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A Generalization of Tensor Calculus and its Applications to Physics

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Abstract

Penrose's abstract index notation and axiomatic introduction of covariant derivatives in tensor calculus is generalized to fields with internal degrees of freedom. The result provides, in particular, an intrinsic formulation of gauge theories without the use of bundles.

§(1): *Introduction*

Traditionally, the subject of tensor calculus was approached in two different ways. Physicists, in general, introduced tensor fields in terms of the transformation properties of their components and defined covariant derivatives using Christoffel symbols; all tensorial operations were reduced to the familiar manipulations of functions. Mathematicians, on the other hand, regarded tensor fields

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as cross sections of suitable bundles and defined covariant derivatives in terms of certain (Lie-algebra-valued) connection 1-forms; the emphasis was on geometry. Although both approaches are of course completely equivalent locally, the difference in orientation can be quite significant in practice. In particular, the first approach is much more convenient in long calculations while the second has the virtue of bringing out the invariant significance of the various operations. Thus, one was always faced with a choice between practical convenience and aesthetic elegance. This state of affairs persisted until Penrose [1] pointed out that one can adopt a third viewpoint which combines the advantages of both traditional approaches. The key ingredient in Penrose's approach is the use of the so-called abstract indices. These are to be thought of as labels indicating the valence of tensor fields and do not take on numerical values. Thus, one can regard a tensor field with abstract indices as an intrinsic object in its own right; no mention need be made of its components and their transformation properties. The availability of indices, however, provides a great flexibility in calculations. The overall viewpoint is a practical one: one introduces the various algebraic operations as well as the derivative operators on tensor fields directly in terms of those properties which one uses in practice. This procedure pinpoints the mathematical structures involved in the operations normally performed, thereby adding a great deal of clarity. As a general rule, proofs, e.g., of algebraic properties of Riemann tensor, of Bianchi identities, etc., are at least as simple as those in the component notation and at least as elegant as the ones in the bundle formalism.

The current situation in the mathematical formulation of gauge theories is analogous to that in tensor calculus prior to the introduction of Penrose's notation: in the particle physics literature, gauge fields are treated in terms of their components in some specified internal frame field, while in the mathematical literature, one regards them as cross sections of suitable bundles. One would therefore like to have a framework which combines the computational facility of the component formalism and the intrinsic character of the bundle description. That is, it is desirable to extend Penrose's framework for tensor calculus to calculus of gauge fields. The purpose of this paper is to present such an extension. The basic idea is to introduce abstract indices also for gauge degress of freedom. These indices will again serve as labels indicating the valence of the field (in the space of internal degrees of freedom) and will not assume numerical values. As in the case of tensors, the algebraic properties of these objects with abstract indices—the generalized tensor fields—will simply mirror the properties of components of gauge fields and novel features will appear only when a derivative operator is introduced. Throughout, gauge and Higgs fields will be described by generalized tensor fields on the space-time manifold itself; we will not need to introduce any bundles. Overall, the approach is more algebraic than geometric: the derivative operators on gauge fields, for example, will be introduced in terms of their "algebraic" properties rather than in terms of, say, parallel transport, or, horizontal subspaces in a principal fiber bundle. Moreover, the gauge group which plays a dominant role in the geometric picture—one uses it in the very first step in the construction of the principal fiber bundle—will play a relatively minor role in the present framework. Nonetheless, the final picture is completely equivalent to the usual one as far as "local" issues are concerned. The difference, as in the case of Penrose's approach to tensor calculus, is in emphasis: computations are simplified by bringing to the forefront the properties of various operations which one needs to use repeatedly while the conceptual issues are made clearer by avoiding the use of frames and maintaining the intrinsic character of various objects.

