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On the Gravitational Field of a Massless Particle†

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Abstract

The gravitational field of a massless point particle is first calculated using the linearized field equations. The result is identical with the exact solution, obtained from the Schwarzschild metric by means of a singular Lorentz transformation. The gravitational field of the particle is nonvanishing only on a plane containing the particle and orthogonal to the direction of motion. On this plane the Riemann tensor has a δ-like singularity and is exactly of Petrov type *N*.

1. *Introduction*

The interest in gravitational fields generated by sources which move with the velocity of light has increased in the last years, because of their close connection to gravitational waves. The best known source of this type is the electromagnetic radiation field. Already Tolman [*1*] in 1934 studied the gravitational field of light beams and pulses in the linearized theory. However, only in 1959, Peres [*2*] proved the existence of exact solutions to the combined Maxwell–Einstein equations for electromagnetic null fields. Recently a series of papers by Bonnor [*3–5*] discussed the gravitational field produced by null fluids, i.e. fluids moving with the velocity of light. The common result of these papers is that the fields produced by null sources are plane fronted gravitational waves.

In this paper we calculate and discuss the gravitational field from a single photon or, to be more precise, the field of a point particle with zero rest mass moving with the velocity of light.

We derive this field in two ways: first, in Section 2 we solve the linearized

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Einstein field equations for a particle with rest mass m moving uniformly with the velocity v. We investigate the field in the limit $v \to 1$, while we let the mass tend to zero in such a way that the energy of the particle remains finite. We show that, when this limit is carefully investigated, the result is the same as solving the field equations directly with an energy momentum tensor for a massless particle, as source term.

In Section 3 we turn to the full Einstein equations and start with the exact metric for a particle at rest, i.e. the Schwarzschild solution. We apply to this metric a Lorentz transformation to obtain the gravitational field as seen by an observer moving uniformly relative to the mass. However, again the limit $v \to 1$ and $m \to 0$ is not properly defined and can only be carried out after a suitable coordinate transformation. We show in some detail how this is achieved. The remarkable result is that both the linearized solution and the exact solution agree completely.

Physically our result is that the gravitational field of a particle moving with the velocity of light leads to a Riemann tensor $R_{iklm}=0$ everywhere, except on the hypersurface which contains the particle. On this hypersurface some components have a δ-like behaviour. Further we show that the resulting metric corresponds to gravitational plane fronted waves. The static Schwarzschild field is thus transformed into a pure radiation field.

Independently, this is also shown by transforming directly the curvature tensor of the Schwarzschild field. Starting with the canonical form of the Riemann tensor, for which the Schwarzschild field is of Petrov type D, we obtain after taking the limit $v \to 1$, a pure Petrov type N field. Again the curvature tensor shows the characteristic δ-like singularity.

In the Appendix we prove a useful relation for obtaining the limit $v \to 1$, which is needed for the above mentioned calculations.

2. *The Linearized Solution*

From the Einstein field equations†

$$R_{ik}-\tfrac{1}{2}g_{ik}R=8\pi T_{ik} \tag{2.1}$$

one derives by the standard approximation

$$g_{ik}=\eta_{ik}+2h_{ik}$$

with $(h_{ik})^2 \simeq 0$, the linearized field equations

$$\Box \psi^{ik}=8\pi T^{ik} \qquad \text{with} \qquad \psi^{ik}=h^{ik}-\tfrac{1}{2}\eta^{ik}h_l{}^l. \tag{2.2}$$

It is our task here to solve equation (2.2) for a uniformly moving point particle source. The energy momentum tensor for a particle of rest mass

† We use the convention $\eta_{ik}=\text{diag}\,(1, -1, -1, -1)$ and indices are raised and lowered with η_{ik} in this section only. Further we take $c=G=1$.

m moving with constant velocity v in the x-direction is given by :

$$T^{ik}(x)=m(1-v^2)^{-1/2}\,\delta(x-vt)\,\delta(y)\,\delta(z)\,s^i s^k \tag{2.3}$$

with

$$s^i=\delta_0{}^i+v\delta_1{}^i. \tag{2.4}$$

Inserting equation (2.3) into (2.2) one can use the retarded Green function to solve for ψ^{ik}

$$\psi^{ik}(x)=[\{(x-vt)^2+(1-v^2)\,(y^2+z^2)\}\,(1-v^2)]^{-1/2}\,m\,s^i\,s^k. \tag{2.4}$$

