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Cosmology with G and Λ Coupling Scalars

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Cosmology with the gravitational and cosmological constants generalized as coupling scalars in Einstein's theory is considered. A general method of solving the field equations is given. Fifteen exact solutions for zero pressure models satisfying $G = G_0(R/R_0)^n$ are given in the Appendix; they are briefly discussed.

1. INTRODUCTION

In the last years (Refs. 1,2 and references therein), a theory of gravitation using G and Λ as no constant coupling scalars have been used. Its motivation was to include a G-varying 'constant' of gravity, as pioneered by Dirac [3]. It is a straightforward generalization of Einstein's equations

$$S^{ab} = -8\pi G T^{ab} - \Lambda g^{ab} \tag{1}$$

where S^{ab} is the Einstein tensor, T^{ab} the matter energy-momentum tensor, g^{ab} the metric tensor, G and Λ are coupling scalars. If we assume the principle of equivalence, as in Einstein's theory, such as the equality of gravitational and inertial mass, and the gravitational time dilation, we must require that G and Λ do not enter in the equations of motion of particles and photons, i.e., only g^{ab} must enter in them. So the interchange of energy between matter and gravitation is given by the usual conservation laws

$$T^{ab}_{;b} = 0. \tag{2}$$

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In this way the role of the scalars G and Λ is confined to the effects on the field equations (1), and, once g^{ab} is determined, the gravitational phenomena are described in the same way as in Einstein's theory.

The covariant divergence of (1) , taking into account the Bianchi identities and (2), gives

$$8\pi G_{;b}T^{ab} + \Lambda_{;b}g^{ab} = 0. \tag{3}$$

Equations (1) and (3) may be considered as the fundamental equations of gravity with G and Λ coupling scalars.

Before the gravitational problem based on (1) and (2), or on (3), is considered, some comments are in order. First, since these equations do not derive from a Hamiltonian principle, they do not contain the propagation equations for the scalar fields. They should be determined to satisfy the conservation relation (3). This may be done on heuristic arguments, or by cosmological constraints as will be the case shown in Sections 2 and 3. An alternative way would be the construction of a tensor-scalar theory of the 'constants' G and Λ, as a generalization of Brans-Dicke equations. Barraco [4] constructed a theory in this sense. However this is another problem and the simplicity of equations (1)-(3) is largely complicated . The cosmological models based on these equations allow the possibility of investigating different cases for G, as in Dirac's cosmology for example, or to solve some cosmological difficulties [5,6]; they may be useful to study the early universe (singular or not) and their relations with particle fields. In any case the strongest constraints are the presently observed G_0 value and the observational limits of Λ_0.

2. COSMOLOGY

Uniform cosmological models are described by the Robertson-Walker metric, which may be written

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-k\,r^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right] \tag{4}$$

where k is the space curvature constant, $k = -1$, 0 or +1, and $R(t)$ the scale factor (the speed of light $c = 1$ and signature $+---$ are used). For perfect fluid cosmology we have the energy-momentum tensor

$$T^{ab} = -pg^{ab} + (p+\rho)U^aU^b. \tag{5}$$

p and ρ are the pressure and density respectively and U^a is the 4-velocity vector ($U_mU^m = 1$). Using co-moving coordinates

$$U^a = (1,0,0,0) \tag{6}$$

in (5) and with the metric (4), Einstein's equations (1) reduce to [2]

$$3\ddot{R} = -4\pi GR(3p + \rho - \Lambda/4\pi G) \tag{7}$$

$$3\dot{R}^2 = \quad 8\pi GR^2(\rho + \Lambda/8\pi G) - 3k. \tag{8}$$

In uniform cosmology $G = G(t)$ and $\Lambda = \Lambda(t)$, so that the conservation equation (3) becomes

$$8\pi\dot{G}\rho + \dot{\Lambda} = 0. \tag{9}$$

Equations (7), (8) and (9) are the fundamental equations. Of course, they trivially reduce to standard Friedmann cosmology when G and Λ are constants. The first two equations, (7) and (8), may be written

$$8\pi Gp = -2\ddot{R}/R - \dot{R}^2/R^2 - k/R^2 + \Lambda \tag{10}$$

$$8\pi G\rho = 3(k + \dot{R}^2)/R^2 - \Lambda. \tag{11}$$

Eliminating Λ between (10) and (11), $\ddot{R}$ with the derivative of (11), and using (9), it is found that

$$\frac{d(\rho R^3)}{dt} + p\frac{d(R^3)}{dt} = 0. \tag{12}$$

Note that (10) and (11) are formally identical to those of usual cosmology [7,8] with G and Λ constants, as must be, since the LHS of (1) depends only on the metric components (4) and G and Λ enter algebraically in the RHS of (1). Also (12) is identical to that of usual cosmology despite the fact that it comes from a differential form of (11) , and from (10), both involving the time dependent scalars G and Λ, however, their derivatives eliminate with (9) , thus leading to (12).