In Section 2, we introduce generalized tensor fields together with their algebraic properties. Section 3 is devoted to calculus. First, we define derivative operators on generalized tensor fields axiomatically, construct the curvature tensors, and derive Bianchi identities. Then, we discuss the issue of uniqueness of derivative operators and show that there is a natural one-to-one correspondence between derivative operators with zero curvature and equivalence classes of frame fields in the internal space, where two fields are considered as equivalent if they differ by a global gauge transformation. In Section 4 we consider the issue of "trivialization." The generalized tensor fields, the derivative operators, and their curvatures are all intrinsic objects. On introducing a basis field, the structure trivializes as follows: first, any generalized tensor field can now be reduced to ordinary tensor fields via contraction of the "internal indices" with the basis. Second, the derivative operator with zero curvature corresponding to the given basis can be used as an "origin" in the affine space of all derivatives, so that any derivative operator can now be represented by (the components of) the generalized tensor field which describes the difference of the given derivative from the "origin." A different choice of basis leads to a different trivialization and the two structures are shown to be related by the familiar gauge transformations. The Appendix considers the special case with one internal degree of freedom and discusses some examples familiar from general relativity.[4]

In view of the widespread use of Penrose's notation in the relativity community, it is hoped that this presentation of gauge theories[5] will be especially useful to relativists.

§(2): *The Algebraic Structure*

Fix a finite-dimensional C^∞ manifold M. Let $t^{\alpha_1 \ldots \alpha_m}{}_{\beta_1 \ldots \beta_n}$ denote a real, C^∞ tensor field of valence $\binom{m}{n}$. Here, $\alpha_1 \ldots \alpha_m$ and $\beta_1 \ldots \beta_n$ are Penrose's [1] "abstract" indices. We now wish to introduce the generalized tensor fields which will

[4] After this work was completed, we learned that a detailed treatment of gauge fields along the lines indicated here appears in [2] (R. Penrose, private communication to GTH).

[5] For the sake of generality, the entire discussion is carried out on n-manifolds which do not carry any preferred structure, e.g., a metric. Therefore, the framework may well have applications besides gauge theories.

carry, in addition to the Greek indices, contravariant and covariant latin indices. Denote by $S^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 \dots a_p}{}_{b_1 \dots b_q}$ the set of symbols

$$t^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 \dots a_p}{}_{b_1 \dots b_q},$$

where the stem letter (t, in the example) varies from one symbol to another but where the index structure of each symbol is the same as that of S. We shall permit only those index structures in which no index appears more than once in each of the four slots. Thus, for example, the set

$$S^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \beta_1 \dots \beta_{n-1}}{}^{a_1 \dots a_p}{}_{b_1 \dots b_q} \quad \text{or} \quad S^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 \dots a_{p-1} a_{p-1}}{}_{b_1 \dots b_q}$$

is not permissible. Denote by S the union of all permissible sets.

Next, we introduce certain structure on S. We require the following.

(i) Each $S^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}$ with no Latin indices is precisely the set of C^∞, real-valued tensor fields on M with the corresponding (abstract) index structure.

(ii) Each $S^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 \dots a_p}{}_{b_1 \dots b_q}$ is an Abelian group inder the operation *addition*, denoted by +.

(iii) Associated with any two generalized tensor fields

$$t^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 \dots a_p}{}_{b_1 \dots b_q} \quad \text{and} \quad u^{\alpha_{m+1} \dots \alpha_{m+i}}{}_{\beta_{n+1} \dots \beta_{n+j}}{}^{a_{p+1} \dots a_{p+k}}{}_{b_{q+1} \dots b_{q+l}}$$

(where no index is repeated in any one of the 4-slots), there exists an element

$$S^{\alpha_1 \dots \alpha_m \alpha_{m+1} \dots \alpha_{m+i}}{}_{\beta_1 \dots \beta_n \beta_{n+1} \dots \beta_{n+j}}{}^{a_1 \dots a_p a_{p+1} \dots a_{p+k}}{}_{b_1 \dots b_q b_{q+1} \dots b_{q+l}},$$

called *outer product* and denoted by the juxtaposition

$$t^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 \dots a_p}{}_{b_1 \dots b_q} \cdot u^{\alpha_{m+1} \dots \alpha_{m+i}}{}_{\beta_{n+1} \dots \beta_{n+j}}{}^{a_{p+1} \dots a_{p+k}}{}_{b_{q+1} \dots b_{q+l}}.$$

Furthermore, we demand that the outer product be associative, commutative, distributive over addition.