For $v=0$ equation (2.4) reduces to the linearized Schwarzschild solution

$$\psi^{ik}=\frac{2m}{r}\,\delta_0{}^i\delta_0{}^k, \qquad \text{with} \qquad r^2=x^2+y^2+z^2.$$

If we let v approach 1, the energy of the particle diverges because of the finite rest mass m. Therefore we write

$$m=p(1-v^2)^{+1/2} \tag{2.5}$$

and keep p constant when taking the limit $v\to 1$. This means that the total energy, p, of the particle is kept constant while its rest mass goes to zero.

However it is not straightforward to obtain ψ^{ik} in the limit $v\to 1$ because

$$\lim_{v\to 1}\{(x-vt)^2+(1-v^2)\,(y^2+z^2)\}^{-1/2} \tag{2.6}$$

is not a tempered distribution [6] and the limit is not defined for all values x^i. To overcome this difficulty one can directly put $v=1$ in the energy tensor equation (2.3) which gives us, taking into account equation (2.5)

$$T^{ik}=p\delta(x-t)\,\delta(y)\,\delta(z)\,\bar{s}^i\bar{s}^k, \qquad \text{with} \qquad \bar{s}^i=s^i\,(v=1). \tag{2.7}$$

But we note that integrating equation (2.2) with equation (2.7) as source term, using the retarded Green function does not lead to the correct result [7]. The reason for this is that the source is moving with the fundamental velocity.

The correct solution for the source term (2.7) is obtained through an ansatz by splitting off the $\delta(x-t)$ function, to give:

$$\psi^{ik}(x)=p\delta(x-t)\,G_2(y,z)\,\bar{s}^i\bar{s}^k+\psi_H{}^{ik} \tag{2.8}$$

where $G_2(y,z)=\ln\,(y^2+z^2)^{1/2}$, is the Green function of the two-dimensional Poisson equation

$$\left(\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)G_2(y,z)=2\pi\,\delta(y)\,\delta(z) \tag{2.9}$$

and $\psi_H{}^{ik}$ is a homogeneous solution of equation (2.2).

In the next section and the Appendix we give a prescription how to obtain a meaningful limiting procedure for (2.6). We prove that this actually leads to the same result given by equation (2.8).

Because all ψ^{ik} are proportional to $\delta(x-t)$ the metric is Minkowskian for $x \neq t$ and

$$R_{iklm}=0$$

i.e. the curvature tensor vanishes everywhere except on the hypersurface $x=t$.

The linearized theory predicts therefore that the gravitational field of a massless point particle has a Riemann tensor which vanishes everywhere except on the hypersurface $t-x=0$ which is determined by the direction of the velocity of the particle. On this hypersurface several components of the Riemann tensor diverge.

However, the linearized theory cannot be applied to the hypersurface $x-t=0$ since the gravitational potentials ψ^{ik} show there a δ-like behaviour. In the next section we shall, therefore, repeat our calculation using the full nonlinear theory of gravitation.

3. *The Exact Solution*

The exact exterior solution for a mass m at rest is the well known Schwarzschild metric, which in isotropic coordinates is given by [8]

$$ds^2=\frac{(1-A)^2}{(1+A)^2}\,dt^2-(1+A)^4\,(dx^2+dy^2+dz^2) \tag{3.1}$$

with $A=m/2r$ and $r^2=x^2+y^2+z^2$.

