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Null Geodesics in Black Hole Metrics with Non-Zero Cosmological Constant

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We study the radial motion along null geodesics in the Reissner-Nordström-de Sitter and Kerr-de Sitter space-times. We analyze the properties of the effective potential and we discuss circular orbits. We find that: 1) the radius of circular photon orbits in the Reissner-Nordström-de Sitter space-times does not depend on the cosmological constant. We show also how this is related to properties of the optical reference geometry. 2) For a specific range of the cosmological constant, photons with high impact parameter may travel radially between the cosmological horizon and the black hole horizon in the equatorial plane of the Kerr-de Sitter space-times.

1. INTRODUCTION

Investigations of the large scale structure of the universe suggest that the cosmological constant Λ is probably non-zero, although very small ($\Lambda \leq 10^{-55}\,\mathrm{cm}^{-2}$). Even the presence of such a small Λ will influence the properties of the geometries describing black holes, because they are then asymptotically de Sitter and not flat. On the other hand, bubbles of false vacuum with large absolute values of the vacuum energy-density are sometimes considered in recent cosmological models [1,2]. Black holes could form inside such bubbles and therefore it is worth considering black

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hole space-times with arbitrarily high values of Λ. The properties of such space-times can be investigated by studying their geodesic structure, that is from the motion of test particles or photons.

The equations describing the geodesic motion in black hole space-times with $\Lambda \neq 0$ were obtained by Carter (Ref. 3, p.57). A detailed discussion of the motion was restricted to the simplest case of the Schwarzschild-de Sitter space-time [4-6]. The latitudinal motion in the Kerr-de Sitter geometry was investigated by Stuchlík [5].

In this paper we consider space-times which are characterized not only by the mass parameter and by the cosmological constant (both $\Lambda > 0$ and $\Lambda < 0$ cases), but also by the charge parameter (Reissner-Nordström-de Sitter geometry) or by the rotation one (Kerr-de Sitter geometry). Our main goal is to describe and analyze the effective potential for radial motion along null geodesics, and to discuss their properties.

2. THE REISSNER-NORDSTRÖM-DE SITTER GEOMETRY

In the standard Schwarzschild coordinates, the Reissner-Nordström-de Sitter geometry is described by the line element

$$ds^2 = -\frac{\Delta_r}{r^2}\,dt^2 + \frac{r^2}{\Delta_r}\,dr^2 + r^2(d\theta^2 + \sin^2\theta\,d\phi^2) \tag{1}$$

where

$$\Delta_r = -\frac{\Lambda}{3}\,r^4 + r^2 - 2Mr + Q^2. \tag{2}$$

M is the mass parameter of the space-time and Q its electric charge. However it is more convenient to use dimensionless coordinates and parameters, defining $y \equiv \Lambda M^2/3$ and expressing all quantities in units of M.

Due to the spherical symmetry of the metric we can always consider the geodesic motion in the equatorial plane $\theta = \pi/2$. The radial motion along null geodesics is then given by (cf. eq (10) in Ref. 5)

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{r^4}\,R(r,y,Q) \equiv \frac{1}{r^4}\,\left[r^4E^2 - \Delta_r\Phi^2\right], \tag{3}$$

E and Φ being the constants of motions. Defining then the impact parameter $\ell = \Phi/E$ it follows from eq. (3) that the r-motion is allowed only in those regions where

$$\ell^2 \leq \ell^2_{\text{eff}}(r,y,Q) \equiv \frac{r^4}{\Delta_r}, \tag{4}$$

ℓ^2_{eff} being the effective potential that determines the turning points of the radial motion.

We now first discuss the existence of horizons which are given by the condition $\Delta_r = 0$. The loci of the horizons can be determined studying the function

$$y = y_h(r, Q) \equiv \frac{r^2 - 2r + Q^2}{r^4}, \tag{5}$$

considering Q as a parameter. The relevant cases are:

a) $Q = 0$
(Schwarzschild-de Sitter geometry) The loci of the horizons can be given in a simple analytic form, see [5]. The subcases are:

$y < 0$: there is only one black hole horizon located at $r_{\text{bh}} < 2$. The space-time is static for $r > r_{\text{bh}}$.