The three equations (9), (11) and (12) are independent and, in the following, they will be used as fundamental. The cosmological problem posed by these equations leaves two degrees of freedom; it may be determined by a physical assumption $p = p(\rho)$, i.e. the 'equation of state', and from an additional explicit adoption on R, ρ, p, G or Λ in terms of t or R which itself depends on t. Once the problem is determined, the integration constants are characterized by the observable parameters

$$H_0 = \frac{\dot{R}_0}{R_0} \tag{13}$$

$$\sigma_0 = \frac{4}{3}\frac{\pi G\rho_0}{{H_0}^2} \tag{14}$$

$$q_0 = -\frac{\ddot{R}_0}{R_0}\,{H_0}^2 \tag{15}$$

$$\epsilon_0 = \frac{p_0}{\rho_0} \tag{16}$$

which must satisfy Einstein's equations at present cosmic time t_0

$$\Lambda_0 = 3{H_0}^2[(1+3\epsilon_0)\sigma_0 - q_0] \tag{17}$$

$$\frac{k}{{R_0}^2} = {H_0}^2[3(1+\epsilon_0)\sigma_0 - q_0 - 1] \tag{18}$$

and the conservation equation

$$\dot{\Lambda}_0 G_0 + 6\dot{G}_0 {H_0}^2 \sigma_0 = 0. \tag{19}$$

3. SOLUTIONS

In this section we will use a method similar to that introduced in [9] for G and Λ constants. We assume the global 'equation of state'

$$p = \tfrac{1}{3}\rho\Phi \tag{20}$$

where Φ is a function of the factor scale R. It should be noted that this assumption makes physical sense when eq. (20) adequately represents the cosmic content; simple cases are $\Phi = 0$ for dust models, $\Phi = 1$ for radiation filled models, or a function going to unity for the early universe and to zero for the present cosmic time, etc.

From eqs. (12) and (20) we obtain

$$\frac{1}{\Psi}\frac{d\Psi}{dR} + \frac{\Phi}{R} = 0 \tag{21}$$

where

$$\Psi = \rho R^3. \tag{22}$$

Equation (21) becomes crucial as a first condition to determine the problem: either Φ or Ψ may be taken to be an arbitrary function. If Φ is a given explicit function of R, then eq. (20) is determined and Ψ follows from (21)

$$\Psi = \Psi_0 \exp\left[-\int \frac{\Phi}{R}\,dR\right]. \tag{23}$$

Conversely, if Ψ is given, Φ immediately follows from (21)

$$\Phi = -\frac{R}{\Psi}\frac{d\Psi}{dR}. \tag{24}$$

The Friedmann equation (11) with (22) becomes

$$3\dot{R}^2 = 8\pi G\Psi R^{-1} + \Lambda R^2 - k. \tag{25}$$

Equations (9) and (22) with $d/dt = \dot{R}(d/dR)$

$$8\pi\frac{dG}{dR} + \Psi^{-1}R^3\frac{d\Lambda}{dR} = 0. \tag{26}$$

Finally, if $G = G(R)$ is given, (26) integrates to give $\Lambda = \Lambda(R)$, (25) determines $R = R(t)$, and the problem is solved; note that $\Lambda = \Lambda(R)$ may be given instead, and that $G(R)$ derives also from (26), giving in turn $R(t)$ from integration of (25) .

