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Spherically Symmetric Solutions in Five-Dimensional General Relativity[1]

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Abstract

The most general time-independent spherically symmetric (in the usual three space dimensions) solution to the five-dimensional vacuum Einstein equations is found, subject to the existence of a Killing vector in the fifth direction. The significance of these solutions is discussed within the context of a previously proposed extension of the Kaluza–Klein model in which the universe, although (4 + 1)-dimensional, has evolved over cosmic times into an effectively (3 + 1)-dimensional one.

§(1): *Introduction*

The idea that the various forces of nature might be unified by enlarging the dimensionality of space-time has a long and generally honorable history that goes back to the work of Nordstrom in 1914 and Kaluza in 1921 [1, 2]. Its earlier adherents were mainly those interested in extending general relativity, while of late increased interest has been evident in the particle physics community, especially among those investigating extended supersymmetry [3–5].

Both the appeal and the frustration of this approach were touched on by Einstein and Pauli [6], who wrote in 1943

When one tries to find a unified theory of the gravitational and electromagnetic fields, he cannot help feeling that there is some truth in Kaluza's five-dimensional theory. Yet its

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foundation is unsatisfactory insofar as, with respect to the group of admissible coordinate transformations, the fifth, space-like coordinate is treated quite differently from the others.

Similar remarks, of course, apply whether one is interested in $4 + n$ dimensions rather than five, or whether one seeks to include the strong and weak interactions in addition to electromagnetism.

In a previous paper [7], we argued that while the universe appears to be (3 + 1)-dimensional, this appearance may be deceiving. We pointed out that there is a simple solution to Einstein's equations (the Kasner solution) which describes the evolution over cosmic times of a space-time with more than three spatial dimensions, such that at the present epoch the extra dimensions have shrunk to a size comparable to the Planck length. The specific example we chose had one extra dimension, whose residual effects could be interpreted as the electromagnetic interaction together with that of a scalar field. Other examples can presumably be chosen to reproduce the effects of other, more complicated, gauge theories.

In this paper, we shall eschew further cosmological speculation in favor of a more detailed look at the properties of solutions to Einstein's equations on a (4 + 1)-dimensional manifold. We shall demand the solutions be asymptotically flat, which is inappropriate for an evolving universe, but which should be relevant to describing our local environment.

Thus, we are chiefly interested in the vacuum Einstein equations

$$R_{\mu\nu} = 0 \tag{1}$$

in (4 + 1)-dimensions. When projected down to an effective (3 + 1)-dimensional manifold, the degrees of freedom in $g_{\mu\nu}$ ($\mu, \nu = 0, 1, 2, 3, 5$) represent the gravitational and electromagnetic fields, and an extra scalar field. For most of this paper, we shall assume that the metric possesses a spacelike Killing vector ξ^a which can be taken to be

$$\xi^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial x^5} \tag{2}$$

We are not forced to demand the existence of this Killing vector, but since the fifth-dimension is so small, it is intuitively reasonable that the gross features of matter can be well described by ignoring any dependence on x^5. (From a quantum mechanical viewpoint, the uncertainty principle tells us that the energy required to produce excitations in the fifth direction is of the order of 10^{20} Mev.) Furthermore, the existence of ξ provides an elegant and unambiguous way to project out the fifth dimension. The formalism for doing this is developed in Section 2.

In Section 3, we find the most general time-independent spherically symmetric (in the usual three space dimensions) solutions to equation (1). These are characterized by three real parameters. In Section 4, we probe equation (1) by introducing a distribution of pressureless charged matter ("charged dust"). This allows us properly to identify the parameters of our vacuum solution as the electric charge, gravitational mass, and scalar charge. Some conclusions are offered in Section 5.

§(2): *Five-Dimensional Manifolds with a Spacelike Killing Vector Field*

Our technique for discussing a five-dimensional manifold with a spacelike Killing vector field follows from a minor extension of Geroch's [8] analysis of the four-dimensional case; details and derivations of the equations used below can be found there.

We assume that the metric g_{ab} of a five-dimensional manifold possesses a spacelike Killing vector field ξ^a, and we let ∇_a be the covariant derivative operator associated with g_{ab}. Then the manifold of the trajectories of ξ^a is four dimensional and has a metric with Lorentzian signature

$$\gamma_{ab} = g_{ab} - \xi_a \xi_b / \phi^2 \tag{3}$$

where

$$\phi^2 = \xi^a \xi_a \tag{4}$$

is the norm of the Killing vector field.

