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Insufficiency of Karmarkar's Condition

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Abstract

It is shown that Karmarkar's condition which has been claimed to be a necessary and sufficient condition for the spherically symmetric (s.s.) space-time to be of class 1 ceases to be sufficient in case $R_{2323} = 0$. An s.s. line element which satisfies Karmarkar's condition but ceases to be of class 1 is given. This line element describes a conformally flat perfect fluid distribution with zero density.

The problem of imbedding of an s.s. space-time into a five-dimensional flat space, has been investigated in great detail by Eiesland [1], Karmarkar [2], and Takeno [3]. Karmarkar considered an s.s. line element in the form

$$ds^2 = -A\,d\rho^2 - B(d\theta^2 + \sin^2\theta\,d\phi^2) + C\,d\tau^2 + 2D\,d\rho\,d\tau \qquad (1)$$

where A, B, C, and D are functions of ρ and τ only. Karmarkar has claimed that the necessary and sufficient condition for (1) to be of class 1 is

$$3F^2 + 8\pi F(4T_2^2 - T_1^1 - T_4^4) - 64\pi^2(T_1^1 T_4^4 - T_1^4 T_4^1) = 0 \qquad (2)$$

where

$$F \equiv -\frac{R_{2323}}{B^2\sin^2\theta} = \frac{1}{B} - \frac{1}{4B^2}(CB'^2 - 2D\dot{B}B' - A\dot{B}^2)(AC + D^2)^{-1} \qquad (3)$$

and T_i^j are the components of the energy–momentum tensor defined through Einstein field equations. The overhead dot and prime denote a partial differentiation with respect to ρ and τ, respectively. The condition (2) was first obtained by Eiesland [1] in the form

$$R_{1414}R_{2323} - R_{1212}R_{3434} - R_{1224}R_{1334} = 0, \qquad (4)$$

in terms of curvature components arising out of (1). The object of this note is to show that the condition (2) is only a necessary condition for the space-time (1) to be of class 1.

By the use of the transformation

$$r = B^{1/2}$$

$$dt = \frac{E(\rho, \tau)}{B'^2} \left[2B^{1/2}(A\dot{B} + DB')\, dr + (CB'^2 - 2D\dot{B}B' - A\dot{B}^2)\, d\tau\right] \qquad (5)$$

where $E(\rho, \tau)$ is an integrating factor, the line element (1) reduces to the form

$$ds^2 = (CB'^2 - 2D\dot{B}B' - A\dot{B}^2)^{-1}\left[-4B(AC + D^2)\, dr^2 + \frac{B'^2}{E^2}\, dt^2\right]$$
$$- r^2(d\theta^2 + \sin^2\theta\, d\phi^2) \qquad (6)$$

When $R_{2323} = 0$, the line element (6) reduces to

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + \frac{B'^2}{E^2}(CB'^2 - 2D\dot{B}B' - A\dot{B}^2)^{-1}\, dt^2 \qquad (7)$$

as a consequence of (3). Thus it is possible to reduce (1) to the form

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + e^{\nu}\, dt^2 \qquad (8)$$

where $\nu \equiv \nu(r, t)$ whenever $R_{2323} = 0$ and $B' \neq 0$. However, when $\dot{B} \neq 0$, a similar reduction is possible. In case $\dot{B}$ and B' both vanish simultaneously the space-time (1) ceases to be of class 1.

It is well known [4] that for a Riemannian space to be of class 1, a necessary and sufficient condition is that a second-order symmetric tensor satisfying the following conditions

$$R_{ijkl} = e(b_{ik}b_{jl} - b_{il}b_{jk}), \qquad e = \pm 1 \qquad (9)$$

$$b_{ij;k} - b_{ik;j} = 0 \qquad (10)$$

should exist. In view of the fact that the only nonzero components of the curvature tensor for (8) are R_{1414}, R_{2424}, and R_{3434}, it is found that the equations (9) imply the following relations:

$$b_{22}b_{33} = 0, \qquad b_{24}b_{33} = 0, \qquad b_{22}b_{44} - b_{24}^2 \neq 0, \qquad b_{33}b_{44} \neq 0$$

which are clearly inconsistent. Hence the space-time (8) is not of class 1. Thus it has been established that the s.s. space-time ceases to be of class 1 when $R_{2323} = 0$.

Calculations show that the only nonzero components of T_i^j for the line element (8) are T_1^1, $T_2^2 = T_3^3$, and $R_{2323} = 0$. Therefore, the condition (2) is identically satisfied. Thus it is shown that Karmarkar's condition (2) holds good and yet the space-time (8) is not of class 1. This establishes the insufficiency of Karmarkar's condition. In light of this result the claim made by Tikekar [5] and Krishna Rao [6] that every conformally flat perfect fluid was of class 1 could not be upheld. In fact the s.s. line element (8) with $e^{\nu} = (H + \frac{1}{4} Gr^2)^2$, where H and G are functions of t alone, describes a perfect fluid with zero density, it is conformal to a flat space, yet it is not of class 1. A more general metric is furnished by Barnes [7].

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