$y = 0$: it corresponds to the Schwarzschild geometry with horizon at $r_{\text{bh}} = 2$.

$0 < y < 1/27$: there are two horizons which coincide for $y = 1/27$. The black hole horizon is at r_{bh}, the cosmological one at $r_{\text{ch}} > r_{\text{bh}}$. The spacetime is static for $r_{\text{bh}} < r < r_{\text{ch}}$.

$y > 1/27$: there are no horizons. The space-time is dynamic everywhere.

b) $0 < Q^2 < 1$.
In this case the extreme points of $y_h(r, Q)$ are located at $r_{c\pm} = (3/2)\left[1 \pm (1-(8/9)Q^2)^{1/2}\right]$. Denoting by $y_{\min}$ and $y_{\max}$ the values of y at the extrema we have the following cases:

$y < y_{\min} < 0$: there are no horizons ($\Delta_r > 0$ at $r > 0$). The geometry is static at $r > 0$ and describes a naked singularity in an anti-de Sitter universe.

$y_{\min} < y < 0$: there are two black hole horizons $r_{\text{bh}\pm}$. The geometry is static for $r < r_{\text{bh}-}$, $r > r_{\text{bh}+}$ and describes a black hole in an anti-de Sitter universe.

$0 < y < y_{\max}$: there are three horizons with $r_{\text{bh}-} < r_{\text{bh}+} < r_{\text{ch}}$. The geometry is static for $r < r_{\text{bh}-}$, $r_{\text{bh}+} < r < r_{\text{ch}}$ and corresponds to a black hole in a de Sitter universe.

$y > y_{\max}$: there is only one cosmological horizon. The geometry is static for $r < r_{\text{ch}}$ and corresponds to a naked singularity in a de Sitter universe.

c) $1 < Q^2 < 9/8$.
Now it is $y_{\min} > 0$. The subcases are:

$y < 0$: no horizons. A naked singularity in an anti-de Sitter universe.

$0 < y < y_{\rm min}$: one horizon ($r_{\rm ch}$). A naked singularity in a de Sitter universe.

$y_{\rm min} < y < y_{\rm max}$: three horizons. A black hole in a de Sitter universe.

$y > y_{\rm max}$: one horizon ($r_{\rm ch}$). A naked singularity in a de Sitter universe.

d) $Q^2 > 9/8$.

There are no local extrema of $y_h(r, Q)$ (if $Q^2 = 9/8$ then $y_{\rm max} = y_{\rm min} = 2/27$). The subcases are:

$y < 0$: No horizons, the geometry is static for all $r > 0$. A naked singularity in an anti-de Sitter universe.

$y > 0$: one horizon ($r_{\rm ch}$). The geometry is static for $0 < r < r_{\rm ch}$. A naked singularity in a de Sitter universe.

The effective potential (4) is well defined only in the static region $\Delta_r > 0$. It diverges at the horizons and it is zero at $r = 0$. In the case of attractive cosmological constant ($y < 0$) it is $\ell^2_{\rm eff}(r \to \infty, y, Q) = -1/y$. The local extrema of the potential, which determine the circular null geodesics, are given by the condition

$$r^2 - 3r + 2Q^2 = 0 \tag{6}$$

which does not depend on the cosmological constant y. It follows that also the radii of the circular null geodesics are also independent of y. Circular geodesics are thus located at r_{c+} and at r_{c-}, that is at the same radii as the extrema of $y_h(r, Q)$. Since r_{c-} is located between the horizons, there will be circular null geodesics at r_{c+} only, if $y_{\rm min} < y < y_{\rm max}$. For $y < y_{min}$ circular geodesics do exist at both r_{c+} and at r_{c-}. Contrary to the radius of the circular photon orbits, the impact parameter of these orbits *depends* on the cosmological constant