A derivative operator D_a can be defined for the four-dimensional metric γ_{ab} by taking the ∇_a derivative of any tensor orthogonal to and Lie transported by ξ^a and then projecting all indices perpendicular to ξ^a. The Riemann tensor of D_a, $\mathcal{R}_{abcd}$, is related to the Riemann tensor of ∇_a, R_{abcd}, by the analog of the Gauss-Codazzi equation

$$\begin{aligned}\mathcal{R}_{abcd} = \gamma^p{}_{[a}\gamma^q{}_{b]}\,\gamma^r{}_{[c}\gamma^s{}_{d]}\,[R_{pqrs} + 2\phi^{-2}(\nabla_p \xi_q)(\nabla_r \xi_s) \\ + 2\phi^{-2}(\nabla_p \xi_r)(\nabla_q \xi_s)]\end{aligned} \tag{5}$$

Below, we will identify the electromagnetic field with an antisymmetric tensor

$$F_{ab} \equiv \phi^{-2} G^{-1/2} \gamma_a{}^p \gamma_b{}^q \nabla_{[p} \xi_{q]} \tag{6}$$

which is orthogonal to and Lie transported by ξ^a. The quantity G is the gravitational constant. Note that F_{ab} vanishes if and only if ξ^a is hypersurface orthogonal. Also, it is easy to show that

$$\nabla_a \xi_b = -2\phi^{-1} \xi_{[a} \nabla_{b]} \phi + \phi^2 G^{1/2} F_{ab} \tag{7}$$

Any Killing vector field satisfies

$$\nabla_a \nabla_b \xi_c = R_{dabc} \xi^d \tag{8}$$

and when antisymmetrized over a, b, and c the right-hand side vanishes. It follows that the curl of F_{ab} also vanishes,

$$D_{[a}F_{bc]} = \gamma^p{}_{[a}\gamma^q{}_b\gamma^r{}_{c]}\nabla_p F_{qr} = 0 \tag{9}$$

which is one-half of Maxwell's equations in four-dimensional space-time.

The four-dimensional implications of the five-dimensional Einstein equations,

$$R_{ab} - \tfrac{1}{2} g_{ab}R = 8\pi G\ {}_5T_{ab}, \tag{10}$$

(${}_5T_{ab}$ is the five-dimensional stress energy tensor) follow from the projection of these indices perpendicular to and parallel to ξ^a and the use of equations (5) and (8). It is first useful to decompose ${}_5T_{ab}$

$${}_5T_{ab} \equiv T_{ab} + \phi^{-3}\xi_{(a}J_{b)} + \phi^{-2}\xi_a\xi_b T_{55} \tag{11}$$

where T_{ab} and J_b are perpendicular to ξ^a and T_{55} is a scalar. Now the 5-5 component of equation (10) implies that

$$\phi D^a D_a \phi = G\phi^4 F_{ab}F^{ab} + (8\pi G/3)\,\phi^2\,[T^a{}_a - 2T_{55}] \tag{12}$$

the 5-$\perp$ component is

$$D^b(\phi^3 F_{ab}) = 4\pi G^{1/2} J_a \tag{13}$$

and the $\perp$-$\perp$ component is

$$\mathcal{R}_{ab} - \tfrac{1}{2}\gamma_{ab}\mathcal{R} = \phi^{-1}D_aD_b\phi - \phi^{-1}\gamma_{ab}D^pD_p\phi + 2G\phi^2\,[F_{ap}F_b{}^p - \tfrac{1}{4}F_{pq}F^{pq}] + 8\pi G T_{ab} \tag{14}$$

The Bianchi identities in the five-dimensional manifold give two conservation laws which are consequences of the above equations,

$$D^a J_a = 0 \tag{15}$$

and

$$D^a(\phi T_{ab}) + G^{1/2}J^a F_{ab} - T_{55}D_b\phi = 0 \tag{16}$$

The great similarity between these equations and the coupled Einstein-Maxwell equations is easiest to see by examining the linearized vacuum version of the above equations. Equation (12) becomes Laplace's equation for a scalar field; equation (13) becomes one of Maxwell's equations with J^a the conserved current (note that ϕ should approach a positive constant in nearly flat space); and equation (14) becomes the four-dimensional vacuum Einstein equation. In fact, in this weak-field limit the scalar field looks only like an addition to the Newtonian gravitational potential and the equations look exactly like Newton's plus Maxwell's.