4. ZERO-PRESSURE MODELS SATISFYING $G = G_0(R/R_0)^n$

Zero-pressure models are defined by $\Phi = 0$; in this case (23) gives

$$\Psi = \Psi_0 = \rho_0 {R_0}^3 = \text{const.} \tag{27}$$

and the density derives from (22) and (27)

$$\rho = \rho_0\left(\frac{R_0}{R}\right)^3 . \tag{28}$$

On the other hand, the condition

$$G = G_0\left(\frac{R}{R_0}\right)^n \tag{29}$$

in (26) with Ψ from (27) implies

$$\Lambda = \Lambda_0 + \mathrm{Cn}\left[1 - \left(\frac{R}{R_0}\right)^{n-3}\right] \tag{30}$$

for $n \neq 3$; Cn is a parameter related to the integration constant of (26) and to Λ_0; its value, using (14), is given by

$$\mathrm{Cn} = 6\frac{n}{n-3}{H_0}^2\sigma_0, \tag{31}$$

and from (17)

$$\Lambda_0 = 3{H_0}^2(\sigma_0 - q_0). \tag{32}$$

The Friedmann equation (25) by means of (27), (28) and (29) takes the form

$$\dot{R}^2 = a_n R^{n-1} + b_n R^2 - k \tag{33}$$

where

$$a_n = \frac{6}{3-n} H_0{}^2 \sigma_0 R_0{}^{3-n} \tag{34}$$

$$b_n = \left(3\frac{n-1}{n-3}\sigma_0 - q_0\right) H_0{}^2 \tag{35}$$

for which (30), (14) and (15) were used.

Finally, the equation for the parameters (18) reduces to

$$\frac{k}{R_0{}^2} = H_0{}^2(3\sigma_0 - q_0 - 1) \tag{36}$$

and (19) is identically satisfied.

It should be noted that the models are completely characterized by the set of parameters (H_0, G_0, σ_0, q_0, n) with $n \neq 3$.

The case $n < 2$ implies $C_n < 0$ in (31) and $a_n > 0$ in (34) and viceversa for $n \geq 2$; $b_n \gtreqless 0$ according to n, σ_0, and q_0 combine in (35); $\Lambda_0 \gtreqless 0$ as $\sigma_0 \gtreqless q_0$ as given by (32) and the curvature parameter k equals $+1$, 0 or -1 according to $\sigma_0 - q_0 - 1 \gtreqless 0$ in (36) . These relations determine the integration conditions of the Friedmann equation (33) and the properties of its solutions.

5. CONCLUSION

The gravity with G and Λ coupling scalars was considered as a simple generalization of Einstein's equations with usual conservation laws for ordinary matter, expressed by $T^{ab}_{;b} = 0$. Its application to cosmology was developed in Section 2. It was shown that the field equations for perfect fluid cosmology are formally identical to Einstein's equations for G and Λ constants including eq. (12). So the evolution of matter is similar to that of Einstein's theory, as to its dependence on space-time geometry given by $R(t)$ in (10), (11), and (12). The coupling of the scalar fields with matter is given by the additional conservation equation (9).

A general method of solving the cosmological field equations was introduced by means of a global equation of state, but without loss of generality . Owing to the freedom mentioned before,the theory allowed us to find several exact solutions for zero-pressure models. It may be used for radiation-like early universe models with $\Phi \to 1$ for $t \to t_i$ (say 0), and matter-like $\Phi \to 0$ for $t \to t_0$, the present cosmic time, even containing matter and background radiation.

The fifteen solutions given in the Appendix illustrate many interesting cases. Most of them show intial singularity $R(0) = 0$ with ever-expanding or a finite cosmic era returning to a future singularity $R(T) = 0$. These, for open or closed space-geometry ($k = 1$, 0 or -1) and for a wide range of the density parameter. Also, the density condition $\sigma_0 = 1/2$ for flatness in standard cosmology is released.

The role of Planck's units $m^p = (\hbar c/G)^{1/2}$, $l^p = (\hbar G/c^3)^{1/2}$ and $t^p = (\hbar G/c^6)^{1/2}$, as well as nucleosynthesis in the early universe should be discused for each solution.

Two interesting cases are A.2c and B.1.3b in the Appendix. They have no initial singularity, and the initial condition may be established $R(0) = R_i$ with $R_i \geq R_m$; where R_m is the minimum value of R at t_m. If $t_m > 0$, the universe starts contracting from R_i until it reaches the minimum R_m and then expansion goes on forever. If $t_m < 0$, expansion starts from R_i. Obviously, for $t_m = 0$ expansion starts from $R_i = R_m$, i.e., from the initial minimum to infinity. A qualitatively similar case was discussed in [2].

APPENDIX: ZERO PRESSURE MODELS SATISFYING $G = G(R/R)^n$

In this Appendix the notation and units of the text are used. To change to conventional units the transformation $R \to R/c$, and $p \to p/c^2$ should be used.

