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LETTER TO THE EDITOR

On Spherically Symmetric Perfect Fluid Distributions and Class One Property

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Abstract

It is shown that for a spherically symmetric perfect fluid solution to be of class one, either (i) $\epsilon=0$, or (ii) $\epsilon+R=0$, ϵ and R being respectively the eigenvalue of the Weyl tensor in Petrov's classification and spur of the Ricci tensor. Hence, it is deduced that whereas every conformally flat perfect fluid solution is of class one, the converse is not true in general. However, the converse does hold for all solutions with $\rho=3p$.

The problem of embedding a four dimensional Riemannian manifold, with signature $(+---)$, in higher dimensional pseudo-Euclidean space has attracted considerable attention in recent years in connection with the symmetry properties of elementary particles. Explicit embedding transformations for a number of well known relativistic Riemannian space-times have been given by Rosen [*3*, *4*]. In the present note we show that every spherically symmetric perfect fluid solution is of class one if and only if either (i) $\epsilon=0$, or (ii) $\epsilon+R=0$, where ϵ is the eigenvalue of the trace free Weyl tensor in Petrov's classification and R is the spur of the Ricci tensor.

We take the spherically symmetric line-element

$$ds^2 = -e^{\alpha}\,dr^2 - r^2\,(d\theta^2+\sin^2\theta\,d\phi^2)+e^{\beta}\,dt^2 \tag{1}$$

where α and β are functions of r and t only. Throughout this note we use relativistic units ($c=1$, $G=1$) and denote partial differentiation with respect

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to r and t by a prime and a dot respectively. The non-vanishing components of the Einstein tensor $G_a{}^b$ for the metric (1) are given by

$$\begin{aligned} G_1{}^1 &= (1/r^2)\,[-1+e^{-\alpha}\,(1+r\beta')] \\ G_2{}^2 = G_3{}^3 &= -\epsilon+(1/r^2)\,[1-e^{-\alpha}\,(1+r\alpha'-r\beta')] \\ G_4{}^4 &= (1/r^2)\,[-1+e^{-\alpha}\,(1-r\alpha')] \\ e^{\alpha}\, G_4{}^1 &= -e^{\beta}\, G_1{}^4 = \dot{\alpha}/r \end{aligned} \tag{2}$$

where

$$\begin{aligned} \epsilon = &-(1/4)\, e^{-\alpha}\, [2\beta''+\beta'(\beta'-\alpha')+2r^{-1}(\alpha'-\beta')] \\ &+(1/4)\, e^{-\beta}\, [2\ddot{\alpha}+\dot{\alpha}(\dot{\alpha}-\dot{\beta})]+(1/r^2)(1-e^{-\alpha}) \end{aligned}$$

is the eigenvalue of the trace free Weyl tensor in Petrov's classification as shown by the author [*2*]. It may be mentioned here that $\epsilon=0$ implies all components of the Weyl tensor vanish identically and hence the space-time is conformally flat.

We now assume that the material filling the sphere is a perfect fluid with only radial motion. The stress energy tensor for such a distribution is given by

$$T_a{}^b=(\rho+p)u_a u^b - p g_a{}^b,\ u_a u^a=1,\ u^2=u^3=0. \tag{3}$$

It was shown by Karmarkar [*1*] that a necessary and sufficient condition for a spherically symmetric space-time described by the line-element (1) to be of class one is

$$3(1-e^{-\alpha})^2+r^2(1-e^{-\alpha})(G_4{}^4+G_1{}^1-4G_2{}^2)-r^4(G_1{}^1G_4{}^4-G_1{}^4G_4{}^1)=0. \tag{4}$$

Combining (2) and (3) with the help of Einstein's field equations and substituting in (4) we get

$$\epsilon[\epsilon+8\pi(\rho-3p)]=0$$

or equivalently

$$\epsilon(\epsilon+R)=0.$$

Therefore, for a spherically symmetric perfect fluid solution to be of class one, either

$$\text{(i)}\quad \epsilon=0, \qquad \text{or (ii)}\quad \epsilon+R=0. \tag{5}$$

Hence we have the following result: Every conformally flat perfect fluid solution is of class one. The converse is not true in general although it does hold for all solutions with $\rho=3p$. A number of solutions obtained by Vaidya [*6*, *7*] which include as particular cases some well known isolated and global distributions of matter all satisfy (5(i)) and therefore are of class one. A solution satisfying (5(ii)) has been given by Tikekar [*5*] recently.

References

1. Karmarkar, K. R. (1948). *Proc. Indian Acad. Sci.*, A **27**, 56.
2. Krishna Rao, J. (1966). *Curr. Sci.*, **35**, 589.
3. Rosen, J. (1965). *Rev. Mod. Phys.*, **37**, 204.
4. Rosen, J. (1965). *Nuovo Cimento*, **38**, 631.
5. Tikekar, R. S. (1970). *Curr. Sci.*, **39**, 460.
6. Vaidya, P. C. (1968). *Phys. Rev.*, **174**, 1615.
7. Vaidya, P. C. (1968). *Curr. Sci.*, **37**, 191.