$$\ell^2_{c\pm}(Q, y) = \frac{r^4_{c\pm}}{-y r^4_{c\pm} + r_{c\pm} - Q^2}\,.$$

3. THE KERR-DE SITTER GEOMETRY: EQUATORIAL NULL GEODESICS

In Boyer-Lindquist coordinates, the Kerr-de Sitter geometry is described by the line element

$$\begin{aligned} ds^2 = &-\frac{\Delta_r}{I^2\rho^2}(dt - a\sin^2\theta\, d\phi)^2 \\ &+ \frac{\Delta_\theta \sin^2\theta}{I^2\rho^2}[a\,dt - (r^2 + a^2)d\phi]^2 + \frac{\rho^2}{\Delta_r}dr^2 + \frac{\rho^2}{\Delta_\theta}d\theta^2, \end{aligned} \tag{7}$$

where

$$\Delta_r = \left(1 - \frac{\Lambda}{3} r^2\right)(r^2 + a^2) - 2Mr, \tag{8a}$$

$$\Delta_\theta = 1 + \frac{1}{3} a^2 \Lambda \cos^2\theta, \tag{8b}$$

$$I = 1 + \frac{1}{3} a^2 \Lambda, \tag{8c}$$

$$\rho^2 = r^2 + a^2 \cos^2\theta; \tag{8d}$$

$a = J/M$ denotes the specific angular momentum of the source.

Since we restrict ourselves to the study of equatorial null geodesics, the radial motion will be governed by (see eq. (10) in Ref. 5)

$$\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{r^4}$$

$$R(r, y, a) \equiv \frac{1}{r^4}\{I^2[((r^2 + a^2)E - a\Phi)^2 - \Delta_r(aE - \Phi)^2]\}. \tag{9}$$

As before we introduce the impact parameter $\ell = \Phi/E$, but it is more convenient to use a new one defined as: $X = (\ell - a)$. Radial motion is then allowed only if

$$X \geq X_+(r, y, a) \quad \text{or} \quad X \leq X_-(r, y, a); \tag{10}$$

the two effective potentials $X_\pm(r, y, a)$ are given by the simple relation

$$X_\pm = \frac{r^2}{a \mp \sqrt{\Delta_r}}. \tag{11}$$

In the following we assume $a \geq 0$.

We again first discuss the existence of horizons in the space-time (7), varying the parameters y and a. We describe only the main steps of the procedure which is straightforward. The loci of the horizons are determined by the condition

$$y = y_{\mathrm{h}}(r, a) \equiv \frac{r^2 - 2r + a^2}{r^2(r^2 + a^2)}. \tag{12}$$

The local extrema of $y_{\mathrm{h}}(r, a)$ are determined by the equation

$$r^3(r - 3) + 2a^2r^2 - a^2r + a^4 = 0 \tag{13}$$

so that for a given a the extrema are determined by

$$a^2 = a_{\rm h}^2(r) \equiv \frac{r}{2}\left[1 - 2r + (8r+1)^{1/2}\right]. \tag{14}$$

The maximum of the function $a_{\rm h}^2(r)$ is at $r_{\rm cr} = (3+2\sqrt{3})/4$ and the corresponding critical value of the parameter a^2 is: $a_{\rm cr}^2 = (1/16)\{2(3+2\sqrt{3})(7+4\sqrt{3})^{1/2} - \sqrt{3}(8+5\sqrt{3})\} \approx 1.212$.

If $a^2 < a_{\rm cr}^2$ the function $y_{\rm h}(r,a)$ has two local extrema $y_{\rm min}(a)$ and $y_{\rm max}(a)$ which can be determined graphically by combining the functions (12) and (14). If $a^2 = a_{\rm crit}^2$ these extrema coincide at $y_{\rm cr} = 16[(7+4\sqrt{3})^{1/2} -3]\{3(7+4\sqrt{3})[1+(7+4\sqrt{3})^{1/2}]\}^{-1} \approx 0.0592$.

The discussion of the existence of horizons and of their character, as a function of the parameters y and a, is formally the same as in the case of the Reissner-Nordström-de Sitter metric.