A. Models with $q_0 = 3(n - 1/n - 3)\sigma_0$

A.1. Models having $n = 1$

The class condition for the sub-class $n = 1$ gives

$$q_0 = 0, \qquad \mathrm{Cn} = -\Lambda_0 = -3{H_0}^2\sigma_0, \qquad a_n = 3{R_0}^2{H_0}^2\sigma_0$$

$$G = G_0(R/R_0), \text{ and } \Lambda = 3{H_0}^2\sigma_0(R_0/R)^2$$

while the solutions for the curvature constant are

A.1a. $k = -1 \quad (\sigma_0 < 1/3)$

$R = (1 - 3\sigma_0)^{-1/2}t, \qquad t_0 = {H_0}^{-1}$

A.1b. $k = 0 \quad (\sigma_0 = 1/3)$

$R = R_0 H_0 t, \quad \text{and} \quad t_0 = {H_0}^{-1}$

A.1c. $k = +1 \quad (\sigma_0 > 1/3)$

$R = (3\sigma_0 - 1)^{-1/2}t, \qquad t_0 = {H_0}^{-1}$

Fig. 1

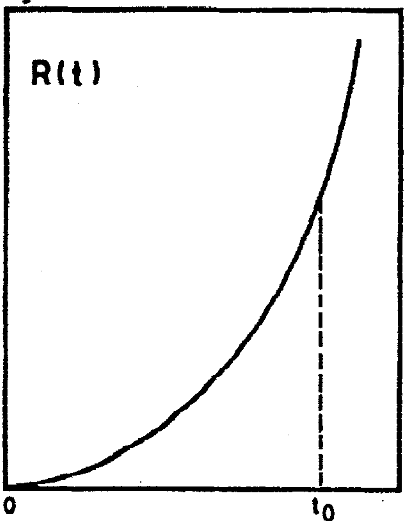


Fig. 2

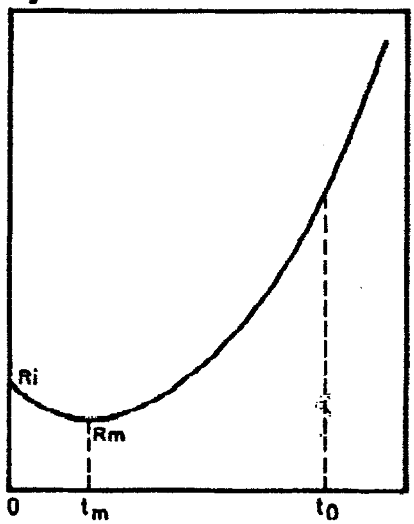


Fig. 3

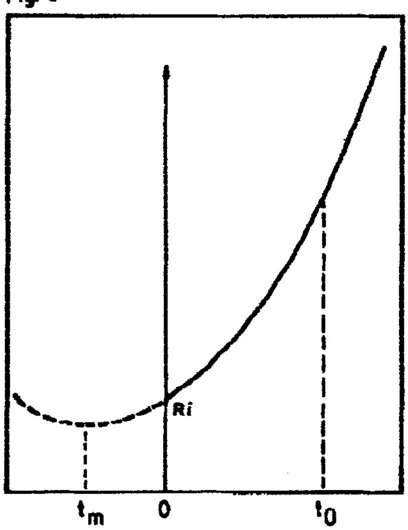
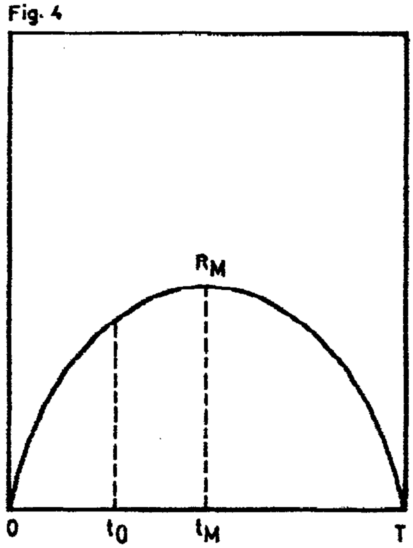


Fig. 4



Figures 1–4. Figure 1: models A.2a, A.2b, A.3a, A.3b, B.1.1b, B.1.2b and B.1.3a. Figures 2 and 3: models A.2.c and B.1.3b. Figure 4: models A.3c and B.1.2a.