The differences between the above equations and the Einstein-Maxwell equations appear in the second order and result from the scalar field which has an effect of magnitude comparable to that of the gravitational field.

§(3): *Spherically Symmetric Solutions*

The most general time-independent metric with spherical symmetry in the three usual space dimensions and a Killing vector in the fifth direction can be put in the form

$$ds^2 = -e^\mu \, dt^2 + 2A \, dt \, dx + \phi^2 \, dx^2 + e^\beta [dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2)] \quad (17)$$

where μ, A, ϕ, and β are functions only of r; this is a generalization of the isotropic form of the Schwarzschild metric. As in the previous section, ϕ^2 is the norm of the Killing vector ξ^a.

The four-dimensional metric on the manifold of trajectories of ξ^a is

$$\gamma_{ab} \, dx^a \, dx^b = -e^\nu \, dt^2 + e^\beta [dr^2 + r^2 (d\theta^2 + \sin^2 \theta \, d\varphi^2)] \quad (18)$$

where

$$e^\nu = e^\mu + A^2/\phi^2 \quad (19)$$

The electromagnetic field tensor is defined in equation (6), and for the metric (17) the only nonvanishing component is

$$F_{rt} = -F_{tr} = \frac{1}{2} G^{1/2} \frac{d}{dr} \left(\frac{A}{\phi^2} \right) \equiv e^{\nu/2} E \quad (20)$$

where E is the radial component of the electric field.

We assume asymptotically flat boundary conditions in conjunction with a normalization of coordinates such that e^ν, e^μ, e^β, and ϕ^2 all approach 1 at infinity.

Although it is possible to study the five-dimensional form of Einstein's equations directly, we prefer to use the formalism of traditional four-dimensional relativity together with the machinery of Section 2 and the four-dimensional metric (18).

The analog of Maxwell's equation, equation (13), is

$$\frac{\partial}{\partial r} (r^2 \phi^3 e^{\beta/2} E) = 4\pi G^{1/2} r^2 e^{(\nu/2 + 3\beta/2)} J^t \quad (21)$$

which in vacuum is integrated immediately to obtain

$$r^2 \phi^3 e^{\beta/2} E \equiv Q \quad (22)$$

where Q is the constant electric charge.

The scalar equation (12) yields

$$\frac{d}{dr} \left(r^2 e^{\nu/2 + \beta/2} \frac{d\phi}{dr} \right) = -2G\phi^3 r^2 e^{\nu/2 + \beta/2} E^2 + \frac{1}{3} 8\pi G r^2 e^{\nu/2 + 3\beta/2} \phi (T^a{}_a - 2T_{55}) \quad (23)$$

and the $\binom{0}{0}$ and $\binom{1}{1}$ components of equation (14) are

$$e^{-\beta}\left(\beta'' + \frac{\beta'^2}{4} + \frac{2\beta'}{r}\right) = \frac{e^{-\beta}}{2\phi}\,\nu'\phi' + G\phi^2 e^{-\beta}E^2 + 8\pi G T^0_0 - \frac{8\pi G\phi^2}{3}\,(T^a{}_a - 2T_{55}) \tag{24}$$

and

$$e^{-\beta}\left(\frac{\beta'^2}{4} + \frac{\beta'\nu'}{2} + \frac{\beta' + \nu'}{r}\right) = \frac{e^{-\beta/2}}{\phi}\,(e^{-\beta/2}\phi')' + G\phi^2 e^{-\beta}E^2 + 8\pi G T^1_1 - \frac{8\pi G\phi^2}{3}\,(T^a{}_a - 2T_{55}) \tag{25}$$

where a prime denotes a derivative with respect to r and we have made use of equation (23). The $\binom{2}{2}$ and $\binom{3}{3}$ components of equation (14) are identical to each other and follow from the others via the Bianchi identities.

The most general vacuum solution to these equations is

$$\phi^2 = a_1\psi^{p_1} + a_2\psi^{p_2} \tag{26}$$

$$e^{\nu} = \psi^2/\phi^2 \tag{27}$$

$$A = (-a_1 a_2)^{1/2}(\psi^{p_1} - \psi^{p_2}) \tag{28}$$

$$e^{\mu} = a_2\psi^{p_1} + a_1\psi^{p_2} \tag{29}$$

and

$$e^{\beta/2} = (1 - B^2/r^2)/\psi \tag{30}$$

where

$$\psi = \left(\frac{r - B}{r + B}\right)^{\lambda/2B} \tag{31}$$

and a_1, a_2, p_1, p_2, λ, and B are constants which satisfy the following relationships.