We are now able to discuss the behaviour of the effective potentials $X_\pm(r,y,a)$. Clearly they are well defined only in the stationary regions $\Delta_r > 0$. However, contrary to the Reissner-Nordström-de Sitter case, they do not diverge at the horizons, but only $X_+(r,y,a)$ diverges somewhere between the black hole and the cosmological horizons, if it diverges at all. The loci of divergence of the potential X_+ are given by

$$y = y_{\rm d} \equiv \frac{r-2}{r(r^2+a^2)}\,. \tag{15}$$

Since the condition $\partial X_\pm/\partial r = 0$ implies that the equation

$$y^2a^4r^3 + 2ya^2r^2(r+3) + r(r-3)^2 - 4a^2 = 0 \tag{16}$$

must be satisfied, the relation

$$\begin{aligned} y &= y_{\rm ex\pm}(r,a) \\ &\equiv \frac{1}{a^2r^2}\left\{-r(r+3) \pm 2[r(3r^2+a^2)]^{1/2}\right\} \end{aligned} \tag{17}$$

gives the local extrema of the effective potentials $X_\pm(r,y,a)$, i.e. the circular photon orbits. At the horizons the effective potentials coincide and their value is

$$X_\pm(r_{\rm b(c)h}) = \frac{r_{\rm b(c)h}^2}{a}. \tag{18}$$

In order to study the behaviour of the functions $X_\pm(r,y,a)$, we must first discuss the functions $y_{\rm d}(r,a)$ and $y_{\rm ex\pm}(r,a)$. The function $y_{\rm d}(r,a)$

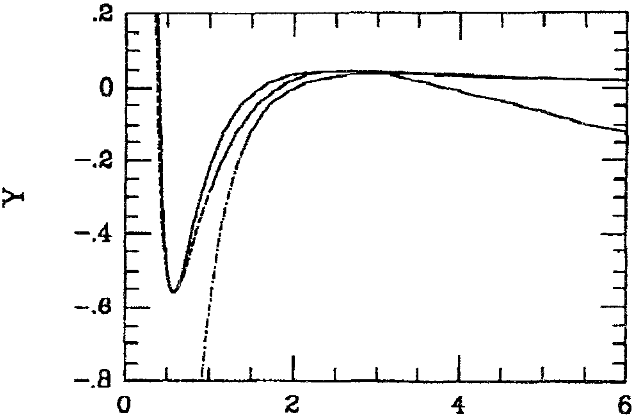

Figure 1. The functions $y_{\rm h}(r,a)$ (solid line), $y_{\rm extr+}(r,a)$ (dashed line) and $y_{\rm d}(r,a)$ (dot–dashed line) are shown for $a = 0.8$. They govern the behaviour of the effective potentials $X_\pm(r,y,a)$. Note the important, although small, region of y in the interval $y_{\rm d-max} < y < y_{\rm h-max}$. These values of y give rise to the most interesting features of the motion. The function $y_{\rm extr-}(r,a)$ is not shown; it monotonically goes from $-\infty$ at $r = 0$ and for $r \to +\infty$ it has the same horizontal asymptot as $y_{\rm extr+}$.

diverges at $r = 0$ to $-\infty$; it is zero at $r = 0$ and goes to zero again for $r \to \infty$. Therefore $X_+(r,y,a)$ can diverge also for naked singularities in a de Sitter space-time and it must always diverge in an anti-de Sitter space-time for both black holes and naked singularities. Its local maximum is determined by the condition

$$a^2 = a_{\rm d}^2 \equiv r^2(r-3). \tag{19}$$

For each value of $a > 0$ we have one maximum of $y_{\rm d}(r,a)$, which we denote as $y_{\rm d-max}(a)$.