A.2. Models $n = 2$

All models of this sub-class have

$$q_0 = -3\sigma_0, \qquad \mathrm{Cn} = -\Lambda_0 = -12{H_0}^2\sigma_0$$
$$G = G_0(R/R_0)^2, \qquad \Lambda = \Lambda_0(R_0/R)$$

and the solutions for the curvature are

A.2a. $k = -1 \quad (\sigma_0 < 1/6)$

$$R = \frac{3}{2}\frac{H_0\sigma_0}{(1-6\sigma_0)^{1/2}}(t+C)^2 - \frac{(1-6\sigma_0)^{1/2}}{6H_0\sigma_0}$$

where

$$R = 0 \text{ for } t = 0, \qquad C = \frac{(1-6\sigma_0)^{1/2}}{3H_0\sigma_0}$$
$$t_0 = \frac{[1-(1-6\sigma_0)^{1/2}]}{3H_0\sigma_0}$$

cf. Figure 1.

A.2b. $k = 0 \quad (\sigma_0 = 1/6)$

$$R = \frac{3}{2}R_0{H_0}^2\sigma_0 t^2, \qquad t_0 = (2/3\sigma_0)^{1/2}{H_0}^{-1}$$

cf. Figure 1.

A.2c. $k = +1 \quad (\sigma_0 > 1/6)$

$$R = \frac{3}{2}R_0{H_0}^2\sigma_0(t-t_m)^2 + \frac{(6\sigma_0-1)^{1/2}}{6\sigma_0}{H_0}^{-1}$$
$$t_m = \pm\left[\frac{2}{3}\frac{(6\sigma_0-1)^{1/2}}{6H_0\sigma_0}R_i - \frac{6\sigma_0-1}{9{H_0}^2{\sigma_0}^2}\right]^{1/2}$$
$$R = R_i \text{ for } t = 0,$$
$$R_i \geq R_m = \frac{(6\sigma_0-1)^{1/2}{H_0}^{-1}}{6\sigma_0} = R(t_m)$$
$$t_0 = \frac{{H_0}^{-1}}{3\sigma_0} \pm t_m; \qquad \text{if } t_m = 0, \text{ then } R_i = R_m.$$

cf. Figures 2 and 3

A.3. Models with $n = -1$

All models for this sub-class have

$$q_0 = \frac{3}{2}\sigma_0, \qquad \mathrm{Cn} = -\Lambda_0 = \frac{3}{2}H_0{}^2\sigma_0$$
$$a_n = \frac{3}{2}R_0{}^4, \qquad \frac{k}{R_0{}^2} = H_0{}^2\left(\frac{3}{2}\sigma_0 - 1\right)$$
$$G = G_0\frac{R_0}{R}, \qquad \Lambda = -\frac{3}{2}H_0{}^2\sigma_0\left(\frac{R_0}{R}\right)^4$$

and for different values of curvature we have

A.3a. $k = -1 \quad (\sigma_0 < 2/3)$

$$R = [(t + \sqrt{a_n})^2 - a_n]^{1/2}$$
$$a_n = \frac{3}{2}\sigma_0 H_0{}^{-2}\left(1 - \frac{3}{2}\sigma_0\right)^{-2}$$
$$t_0 = \left[1 - \left(\frac{3}{2}\sigma_0\right)^{1/2}\right]\left(1 - \frac{3}{2}\sigma_0\right)H_0{}^{-1}$$

$R = 0$ for $t = 0$.

cf. Figure 1.

A.3b. $k = 0 \quad (\sigma_0 = 2/3)$

$$R = (2H_0)^{1/2} R_0 t^{1/2}$$

$$t_0 = \frac{1}{2} {H_0}^{-1}.$$

cf. Figure 1.

A.3c. $k = +1 \quad (\sigma_0 > 2/3)$

$$R = [a_n - (t - \sqrt{a_n}\,)^2]^{1/2}$$

$$a_n = \frac{3}{2} \frac{\sigma_0 {H_0}^{-2}}{(3/2)\sigma_0 - 1} = {R_M}^2 = {t_M}^2$$

$$t_0 = \left[\left(\frac{3}{2} \sigma_0 \right)^{1/2} - 1 \right] \left(\frac{3}{2} \sigma_0 - 1 \right) {H_0}^{-1}$$

$R = 0$ for $t = 0$ and $t = 2t_M$.

cf. Figure 4.