Define

$$\kappa \equiv 4(4B^2 - \lambda^2)/\lambda^2 \tag{32}$$

then

$$\begin{aligned} p_1 &= 1 + (1 + \kappa)^{1/2} \\ p_2 &= 1 - (1 + \kappa)^{1/2} \end{aligned} \tag{33}$$

and

$$a_1 + a_2 = 1 \tag{34}$$

The electric field E comes from equation (22) where the charge Q is related to the other constants by

$$Q^2 = -a_1 a_2 \lambda^2 (1 + \kappa) G^{-1} \tag{35}$$

Only three of these parameters are independent, for example, κ, a_1, and B; the others are then determined from the above equations.

In the next section, we discuss the ratio of the electrostatic repulsive force to the sum of the gravitational and scalar attractive forces, as measured at large distances, for two identical objects of this sort. From equations (49)-(51) and (55) this charge-to-mass ratio is

$$\frac{Q^2}{GM^2} = \frac{4a_1(a_1 - 1)(\kappa + 1)}{\kappa + 4 + 4a_1(a_1 - 1)(\kappa + 1)} = \frac{Q^2}{4B^2 + Q^2} \tag{36}$$

So two identical objects will feel a net attractive or repulsive force depending upon the sign of B^2. Although it is not evident from equation (36), the condition that the metric be purely real places restrictions on B^2 and Q^2 which keeps Q^2/GM^2 positive.

For the metric described by equations (26) to (35) κ may be freely chosen as any real number. This choice then determines the range allowed for a_1 and B such that the metric is real. We find that the resulting metric falls into one of three distinct classes depending on the value of κ.

Class I. If $\kappa > -1$, then B is any positive real number,[2] and a_1 and a_2 are also real but must lie outside the interval [0, 1]. This class includes five-dimensional versions of the Schwarzschild and Brans (with his $\omega = 0$) metrics as special limits. And a boost in the fifth direction can always be found which removes the g_{xt} part of the metric. The charge-to-mass ratio, $(Q^2/GM^2)^{1/2}$, takes on values from 0 to 1.

A metric of this class contains a naked singularity at $r = B$, unless $\lambda = 2B$. For this special case the metric comes from a boost in the fifth direction of one of the uncharged solutions with either $a_1 = 0$ or $a_2 = 0$. If $a_1 = 0$ and $\lambda = 2B$, the metric is just the Schwarzschild metric with dx^2 tacked on the end.[3] So its boosted version has a singularity inside an event horizon. And with $a_2 = 0$ and $\lambda = 2B$ the metric comes from the Schwarzschild metric but with $t \leftrightarrow ix$. This metric, curiously, has no singularity at $r = B$ if the points at x and $x + 4\pi B$ are

[2] The metric is invariant under $B \to -B$.

[3] If $g_{ab}\, dx^a\, dx^b$ is a four-dimensional solution of Einstein's equations, then $g_{ab}\, dx^a\, dx^b + dx^2$ is always a five-dimensional solution, when x is a new fifth coordinate.

identified[4]; but for the boosted version this identification is not sufficient for removing the singularity and all such solutions have naked singularities.

Class II. If $-4 < \kappa < -1$ then B is any real number; but a_1 and a_2 are complex conjugates with a real part of $\frac{1}{2}$. It is then convenient to define

$$a_1 = \tfrac{1}{2} + i\alpha \tag{37}$$

where α is now any real number. Again $(Q^2/GM^2)^{1/2}$ varies from 0 to 1. The singularities in this class of metrics are essentially the same as those in the previous class.

Class III. If $\kappa < -4$ then the metric is real only if both B is an imaginary number and a_1 and a_2 are complex conjugates, so that α may be defined as in equation (37). For this case Q^2/GM ranges from 1 to ∞.

This is an interesting class of solutions. Neither ψ nor e^{β} vanishes for any finite value of r. As a result, ***in this class the five-dimensional metric is always invertible and contains no singularities***. A coordinate change to $R = r^{-1}$ easily shows that the region $r \to 0, R \to \infty$ is another asymptotically flat region. So the metric represents a static, nonsingular wormhole joining two asymptotically flat regions which may (or may not) be distinct.

The existence of such a solution is particularly interesting in the light of a theorem proved by Lichnerowicz [9] which states that the only stationary, asymptotically flat, singularity-free solution of Einstein's equations in four dimensions is flat space. His proof relies on the presence of just four dimensions. Also Einstein and Pauli [6] considered stationary, singularity-free solutions to equation (1) in five dimensions and in fact proved that no such solution can be imbedded in Minkowski space.