The functions $y_{\rm ex\pm}(r,a)$ diverge at $r = 0$ ($y_{\rm ex+} \to +\infty$, $y_{\rm ex-} \to -\infty$). The relation

$$a^2 = a^2_{{\rm ex}(z)}(r) \equiv \frac{r}{4}(r-3)^2 \tag{20}$$

yields their zeros, i.e. the loci of the photon circular orbits in the Kerr space-time [7,8]. Moreover, since

$$\frac{\partial y_{\rm ex}}{\partial r} = \frac{3r}{a^2 r^3[r(3r^2+a^2)]^{1/2}}\left\{[r(3r^2+a^2)]^{1/2} \mp (r^2+a^2)\right\} \tag{21}$$

only $y_{\rm ex+}(r,a)$ has local extrema, which are given by:

$$a^2 = a^2_{{\rm ex}(e)}(r) \equiv \frac{r}{2}\left\{1 - 2r \pm (8r+1)^{1/2}\right\} \equiv a_{\rm h}^2(r). \tag{22}$$

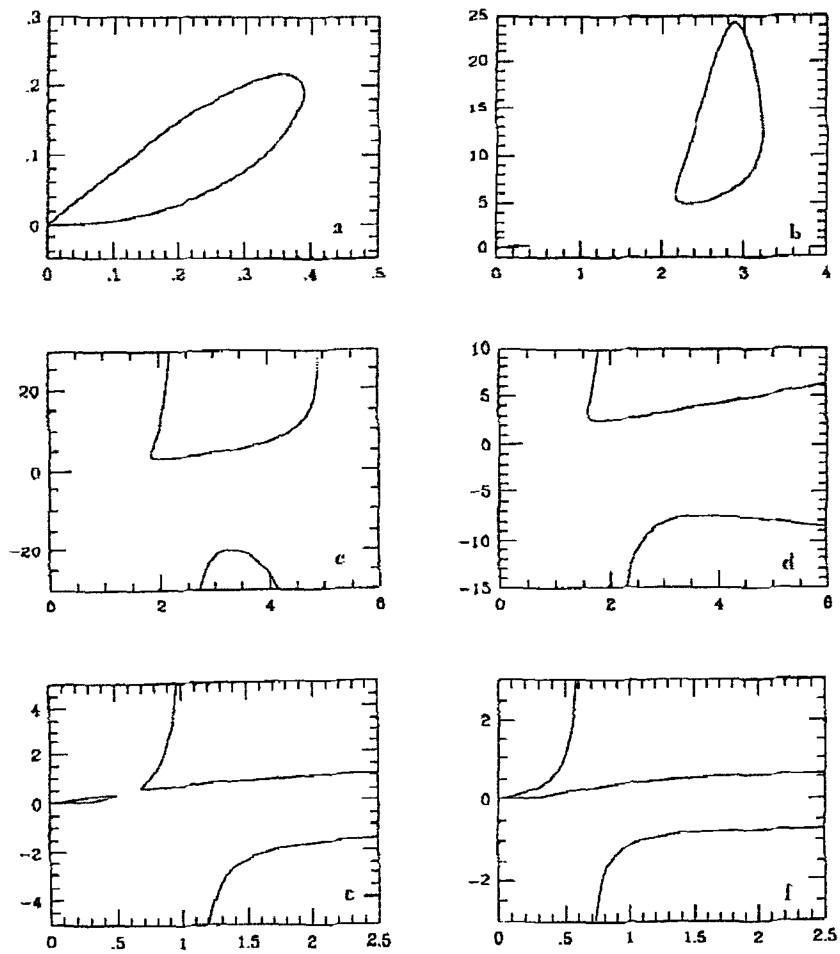

Figure 2. The effective potentials $X_{\pm}(r, y, a)$ are plotted for typical values of the parameter y and for fixed $a = 0.8$. The various figures correspond to: $y = 0.1$ (a); $y = 0.04$ (b); $y = 0.025$ (c); $y = 0$ (d); $y = -0.5$ (e) and $y = -2$ (f). In the most interesting case (b) the minimum of the curve is denoted with $X_{\rm cr1}$ in the text, while the maximum with $X_{\rm cr2}$. The repulsing barriers between the black hole and the cosmological horizon disappear for photons with $X < X_{\rm cr1}$ and $X > X_{\rm cr2}$.