B. Models with $q_0 \neq 3[(n-1)/(n-3)]\sigma_0$

B.1. Models having $n = 1$

These models are characterized by

$$q_0 \neq 0, \quad \mathrm{Cn} = -3{H_0}^2\sigma_0, \quad a_n = 3\sigma_0 {R_0}^2 {H_0}^2, \quad b_n = -q_0 {H_0}^2,$$
$$\Lambda_0 = 3{H_0}^2(\sigma_0 - q_0), \quad k/{R_0}^2 = {H_0}^2(3\sigma_0 - q_0 - 1),$$
$$G = G_0(R/R_0) \quad \text{and} \quad \Lambda = 3{H_0}^2[\sigma_0(R_0/R)^2 - q_0]$$

and the solutions are

B.1.1. $k = -1 \quad (q_0 > 3\sigma_0 - 1)$

B.1.1a. $q_0 > 0 \quad (\sigma_0 > 1/3)$

$$R = \frac{(1+q_0)^{1/2} H_0{}^{-1}}{(1+q_0-3\sigma_0) q_0{}^{1/2}} \sin(q_0{}^{1/2} H_0\, t)$$

$$t_0 = \frac{H_0{}^{-1}}{q_0{}^{1/2}} \sin^{-1}\left[\frac{q_0}{1+q_0}\right]^{1/2}$$

$$T = \frac{\pi H_0{}^{-1}}{q}, \qquad R = 0 \text{ for } t = 0 \text{ and } t = T$$

cf. Figure 4.

B.1.1b. $q_0 < 0 \quad (\sigma_0 < 1/3)$

$$R = \frac{H_0{}^{-1}}{(1+q_0-3\sigma_0)\,|q_0|^{1/2}} \sinh(\,|q_0|^{1/2} H_0\, t)$$

$$t_0 = \frac{H_0{}^{-1}}{|q_0|^{1/2}} \sinh^{-1} |q_0|^{1/2}.$$

cf. Figure 1.

B.1.2. $k = 0 \quad (q_0 = 3\sigma_0 - 1)$

B.1.2a. $\sigma_0 > 1/3 \quad (q_0 > 0)$

$$R = (3\sigma_0 - 1)^{-1/2} (3\sigma_0)^{1/2} R_0 \sin[\,(3\sigma_0 - 1)^{-1/2} H_0\, t]$$

$$t_0 = H_0{}^{-1} (3\sigma_0 - 1)^{-1/2} \sin^{-1}\left[\frac{3\sigma_0 - 1}{3\sigma_0}\right]^{1/2}$$

$$T = \pi (3\sigma_0 - 1)^{1/2} H_0{}^{-1}$$

$R = 0$ at $t = 0$ and $t = T$.

cf. Figure 4.

B.1.2b. $\sigma_0 < 1/3 \quad (q_0 < 0)$

$$R = (1 - 3\sigma_0)^{-1/2}(3\sigma_0)^{1/2} R_0 \sinh[(1 - 3\sigma_0)^{1/2} H_0 t]$$

$$t_0 = (1 - 3\sigma_0)^{-1/2} H_0^{-1} \sinh^{-1} \left[\frac{1 - 3\sigma_0}{3\sigma_0}\right]^{1/2} .$$

cf. Figure 1.

B.1.3. $k = +1 \quad (q_0 < 3\sigma_0 - 1)$

B.1.3a. $q_0 > -1$

$$R = \frac{H_0^{-1}(q_0 + 1)^{1/2}}{(3\sigma_0 - q_0 - 1)} \sinh[|q_0|^{1/2} H_0 t]$$

$$t = \frac{H_0^{-1}}{|q_0|^{1/2}} \sinh^{-1} \left[\frac{q_0}{q_0 + 1}\right]^{1/2} .$$

cf. Figure 1.

B.1.3b. $q_0 < -1$

$$R = \frac{H_0^{-1}|q_0 + 1|^{1/2}}{(3\sigma_0 - q_0 - 1)^{1/2}|q_0|^{1/2}} \cosh[|q_0|^{1/2} H_0 (t - t_m)]$$

$$R_m = R(t_m) = \frac{H_0^{-1}|q_0 + 1|^{1/2}}{(3\sigma_0 - q_0 - 1)^{1/2}|q_0|^{1/2}}$$

$$R_i = R(0) \geq R_m .$$

cf. Figures 2 and 3.

$$t_0 = t_m + \frac{H_0^{-1}}{|q_0|^{1/2}} \cosh^{-1} \frac{|q_0|^{1/2}}{|q_0 + 1|^{1/2}} .$$

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