An observer moving uniformly relative to this mass will see the metric deformed by a Lorentz transformation. If we apply a Lorentz transformation in the x-direction

$$\begin{aligned} \bar{t}&=(1-v^2)^{-1/2}\,(t+vx), & \bar{y}&=y \\ \bar{x}&=(1-v^2)^{-1/2}\,(x+vt), & \bar{z}&=z \end{aligned} \tag{3.2}$$

the line element (3.1) changes to

$$ds^2=(1+A)^2\,(d\bar{t}^2-d\bar{x}^2-d\bar{y}^2-d\bar{z}^2)-\left\{(1+A)^4-\left(\frac{1-A}{1+A}\right)^2\right\}\frac{(d\bar{t}-vd\bar{x})^2}{1-v^2} \tag{3.3}$$

with

$$A=\frac{m}{2r}=\frac{p(1-v^2)}{2\{(\bar{x}-v\bar{t})^2+(1-v^2)\,(\bar{y}^2+\bar{z}^2)\}^{1/2}}.$$

Here we see the same characteristic denominator as in equation (2.6). However, the factor $(1-v^2)$ in A guarantees that

$$\lim_{v\to 1} A=0$$

for all values of t, x, y and z.

Again

$$\lim_{v\to 1}\frac{p}{2\{(\bar{x}-v\bar{t})^2+(1-v^2)\,(\bar{y}^2+\bar{z}^2)\}^{1/2}}=\begin{cases}\dfrac{p}{2\,|\,\bar{x}-\bar{t}\,|} & \text{for} \quad \bar{x}\neq\bar{t} \\ \text{divergent} & \phantom{\text{for}} \quad \bar{x}=\bar{t}\end{cases} \tag{3.4}$$

is only defined for space time points $\bar{t} \neq \bar{x}$, while for $\bar{t} = \bar{x}$ it cannot be written even in terms of generalized functions.

For $\bar{t} \neq \bar{x}$ one easily sees that the line element becomes

$$ds^2 = d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - d\bar{z}^2 - \frac{4p}{|\bar{t}-\bar{x}|}(d\bar{t}-d\bar{x})^2 \tag{3.5}$$

for which

$$\bar{R}_{iklm}(\bar{x}) = 0.$$

In order to carry out the limit $v \to 1$ also for space time points $\bar{t} = \bar{x}$ we make use of the following relation, which we prove in the Appendix

$$\lim_{v\to 1} \{[(\bar{x}-v\bar{t})^2 + (1-v^2)\,\rho^2]^{-1/2} - [(\bar{x}-v\bar{t})^2 + (1-v^2)]^{-1/2}\} = -2\delta(\bar{x}-\bar{t}) \ln \rho$$

with

$$\rho^2 = \bar{y}^2 + \bar{z}^2. \tag{3.6}$$

To apply this relation we need to generate in equation (3.3) a term equal to the second one on the l.h.s. of relation (3.6). This can be achieved by the coordinate transformation $T(v)$

$$T(v):\quad x' - vt' = \bar{x} - v\bar{t} \tag{3.7}$$
$$x' + vt' = \bar{x} + v\bar{t} - 4p \ln [\sqrt{(\bar{x}-v\bar{t})^2 + (1-v^2)} - (\bar{x}-\bar{t})].$$

This transformation leaves the coordinates $\bar{x}$ and $\bar{y}$ and the function A invariant. Since we are interested in the limit $v \to 1$ we may expand the metric in powers of A:

$$\left(\frac{1-A}{1+A}\right)^2 - (1+A)^4 = -4A + 2A^2 - 16A^4 + \ldots$$

When transforming the metric equation (3.3) with $T(v)$ we retain only those powers in $1-v$, which contribute to the metric as v tends to 1. After some calculations we are left with the following metric:

$$ds^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2$$
$$-4p\left\{\frac{1}{\sqrt{(x'-vt')^2 + \rho^2(1-v^2)}} - \frac{1}{\sqrt{(x'-vt')^2 + (1-v^2)}}\right\}(dt' - dx')^2. \tag{3.8}$$

It is now possible to take the limit $v \to 1$ which is given by relation (3.6), to obtain

$$ds^2 = dt'^2 - dx'^2 - dy'^2 - dz'^2 + 8p\delta(t'-x') \ln (y'^2 + z'^2)^{1/2} (dt' - dx')^2. \tag{3.9}$$