(iv) A generalized tensor field (such as $t^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 a_2 \dots a_p}{}_{b_1 a_2 \dots b_q}$) in which the latin index appears both in the contravariant and the covariant slot is identified with the generalized tensor field with the same stem letter and with the index structure obtained by omitting the repeated indices

$$(t^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 a_3 \dots a_p}{}_{b_1 b_3 \dots b_q}).$$

We shall say that

$$t^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 a_3 \dots a_p}{}_{b_1 b_3 \dots b_q}$$

is obtained from

$$t^{\alpha_1 \dots \alpha_m}{}_{\beta_1 \dots \beta_n}{}^{a_1 a_2 a_3 \dots a_p}{}_{b_1 b_2 b_3 \dots b_q}$$

by *contraction* of indices a_2 and b_2. We demand that the operation of contraction commute with addition and (if carried out between the indices of the same generalized tensor field, also with) outer product.

(v) Given a generalized tensor field (such as $t^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q}$) and two Latin indices one of which appears either in the contravariant or the covariant slot of this field (say, a_1) and the other of which appears in neither (say, c_1), there exists a unique generalized tensor field ($t^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{c_1a_2\ldots a_p}{}_{b_1\ldots b_q}$) in which the second index replaces the first. This operation will be called *index substitution*. We demand that it commute with addition, outer product and contraction (provided these operations are well defined both before and after the desired index substitution.)

Remarks. (1) as with tensors [1], addition, outer-product, contraction, and index substitution are the only algebraic operations that we shall need while dealing with generalized tensor fields.

(2) The requirement that the outer product be commutative is a key one. This requirement may seem to be contrary to the usual conditions imposed on the outer product in the mathematical literature. However, it is not: as is explained in [1], it is the availability of indices that enables the imposition of this condition. Thus, for example, the generalized tensor field V^aW^b is being identified with W^bV^a (*not* with V^bW^a!).

(3) Contraction and index substitution are of course well defined also on Greek indices. We did not introduce these requirements explicitly simply because they follow from condition (i). Each of these operations on Greek indices is required to commute with contraction and index substitution operations on Latin indices. Finally, note that the operation of index substitution may be denoted by δ: $V^a \to \delta^b{}_a V^a = V^b$. This operation merely sets up isomorphisms. Thus V^b in S^b is the image under index substitution of V^a in S^a; V^b is *not* identified with V^a.

As a consequence of the requirements imposed on S, one readily obtains the following result:

Theorem 1. Each set $S^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q}$ is a module over the ring of C^∞ functions on M.

Using this result, we impose the last condition on S.

(vi) $S^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q}$ consists precisely of the (functionally) linear mappings from $S^{b_1}{}_{a_1}$ to $S^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_2\ldots a_p}{}_{b_2\ldots b_q}$ and similarly for other index combinations.

We can now introduce the basic definition which will make the physical interpretation of S evident.

Definition 1. S will be called a *system of generalized tensor fields with N internal degrees of freedom* provided the module S^a is N dimensional, i.e., provided the maximal number of linearly independent, nowhere vanishing[6] elements of S^a is N.

Thus, while the Greek indices are "space-time" (or tensor) indices, the Latin ones are the "internal"; a generalized tensor field represents, e.g., a tensor-valued gauge or Higgs field.

To probe the structure available on S, it is convenient to introduce two additional definitions.

Definition 2. A collection $\{e^a{}_{\mathbf{a}}\}$, $\mathbf{a} = 1, 2, \ldots, N$, of nowhere vanishing, linearly independent elements of S^a will be said to constitute a *basis* in S^a.

Definition 3. The group of $N \times N$ invertible matrices whose entries are C^∞ functions on M will be called the *local general linear* group and will be denoted by $GL(N, R)_{\text{loc}}$.

