour is demonstrated in a direct and simple way through the method of the effective potential introduced earlier to reduce the analysis of the geodesics to the dynamics of a single particle submitted to a central force.

After a review of the geodesics we present a set of gaussian system of coordinates in such a way that their union provides a complete cover for the whole manifold. We have identified the existence of closed time-like lines (CTL) with the property that forbids the extension of a local gaussian frame beyond a certain range, Thus, we recognize the impossibility of establishing a global Cauchy surface. This yields two alternatives; either we abandon the project of constructing a causal chain of events from a given set of initial data, or we should restrict the description of the evolution of a field to a finite region inside the frontiers of a causally related bounded domain of space-time. This has led us to contruct a convenient frame which is the generalization of Milne's idea of special observers. This generalized Milne's observers constitutes a special frame to which a causal field theory can be constructed.

We are thus led to conjecture that it is possible to use this kind of frame in any space-time which contains closed time-like lines in order to provide for a formulation of a causal field theory.

On the other hand, we have presented a mechanism which consists of a combined action of long range fields (electrodynamics and gravity) that induces a real particle—say, an electron—to undergo backwards time travel.

Finally, we argue that going into a quantum version of the theory does not modify the confining character of Gödel's geometry.

APPENDIX. SCHRÖDINGER'S QUANTUM EQUATION

In this appendix we will present an overview of the arguments presented in Section 4 and which aim to anser the following question: Can a quantum mechanism induce a particle — interacting only via the gravitational process — to go beyond the classical confined region [6]?

We have seen in preceding sections that the mechanism of confinement depends only on the behaviour of geodesics under an r-displacement (we will work here in the cylinder coordinate system).

Using the momentum P_μ defined previously we can re-write the equation for the r variable (7). It is sufficient to consider the case in which $B_0 = C_0 = 0$ [cf. eq. (15)]. We then have

$$P_r^2 + D_0^2 + A_0^2 \left[1 - \frac{2}{\cosh^2 r}\right] = 0. \tag{A.1}$$

Following the standard prescription of the first quantization, we will treat P_r as the operator,

$$P_r \to -i\left(\frac{d}{dr} + \frac{1}{r}\right) \tag{A.2}$$

to obtain Schrödinger's equation

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - D_0^2 + \frac{2A_0^2}{\cosh^2 r}\right]\Psi = A_0^2\Psi \tag{A.3}$$

or, setting

$$2mV(r) = D_0^2 - \frac{2A_0^2}{\cosh^2 r} \tag{A.4}$$

$$2mE = -A_0^2 \tag{A.5}$$

thus,

$$-\left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\Psi = 2m(E - V)\Psi. \tag{A.6}$$

We set

$$\Psi = \frac{1}{r}\, u(r) \tag{A.7}$$

to arrive at

$$\left| \frac{d^2}{dr^2} + k^2 + \frac{\lambda(\lambda-1)}{\cosh^2 r} \right| u(r) = 0 \tag{A.8}$$

in which

$$\begin{aligned} \lambda(\lambda-1) &= 2A_0^2 \\ k^2 &= -(D_0^2 + A_0^2). \end{aligned} \tag{A.9}$$

Thus, the dynamical behaviour of a particle in Gödel's background under the standard Schrödinger quantization is reduced to the problem of a particle in a Posch–Teller potential ($A.8$). This potential has a well-known structure, and appears in connection with completely integrable many-body systems in one dimension. It appears also in soliton solutions to the Kortweg–de Vries equations, etc.

Changing to a new variable $y = \cosh^2 r$ (which maps the domain $0 < r < \infty$ into the transformed one $1 < y < \infty$) eq. ($A.8$) can be written in the more fashionable form

$$y(1-y)\frac{d^2u}{dy^2} + \left(\frac{1}{2} - y\right)\frac{du}{dy} - \left(\frac{k^2}{4} + \frac{\lambda(\lambda-1)}{4y}\right) u = 0. \tag{A.10}$$

Write

$$u = y^{1/2} v(y) \tag{A.11}$$

to obtain the equation for v as

$$y(1-y)\frac{d^2v}{dy^2} + [(\lambda+1/2) - (\lambda-1)y]\frac{dv}{dy} - \frac{1}{4}(\lambda^2 + k^2)v = 0. \tag{A.12}$$

Re-defining

$$\begin{aligned} a &= \tfrac{1}{2}(\lambda + ik) \\ b &= \tfrac{1}{2}(\lambda - ik), \end{aligned} \tag{A.13}$$

we re-write this equation for v as

$$y(1-y)\frac{d^2v}{dr^2} + [c - (a+b+1)y]\frac{dv}{dr} - abv = 0 \tag{A.14}$$

with $c = a + b + 1/2$.

This procedure led us to eq. ($A.14$) which can be recognized as a typical equation for hypergeometric functions. The radius of convergence of this series is given by $|y| < 1$.

It is possible to extend analytically this function for the whole complex plane through the cut $[1,\infty)$. Define the variable $z = 1 - y$; then

$$z(1-z)\frac{d^2v}{dz^2} + [1/2 - (a+b+1)z]\frac{dv}{dz} - zbv = 0, \qquad (A.15)$$

the solutions of which are given by

$$v = c_1 F(a,b,1/2,z) + c_2(1-y)^{1/2}F(a+1/2,b+1/2,3/2,z). \qquad (A.16)$$

In order for Ψ to be finite at the origin $r = 0$ we must set $c_1 = 0$. In the other limit $r \longrightarrow \infty$ we can write

$$\Psi \approx \frac{c}{r}e^{-(\lambda+1)}e^{(\lambda+1)r}\Gamma\frac{3}{2}\left[\frac{\Gamma(b-a)}{\Gamma(b+1/2)\Gamma(1-a)}2^{2a+1}e^{-(2a+1)r} + \frac{\Gamma(a-b)}{\Gamma(a+1/2)\Gamma(1-b)}2^{2b+1}e^{-(2b+1)r}\right]$$

and thus

$$\Psi \approx \frac{e^{\lambda r}}{r}e^{-2ar} + \frac{e^{\lambda r}}{r}e^{-2br}. \qquad (A.17)$$

In order to normalize $\Psi(r)$, the first term of the expression above should be null. Thus,

$$1 - a = -n \qquad (A.18)$$

or

$$\tfrac{1}{2} + b = -n. \qquad (A.19)$$

Thus, the Γ function has poles in these values. Setting

$$a = \frac{1}{2}\left(\lambda - \sqrt{D_0^2 + A_0^2}\right)$$
$$\lambda = \frac{1}{2}\left(1 + \sqrt{1 + 8A_0^2}\right) \qquad (A.20)$$
$$E = \frac{-A_0^2}{2m},$$

we can find a functional relation

$$E_n = f(n, D_0, m) \qquad (A.21)$$

where n is an integer and > 0.

We can now interpret particles of mass m following geodesics in Gödel's Universe as quantum particles submitted to a Posch–Teller potential. The functions Ψ, being normalized, permit us to achieve the expression of the energy which depends only on D_0, the mass m and an integer number n.

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