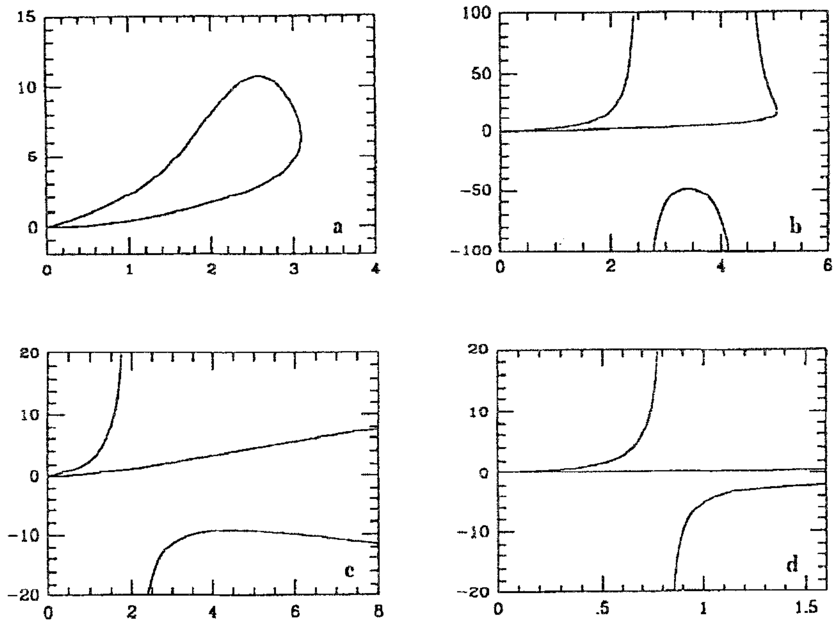

Figure 3. The same as in Fig. 2 but for $a = 1.5$. The various figures correspond to: $y = 0.05$ (a); $y = 0.025$ (b); $y = 0$ (c) and $y = -0.5$ (d).

Therefore the local extrema of the two functions $y_{\mathrm{h}}(r, a)$ and $y_{\mathrm{ex}+}(r, a)$ coincide.

The functions $y_{\mathrm{h}}(r, a)$, $y_{\mathrm{d}}(r, a)$ and $y_{\mathrm{ex}\pm}(r, a)$ are plotted in Fig. 1 for two values of the parameter a, namely $a = 0.8$ and $a = 1.5$, which give qualitatively different behaviour of y_{h} and $y_{\mathrm{ex}+}$. By means of Fig. 1 we can now easily find out the qualitatively different cases for the behaviour of the effective potentials $X_{\pm}(r, y, a)$. They are plotted in two sequences characterized by the value of y. The first sequence is drawn in Fig. 2 for $a = 0.8$ and the second one in Fig. 3 for $a = 1.5$. For comparison we have included also the *pure* Kerr cases $y = 0$.

4. NULL GEODESICS AND THE OPTICAL REFERENCE FRAME

In Section 2 we have shown that the radii of circular null geodesics in the Reissner-Nordström-de Sitter geometry do not depend on the value of the parameter y. This fact can be explained in a way similar to the one used in [9] for the Schwarzschild-de Sitter geometry, by means of the optical reference geometry studied in [10,11].

In static space-times with a time Killing vector $\xi_{(t)} = \partial/\partial t$ the optical reference geometry with metric components $\widetilde{g}_{ik}$ is defined by the relation

$$ds^2 = \Psi(-dt^2 + \widetilde{g}_{ik}dx^i dx^k), \tag{23}$$

where the conformal factor Ψ is given by:

$$\Psi = -\xi_{(t)} \cdot \xi_{(t)} = -g_{tt}. \tag{24}$$

Spacelike projections of the photon trajectories are geodesic lines in the optical reference geometry, which means that the geodesic curvature radius $\widetilde{\mathcal{R}}$ is infinite in each point of the projected photon trajectories in the optical geometry [11].