Remarks. (1) In the terminology of [1], $a_1, b_1, \ldots$ are the "abstract indices," while $\mathbf{a}, \mathbf{b}, \ldots$ take on numerical values $1, 2, \ldots, N$. Thus, $\{e^a{}_{\mathbf{a}}\}$ stands for a collection of N elements of S^a. Elements of $GL(N, R)_{\text{loc}}$ can be denoted by symbols of the type $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}$. Clearly, $GL(N, R)_{\text{loc}}$ has a well-defined action on the collection of bases in S^a: $e^a{}_{\mathbf{a}} \to \hat{e}^a{}_{\mathbf{a}} = \Lambda^{\mathbf{b}}{}_{\mathbf{a}} e^a{}_{\mathbf{b}}$. Here, and in what follows, Einstein's summation convention is used for the numerical indices $\mathbf{a}, \mathbf{b}, \ldots$.

(2) It follows from the last requirement on S that S_a—being the space of linear mappings from S^a to C^∞ functions on M—is again N dimensional. Given a basis $\{e^a{}_{\mathbf{a}}\}$ in S^a, we acquire a natural dual basis $\{e^{\mathbf{a}}{}_a\}$ in S_a: $e^a{}_{\mathbf{a}} e^{\mathbf{b}}{}_a = \delta_{\mathbf{a}}{}^{\mathbf{b}}$, the identity element of $GL(N, R)_{\text{loc}}$. The two bases also satisfy $e^a{}_{\mathbf{a}} e^{\mathbf{a}}{}_b = \delta^a{}_b$, the index substitution operator.

(3) From the requirements (ii)-(vi), it follows that any generalized tensor field $t^{a_1 \ldots a_p}{}_{b_1 \ldots b_q}$ can be obtained, as a sum of outer products of elements of $S^{a_1}, \ldots, S^{a_p}, S_{b_1}, \ldots, S_{b_q}$. Consider, for example, a field $t^a{}_b$. By (vi), $t^a{}_b$ is a linear mapping from S^b to S^a. Hence, given a basis $\{e^a{}_{\mathbf{a}}\}$ in S^a, we can set $t^{\mathbf{a}}{}_{\mathbf{b}} = (t^a{}_b e^b{}_{\mathbf{b}})\, e^{\mathbf{a}}{}_a$; $t^{\mathbf{a}}{}_{\mathbf{b}}$ is an $N \times N$ matrix whose entries are C^∞ functions on M. Therefore, it can be expressed as a sum of decomposable matrices of the type $\Lambda^{\mathbf{a}} \tau_{\mathbf{b}}$, whence $t^a{}_b$ can be expressed as a sum of terms of the type $\Lambda^{\mathbf{a}} \tau_{\mathbf{b}} e^a{}_{\mathbf{a}} e^{\mathbf{b}}{}_b \equiv \Lambda^a \tau_b$, say. Since the expression of $t^{\mathbf{a}}{}_{\mathbf{b}}$ in terms of decomposable matrices is not unique, neither is the expression of $t^a{}_b$ as a sum of outer products.

Lemma 2.1. $GL(N, R)_{\text{loc}}$ acts simply and transitively on the collection of bases in S^a.

Proof. Let $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}$ in $GL(N, R)_{\text{loc}}$ be such that $\Lambda^{\mathbf{a}}{}_{\mathbf{b}} e^a{}_{\mathbf{a}} = e^a{}_{\mathbf{b}}$ for a basis $\{e^a{}_{\mathbf{a}}\}$. Then, by transvecting with $e^{\mathbf{c}}{}_a$ we obtain $\Lambda^{\mathbf{c}}{}_{\mathbf{b}} = \delta^{\mathbf{c}}{}_{\mathbf{b}}$. Thus the only element of

[6] k^a is said to vanish at a point p of M if the function $k^a t_a$ vanishes at p for all elements t_a of S_a.

$GL(N, R)_{\text{loc}}$ which leaves a basis invariant is the identity, whence the action is simple. Next, consider any two bases $\{e^a{}_{\mathrm{a}}\}$ and $\{\hat{e}^a{}_{\mathrm{a}}\}$ in S^a. It is easy to check that $\Lambda^{\mathrm{a}}{}_{\mathrm{b}} := \hat{e}^b{}_{\mathrm{b}} e^{\mathrm{a}}{}_b$ is in $GL(N, R)_{\text{loc}}$ and $\hat{e}^a{}_{\mathrm{a}} = \Lambda^{\mathrm{b}}{}_{\mathrm{a}} e^a{}_{\mathrm{b}}$. Hence the action is transitive. ∎

Lemma 2.1, leads to the following result.