Finally, we return to the coordinates $\bar{x}$ with the inverse transformation $[T(v=1)]^{-1}$ which can be read off from equation (3.7). The result is

$$ds^2 = d\bar{t}^2 - d\bar{x}^2 - d\bar{y}^2 - dz^2 - 4p\left\{\frac{1}{|\bar{t}-\bar{x}|} - 2\delta(\bar{t}^2 - \bar{x}^2) \ln (\bar{y} + \bar{z})^{1/2}\right\}(d\bar{t} - d\bar{x})^2. \tag{3.10}$$

For $\bar{t} \neq \bar{x}$ this is exactly the metric equation (3.5) for which $R_{iklm}=0$, since the term $|\bar{t}-\bar{x}|^{-1}$ does not contribute to the Riemann tensor. Because of the appearance of δ-functions in the g_{ik} one should be cautious when working with this metric. However, $\det(g_{ik})=-1$, and the g^{ik} are well defined. Moreover, one notices that the metric (3.10) has the form of a gravitational plane fronted wave, which has been extensively studied [9, 10].

For the metric (3.10) one shows that the non-vanishing components of the curvature tensor are given by:

$$R_{abcd}=\begin{cases} H,_{ac} & \text{for} \quad b=d \\ -H,_{ac} & \text{for} \quad b\neq d \end{cases} \tag{3.11}$$

where a, c can take the values 2 and 3 only while b and d are restricted to 0 and 1.

Calculating R_{iklm} from the metric (3.10) with the help of (3.11) leads to†

$$R_{0202}=4p\ \delta(\bar{t}-\bar{x})\left[\frac{\bar{y}^2-\bar{z}^2}{(\bar{y}^2+\bar{z}^2)^2}+\pi\delta(\bar{y})\ \delta(\bar{z})\right]$$

$$R_{0303}=4p\ \delta(\bar{t}-\bar{x})\left[\frac{\bar{y}^2-\bar{z}^2}{(\bar{y}^2+\bar{z}^2)^2}-\pi\delta(\bar{y})\ \delta(\bar{z})\right]$$

$$R_{0203}=-4p\ \delta(\bar{t}-\bar{x})\frac{2\bar{y}\bar{z}}{(\bar{y}^2+\bar{z}^2)^2}. \tag{3.12}$$

All other components are either vanishing or related to the ones given above by symmetry.

For the Ricci tensor the nonzero components are given by

$$R_{00}=R_{11}=-R_{01}=-(H,_{22}+H,_{33})=8\pi p\delta(\bar{t}-\bar{x})\ \delta(\bar{y})\ \delta(\bar{z})$$

so that the energy tensor has the required form

$$\bar{T}^{ik}(\bar{x})=p\ \delta(\bar{t}-\bar{x})\ \delta(\bar{y})\ \delta(\bar{z})\ \bar{s}^i\bar{s}^k.$$

The reason why it is possible to carry out the limit $v\rightarrow 1$, in the metric equation (3.8) after transforming with $T(v)$ is

$$\lim_{v\rightarrow 1}\frac{\partial}{\partial x^i}T(v)\neq\frac{\partial}{\partial x^i}\lim_{v\rightarrow 1}T(v)=\frac{\partial}{\partial x^i}T(1).$$

Next we shall rederive (3.12) in a different manner.

Instead of transforming the Schwarzschild line element (3.1) we apply the Lorentz transformation (3.2) directly to the components of the Riemann tensor and investigate their behaviour for $v\rightarrow 1$.

For this we work in the Vierbein formalism. The Riemann tensor of the Schwarzschild metric (3.1) can be transformed into canonical form

† When differentiating we use the method of generalized functions [6].

by the choice of a Vierbein λ_α:

$$\lambda_0=\begin{pmatrix}\frac{1+A}{1-A}\\0\\0\\0\end{pmatrix}\qquad \lambda_1=\frac{1}{(1+A)^2\sqrt{x^2+y^2}}\begin{pmatrix}0\\-y\\x\\0\end{pmatrix}$$

$$\lambda_2=\frac{1}{r(1+A)^2\sqrt{x^2+y^2}}\begin{pmatrix}0\\zx\\zy\\-(x^2+y^2)\end{pmatrix}\qquad \lambda_3=\frac{1}{r(1+A)^2}\begin{pmatrix}0\\x\\y\\z\end{pmatrix}. \tag{3.13}$$