In the case of the Reissner-Nordström-de Sitter geometry we have $\Psi = \Delta_r/r^2 = \left(1 - 2/r + Q^2/r^2 - yr^2\right)$, and $\widetilde{g}_{rr} = 1/\Psi^2$, $\widetilde{g}_{\theta\theta} = r^2/\Psi$, $\widetilde{g}_{\phi\phi} = r^2 \sin^2\theta/\Psi$.

Circles in the optical geometry, i.e. trajectories of the axial Killing vector $\widetilde{\xi}_{(\phi)}$, have a proper radius

$$\widetilde{r} = (\widetilde{\xi}_{(\phi)} \cdot \widetilde{\xi}_{(\phi)})^{1/2} = \frac{r\sin\theta}{\Psi^{1/2}}. \tag{25}$$

The geodesic curvature $\widetilde{\mathcal{R}}$ of the circles $\widetilde{r} = \text{const.}$ is given by [11]:

$$\widetilde{\mathcal{R}}^{-2} = \frac{1}{\widetilde{r}^2}(\widetilde{\nabla}_i\widetilde{r})(\widetilde{\nabla}_k\widetilde{r})\widetilde{g}^{ik}. \tag{26}$$

Considering *main circles*, i.e. circles in the equatorial plane ($\theta = \pi/2$) which have the origin of coordinates as their center, we find that

$$\widetilde{\mathcal{R}} = r\left(1 - \frac{3}{r} + \frac{2Q^2}{r^2}\right)^{-1}. \tag{27}$$

We see that the geodesic curvature of main circles does not depend on y. Therefore it follows that also the radius of the circular photon orbits (which, we stress, are just main circles with $\widetilde{\mathcal{R}} \to \infty$) do not depend on y. So the property of the optical geometry, namely the independence of $\widetilde{\mathcal{R}}$ on y, can be considered as an explanation of the independence of $r_{c\pm}$ on y itself. On the other hand circles outside the equatorial plane (*not* main circles) have geodesic curvature which depends on the cosmological constant

$$\widetilde{\mathcal{R}} = r\left[\left(1 - \frac{3}{r} + \frac{2Q^2}{r^2}\right)^2 + \left(1 - \frac{2}{r} + \frac{Q^2}{r^2} - yr^2\right)\cot^2\theta\right]^{-1/2}. \tag{28}$$

However such circles are not geodesics of the optical geometry and cannot correspond to photon circular orbits.[5]

For a stationary (rotating) space-time the optical reference geometry is given in a more complicated way [10]:

$$ds^2 = \Psi\left[-(dt + 2\alpha_i dx^i)^2 + \widetilde{g}_{ik}dx^i dx^k\right], \tag{29}$$

where the off-diagonal terms $\alpha_i = -g_{ti}/2\Psi$ are the contribution of the rotation of the source to the gravitational field. In the equatorial plane of the *stationary regions* of the Kerr-de Sitter space-time we have: $\Psi = (\Delta_r - a^2)/I^2a^2$, $\widetilde{g}_{rr} = I^2r^4/\Delta_r(\Delta_r - a^2)$, $\widetilde{g}_{\theta\theta} = I^2/(\Delta_r - a^2)$, $\widetilde{g}_{\phi\phi} = [(r^2 + a^2 - a^2\Delta_r)(\Delta_r - a^2) + a^2(r^2 + a^2 - \Delta_r)^2]/(\Delta_r - a^2)^2$.