A metric of this class does have one pathological property. There always exists some value of r for which ϕ^2 is negative; in this region the Killing vector ξ^a is timelike. The pathology arises when, as in Paper I [7], we choose the topology of the extra dimension to be a small circle to explain the unobserved nature of the extra dimension. Thus a curve with constant t, r, θ, and φ, with x running from 0 to $2\pi L$, is a closed timelike line.

Another class of asymptotically flat solutions of the five-dimensional vacuum Einstein equations has recently been found by Belinsky and Ruffini [10]. Their class has three free parameters which might be chosen to be the mass, electric charge, and angular momentum; the scalar charge is uniquely related to these others. These solutions appear to us to be the Jordan-Brans-Dicke Kerr solutions found by McIntosh [11] coupled with a boost in the fifth direction. Our solutions overlap with the Belinsky and Ruffini solutions only when our electric charge vanishes. And, in particular, their solutions never allow a charge-to-mass ratio greater than unity.

[4]In this case the surface $r = B$ is an axis of the x coordinate and the topology of the extra dimension arises rather naturally.

§(4): *Weak-Field Limit*

It is useful to analyze weak-field, nonvacuum solutions of equations (21)-(25) in order to make some correspondence between the electric charge, gravitational mass, and inertial mass of this theory with those of the traditional Einstein-Maxwell equations. We let the source of the metric be a static, spherically symmetric, perfect fluid of radius R, density ρ, and pressure p with $p \ll \rho$, so that

$$T^{ab} = (\rho + p)\, U^a U^b + p g^{ab} \tag{38}$$

where U^a, the four-velocity of the fluid, has only the t component nonvanishing. We also assume that

$$J^a = q\rho U^a \tag{39}$$

where q is the electric charge per unit mass and that

$$T_{55} = s\rho \tag{40}$$

for some quantity s which we call the scalar charge per unit mass. If there is no magnetic field the only component of F_{ab} will be the rt component which is the electric field.

In the weak-field limit, $e^{\nu} \approx 1 + \delta\nu$, $e^{\beta} \approx 1 + \delta\beta$, and $\phi \approx 1 + \delta\phi$ where $\delta\nu$, $\delta\beta$, and $\delta\phi$ are presumed small. In this limit and with the above T_{ab}, equations (21)-(25) become

$$\frac{1}{r^2}\frac{d}{dr}(r^2 E) = 4\pi G^{1/2} q\rho \tag{41}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\delta\phi}{dr}\right) = \frac{-8\pi G\rho}{3}(1 + 2s) \tag{42}$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\delta\beta}{dr}\right) = \frac{-16\pi G\rho}{3}(1 - s) \tag{43}$$

and

$$\frac{1}{r}\frac{d}{dr}(\delta\beta + \delta\nu) = \frac{d^2\delta\phi}{dr^2} + \frac{8\pi G\rho}{3}(1 + 2s) \tag{44}$$

For simplicity we assume that ρ, q, and s are all constant throughout the source. Then, with the definition $m = \frac{4}{3}\pi G\rho R^3$, equations (41)-(44) are immediately integrated to yield

$$E = \frac{4\pi G^{1/2}}{r^2}\frac{q\rho R^3}{3} \equiv G^{1/2}\frac{qm}{r^2} \tag{45}$$

$$\delta\phi = \frac{2}{3}(1 + 2s)\frac{m}{r} \tag{46}$$

$$\delta\beta = \frac{4}{3}(1 - s)\frac{m}{r} \tag{47}$$

and

$$\delta\nu = -\frac{4}{3}(2 + s)\frac{m}{r} \tag{48}$$

Note that if $s = -\frac{1}{2}$ then $\delta\phi = 0$ and the rest of the solution looks exactly like the leading terms of the Reissner-Nordström solution.