The Vierbein components $R_{\alpha\beta\gamma\delta}=R_{iklm}\,\lambda_\alpha{}^i\lambda_\beta{}^k\lambda_\gamma{}^l\lambda_\delta{}^m$ of the Riemann tensor take then, written in six-dimensional space,

$$R_{\alpha\beta\gamma\delta}=R_{AB}\quad \alpha\beta\to A,\quad \gamma\delta\to B,\quad A,B=0,1,\ldots,6$$

the form

$$R_{AB}=\begin{pmatrix}P & Q\\ Q & -P\end{pmatrix}$$

$$P=\begin{pmatrix}\alpha & & 0\\ & -\alpha/2 & \\ 0 & & -\alpha/2\end{pmatrix}\qquad Q=[0] \tag{3.14}$$

where the scalar invariant $\alpha=2m/r^3$. Thus the field is of type D [9]. From equation (3.14) we deduce that the Ricci tensor vanishes everywhere. However, equation (3.14) is only valid for space-time points where $r\neq 0$; for $r=0$ the curvature tensor has an essential singularity and δ-like terms should be added in order that the Einstein equations are satisfied for a point source.

For simplicity we restrict ourselves to the exterior field, i.e. $r\neq 0$.

The Vierbein frame (3.12) is at rest relative to the Schwarzschild singularity and on the plane $x=0$, λ_1 is parallel to the x-axis. In order to calculate the Riemann tensor as seen by an observer moving in the x-direction with velocity v, we have to apply a Lorentz rotation to (3.13), given by

$$\lambda_{\bar\alpha}=\Lambda_{\bar\alpha}{}^\beta\,\lambda_\beta\qquad \Lambda_{\bar\alpha}{}^\beta=\begin{pmatrix}1 & v & & \\ v & 1 & & \\ & & \sqrt{1-v^2} & \\ & & & \sqrt{1-v^2}\end{pmatrix}\frac{1}{\sqrt{1-v^2}}. \tag{3.15}$$

The components of R_{AB} transform under this rotation according to

$$R_{AB}=L_A{}^C L_B{}^D R_{CD} \tag{3.16}$$

where

$$L_A{}^B=\tfrac{1}{2}[\Lambda_{\bar\alpha}{}^\gamma\Lambda_{\bar\beta}{}^\delta-\Lambda_{\bar\alpha}{}^\delta\Lambda_{\bar\beta}{}^\gamma].$$

The local coordinates $\bar{t}$ and $\bar{x}$ of an observer who is at rest with respect to the transformed Vierbein are parallel to $\alpha_{\bar{0}}$ and $\lambda_{\bar{1}}$. The quantity $\alpha(x)$ contained in (3.14) becomes a function of the new coordinates $\bar{x}$:

$$\alpha(\bar{x}) = 2p(1-v^2)\,[(\bar{x}-v\bar{t})^2+(1-v^2)\,(\bar{y}^2+\bar{z}^2)]^{-3/2}.$$

Now $\lim_{v\to 1} \alpha(\bar{x})$ is well defined (see App.) and gives:

$$\lim_{v\to 1} \alpha(\bar{x}) = -2p\,\delta(\bar{t}-\bar{x})\,[\bar{y}^2+\bar{z}^2]^{1/2}. \tag{3.17}$$

Performing the Vierbein transformation (3.15) with the aid of (3.16) and inserting for $\alpha(\bar{x})$ in (3.14), we finally obtain the Riemann tensor for points where $\bar{y}^2+\bar{z}^2 \neq 0$:

$$\bar{P} = \frac{p\delta(\bar{t}-\bar{x})}{(\bar{y}^2+\bar{z}^2)^{1/2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad \bar{Q} = \frac{p\delta(\bar{t}-\bar{x})}{(\bar{y}^2+\bar{z}^2)^{1/2}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{3.18}$$

In accordance with (3.12) we obtain the δ-like behaviour of R_{iklm} on the hypersurface $\bar{x}=\bar{t}$. The functions $\delta(\bar{y})\,\delta(\bar{z})$ are missing in equation (3.18) because of the restriction to values $\bar{y}^2+\bar{z}^2 \neq 0$. Therefore $\bar{R}_{ik}(\bar{x})=0$, i.e. we have only the exterior solution here. From equation (3.18) we see that the field is exactly Petrov type N.