The proper circumferential radius of the main circles ($\theta = \pi/2$) is given by

$$\tilde{r} = \frac{r^2\Delta_r^{1/2}}{(\Delta_r - a^2)}.$$

One can then show that

$$\widetilde{\mathcal{R}} = \frac{Ir^2\{[-yr^2(r^2 + a^2) + r(r-2)][-yr^2(r^2 + a^2) + r^2 - 2r + a^2]\}^{1/2}}{[y^2a^2r^3(r^2 + a^2) - yr^2(r^3 - 3r^2 - 5a^2) + r^3 - 5r^2 + 6r - 2a^2]}. \tag{30}$$

In this case even the geodesic curvature of the main circles depends on the cosmological constant. However this is not too surprising as *in stationary space-times the photon trajectories do not coincide with the geodesics of the optical reference geometry.* Therefore circles with $\widetilde{\mathcal{R}} \to \infty$ do not correspond to circular photon orbits. As shown in [11] this is due to the fact that in the generalized version of the force equation, a new term arises (that corresponds to a Coriolis type force) as a result of the rotation of the source.

Note that, contrary to the case of static space-times, for *stationary* space–times there is no simple connection between the optical geometry and the motion of photons.

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[5] The connection between the photon motion and the optical geometry can be presented in a more obvious form, as shown by Abramowicz, Miller and Stuchlik (1990, in preparation). For static, spherically symmetric space-times

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2,$$

the photon motion is determined by the effective potential $\mathcal{V} = 1/\tilde{r}^2 = -g_{tt}/g_{\phi\phi}$. Therefore the derivative of the effective potential and the geodesic curvature of main circles $\tilde{r} =$ const are related by: $(d\mathcal{V}/dr)^2 = 4g_{tt}g_{rr}/(g_{\phi\phi}\widetilde{\mathcal{R}}^2) = (4/\tilde{r}^6)(d\tilde{r}/dr)^2$.

5. CONCLUSIONS

We have shown that the radius of circular photon orbits in a spherically symmetric Reissner-Nordström (Schwarzschild)-de Sitter space-time does not depend on Λ. We have moreover described the interesting relation between this property and the fact that in the optical reference geometry, defined in the static regions of these space-times, the geodesic curvature $\widetilde{\mathcal{R}}$ of the main circles $\tilde{r} = \text{const.}$, $\theta = \pi/2$ (which have the origin of coordinates as their center) does not depend on the cosmological constant.

In the Kerr-de Sitter space-times the loci of circular photon orbits depend on the cosmological constant (as also the geodesic curvature of main circles does). Contrary to the Reissner-Nordström-de Sitter case, in Kerr-de Sitter space-times circular photon orbits do exist for each value of the parameters y and a. Since the functions $y_{\mathrm{h}}(r,a)$ and $y_{\mathrm{ex+}}(r,a)$ have common local extrema, we can conclude that three circular photon orbits exist in metrics describing black holes, while only one such orbit exists in the case of naked singularities (for both de Sitter and anti-de Sitter cases). All circular photon orbits are unstable to radial perturbations.

The character of the radial motion can be easily inferred from Fig. 2 and Fig. 3. The most interesting and surprising feature appears for Kerr black holes in a de Sitter universe: the effective potential X_- does not diverge at r_{ch}, but it diverges at some $r_{\mathrm{d}} < r_{\mathrm{ch}}$ (we recall that for static holes the potential diverges just at r_{ch}). By increasing the impact parameter X up to the value $X_{\pm}(r_{\mathrm{ch}}) = r_{\mathrm{ch}}^2/a$, photons will be repelled by the barrier at r_{t} increasing up to r_{ch}. However as we further increase X, r_{t} starts decreasing to r_{d}. In the case of a very fine tuning of the parameters y and a, namely when the condition: $y_{\mathrm{d-max}}(a) < y < y_{\mathrm{max}}(a)$ is satisfied (and such interval exists for each $a^2 < a_{\mathrm{cr}}^2$), *photons with impact parameter high enough ($X > X_{\mathrm{cr2}}$) can travel between the horizons* r_{bh} *and* r_{ch} (the repulsive barrier disappears in this case), in addition to the *standard* photons with a low enough impact parameter ($X < X_{\mathrm{cr1}}$) and moreover, for photons with negative values of the impact parameter there is no repulsive barrier at all (Fig. 2, case $y = 0.04$). Therefore the *cooperation* of both black hole rotation and cosmological repulsion can lead to rather unexpected features of the geodesic structure of the Kerr-de Sitter space-times.

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