The spherically symmetric metric of the last section may be expanded in powers of r^{-1}. This leads to a one-to-one correspondence between the strong-field spherically symmetric solutions and the weak-field solutions of this section. The quantities m, q, and s are related to the parameters of equations (32)-(35) by

$$m = \lambda[1 - \tfrac{1}{4}(a_1 p_1 + a_2 p_2)] \tag{49}$$

$$s = -\left[\frac{2 + a_1 p_1 + a_2 p_2}{2 + a_1 p_2 + a_2 p_1}\right] \tag{50}$$

and

$$Gqm = [-a_1 a_2 \lambda^2 (1 + \kappa)]^{1/2} \tag{51}$$

The Bianchi identities determine the equations of motion of the matter. In the weak-field background geometry of equations (45)-(48) distribute some pressureless dust with mass density ρ^* and scalar and electric charges per unit mass s^* and q^*. If only small velocity radial motion is allowed then equation (16) implies that the dust moves along a trajectory governed by

$$\frac{d^2 r}{d\tau^2} = \frac{Gqq^* m}{r^2} - \frac{2}{3}(2 + s)\frac{m}{r^2} - \frac{2s^*}{3}(1 + 2s)\frac{m}{r^2} \tag{52}$$

which illustrates, from left to right, the electrostatic force and traditional gravitational attraction from g_{00} along with an additional attraction caused by the scalar field. Equation (52) can also be written in the more symmetrical form

$$r^2 m^* \frac{d^2 r}{dt^2} = Gq^* q m^* m - m^* m - \frac{1}{3}(1 + 2s^*)(1 + 2s)\, m^* m \tag{53}$$

which demonstrates that Newton's third law is satisfied.

The equivalence principle asserts that the path along which an object falls in a gravitational field is independent of the composition of the object. It is clear from equation (53) that this principle forces s to be the same for all types of matter and the gravitational mass as determined by the Kepler orbits to be

$$M \equiv mG^{-1/2}[1 + \tfrac{1}{3}(1 + 2s)^2]^{1/2} \tag{54}$$

In this weak-field limit the ratio of the electrostatic repulsive force to the sum of the gravitational and scalar attractive forces for two identical objects can be interpreted as the charge-to-mass ratio squared for the object. So Q^2/GM^2, which was used in Section 3, is

$$\frac{Q^2}{GM^2} = \frac{q^2}{1 + \frac{1}{3}(1 + 2s)^2} \tag{55}$$

A theory based upon equation (10) is incomplete until the stress-energy tensor of five-dimensional matter, ${}_5T_{ab}$, is specified. We know of no way to choose ${}_5T_{ab}$ a priori. It is clear, however, that the 16 components of ${}_5T_{ab}$ which correspond to the usual four-dimensional space-time components should look like the four-dimensional stress energy of traditional matter. In addition, in equations (13) and (16) it is clear that the k-5 components of ${}_5T_{ab}$, where k runs from 0 to 3, look like the electric current. Only the 5-5 component seems to have no traditional analog in general relativity; however, in the five-dimensional theory it plays an important role as part of the source of the scalar field which contributes to the gravitational attraction in the Newtonian limit.

If we arbitrarily set ${}_5T_{55} = 0$ then, with no electromagnetic field, equations (12) and (14) reduce to the Brans-Dicke theory of gravity with their constant $\omega = 0$, which is not consistent with experimental evidence. On the other hand if we set $T_{55} = \frac{1}{2} T^a{}_a$, which is equivalent to $s = -\frac{1}{2}$ for the weak-field fluid, then, in the absence of an electromagnetic field, the source of the scalar field vanishes and equation (14) reduces to the traditional Einstein equations.

Today all experimental tests of gravity examine the weak gravitational field exterior to a spherically symmetric, uncharged object and are consistent with Einstein's equations, sometimes to within a few percent [12, 13]. Hence we can interpret these tests as putting a small upper limit on the quantity $s + \frac{1}{2}$ for the weak-field fluid.

It would be satisfying to have a five-dimensionally covariant argument for why s should be approximately $-\frac{1}{2}$. Unfortunately, such an argument is lacking at the present time.

§(5): *Conclusions*

In this paper we have displayed the most general time-independent spherically symmetric solution to Einstein's field equations in five dimensions, subject to the existence of a Killing vector in the fifth direction. With the appropriate choice of parameters, this solution can be expected to play much the same role in our theory that the Schwarzschild solution does in ordinary general relativity as the unique, spherically symmetric, static solution. And the charged nonrotating black holes of this theory are just the uncharged solutions with a boost in the fifth direction.

Of special interest are those solutions with $Q > M$, which may serve, at the classical level, as candidates for models of elementary particles. It is noteworthy that they are entirely singularity-free and only suffer from the blemish of closed timelike lines which result from the global topology (specifically, the fact that we have chosen to wrap the fifth dimension up on itself) and not because of any local property of the manifold.

On the classical level this theory is still incomplete. Why the scalar field is absent in experiments is one conundrum we face and is intimately related to the nature of five-dimensional matter.

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