4. *Discussion*

We first note that transforming the exact Schwarzschild metric (3.1) has led us to the same result as obtained from the linearized theory, i.e. equation (3.10) and equation (2.8) respectively. This result is mathematically expected because $\lim A \to 0$ for $v \to 1$, which corresponds to a linearization of the metric (3.1).

Secondly, we see that the metric (3.10) is of the type of a plane fronted gravitational wave. This result is confirmed by showing that the Lorentz transformed Riemann tensor of the exterior Schwarzschild field becomes a pure Petrov type N field. Pirani [*11*] has previously pointed out that the leading term of the gravitational field of a fast moving particle, is of type N, although the exact type remains D. In our case the type changes from D to N, which is due to the singular character of the Lorentz transformation for $v \to 1$.

Physically the gravitational field of a rapidly moving particle shows the same characteristic behaviour as its electromagnetic field: it is dilated in the direction orthogonal to the particles motion and compressed in the direction of the motion. This can be seen by investigation of the quantity $\alpha(x)$ in equation (3.17) or in equation (2.7) for large values of v. In the limit of a massless point particle moving with the speed of light, this compression becomes extreme and the field is non-vanishing only on a plane containing the particle.

The metric (3.10) represents therefore a pulse of a plane fronted gravitational wave. The gravitational field travels along with the particle and being zero everywhere except at the hypersurface $t=x$. We should mention that Bonnor [*3*] and Penrose [*12*] have independently discussed this type of a gravitational wave pulse.

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Appendix

In this appendix we sketch the proof of relation (3.6) which is essential for carrying out the limit $v\to 1$ in the metric (3.1).

If one defines

$$f_v(x^i)\equiv[(x-vt)^2+(1-v^2)(y^2+z^2)]^{-1/2}-[(x-vt)^2+(1-v^2)]^{-1/2} \quad (A.1)$$

the relation (3.6) reads

$$\lim_{v\to 1} f_v(x^i)=-2\delta(t-x)\ln(y^2+z^2)^{1/2} \quad (A.2)$$

to be valid in the sense of generalized functions for all values of t, x, y and z. We prove this by showing that

(i) $$\lim_{v\to 1} F_v(x^i)=-2\theta(t-x)\ln(y^2+z^2)^{1/2} \quad (A.3)$$

where

$$F_v(x^i)=\int_{-\infty}^{x}\mathrm{d}x'\,f_v(t,x',y,z).$$

This relation has to be valid pointwise, i.e. for almost all points.

(ii) There exists a local integrable function $h(x^i)$ independent of v for which

$$|F_v(x^i)|<h(x^i)$$

for all values if v is valid.

Integrating $f_v(x^i)$ with respect to x we obtain for $F_v(x^i)$

$$F_v(x^i)=\ln\left\{\frac{(x-vt)+[(x-vt)^2+(1-v^2)(y^2+z^2)]^{1/2}}{(x-vt)+[(x-vt)^2+(1-v^2)]^{1/2}}\right\}-\ln(y^2+z^2) \quad (A.4)$$

thus

$$\lim_{v\to 1} F_v(x^i)=\begin{cases}-\ln(y^2+z^2) & \text{for } x-t>0\\ 0 & \text{for } x-t<0\end{cases} \quad (A.5)$$

which proves (i) for all values except

$$x=t \quad \text{and} \quad y^2+z^2=0.$$

Further, one shows that

$$|F_v(x^i)|\le|\ln(y^2+z^2)|. \quad (A.6)$$

This is seen from relation (A.4) by discussing the four possible cases: $x-vt \gtrless 0$ and $y^2+z^2 \gtrless 1$. The extrema of $F_v(x^i)$ are at the boundary of the v interval, i.e. for $v=0$ and $v=1$. Because $\ln(y^2+z^2)$ is locally integrable this completes the proof of (ii) and thus of relation (A.2). In the same way one verifies equation (3.17).

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