Theorem 2. Any two systems S and S' of generalized tensors with N internal degrees of freedom over the manifold M are isomorphic and $GL(N, R)_{\text{loc}}$ acts simply and transitively on the collection of isomorphisms.

Proof. Fix a basis $\{e^a{}_{\mathrm{a}}\}$ in S^a and $\{e'^a{}_{\mathrm{a}}\}$ in S'^a. It follows from the requirements (i)-(vi) that an isomorphism between these bases extends uniquely to an isomorphism between S and S'. Hence S and S' are isomorphic. Next, any isomorphism between S and S' provides, in particular, an isomorphism between the collection of bases. Hence, by Lemma 2.1, $GL(N, R)_{\text{loc}}$ acts simply and transitively on the set of isomorphisms between S and S'. ∎

Remarks. (1) Theorem 2 says that there are "as many isomorphisms between S and S' as there are elements of $GL(N, R)_{\text{loc}}$." This, then, is the freedom one has in constructing a system of generalized tensors with N internal degrees on a given manifold M. We shall see, in the next section, that the introduction of a derivative operator reduces this freedom considerably.

(2) Note that the basic objects in the discussion are fields, rather than generalized tensors evaluated at a point. It is this fact that led us directly to the local gauge groups without using bundles.

(3) The restriction to real fields is for convenience only: the use of complex fields would have led to two types of indices, $a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$, and made the general formalism in this section more cumbersome. The extension to complex fields of all our results is straightforward.

(4) Sometimes, physical considerations lead to the introduction of certain preferred generalized tensor fields such as a positive definite metric g_{ab} or an "alternating tensor field" $\epsilon_{a_1 \ldots a_N}$. Then, one can restrict oneself only to those bases which are adapted to this structure, e.g., orthonormal, right/left-handed bases. This restriction causes a reduction of the gauge group from $GL(N, R)_{\text{loc}}$ to, e.g., $O(N, R)_{\text{loc}}$ or $SO(N, R)_{\text{loc}}$.

(5) We must emphasize, however, that the framework of generalized tensors itself does *not* require the use of a basis: We have introduced bases only to make contact with the more familiar frameworks. The primary objects are the generalized tensor fields with abstract indices; derivative operators, for example, will be introduced *directly* on these objects. If the introduction of bases is avoided, the local gauge group $GL(N, R)_{\text{loc}}$ makes its appearance at the algebraic level only via Theorem 2.

(6) The algebraic structure on generalized tensors is exactly the same as that normally used in the particle physics literature and the "abstract" indices have all the familiar properties of "component labels." One might therefore feel that

nothing new has been achieved. Recall, however, the motivation behind the formalism: one wishes to obtain the flexibility and manipulation power of the component notation without having to use components. Thus, the fact that abstract indices appear to have the algebraic properties of components is *not* a drawback; this is precisely what is aimed at.

§(3): *Calculus*

Fix a system S of generalized tensor fields with N internal degrees of freedom on M. A *derivative operator* ∇_α on S is a mapping from sets of the type $S^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q}$ to those of the type $S_\alpha{}^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q}$ satisfying the following conditions. (i) Linearity:

$$\nabla_\alpha(t^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots} + u^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots}) = \nabla_\alpha(t^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots}) + \nabla_\alpha(u^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots})$$

(ii) Liebnitz Rule:

$$\nabla_\alpha(t^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots}v^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots}) = (\nabla_\alpha t^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots})\, v^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots} + t^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots}\nabla_\alpha v^{\cdots}{}_{\cdots}{}^{\cdots}{}_{\cdots}$$

(iii) Given any (C^∞) function f and a vector field V^α on M,

$$V^\alpha\nabla_\alpha f = \mathcal{L}_V f \qquad \text{and} \qquad \nabla_{[\alpha}\nabla_{\beta]}f = 0$$

where, as usual, $\mathcal{L}_V$ denotes the Lie derivative with respect to V and square brackets around α and β denote skew symmetrization.

An immediate consequence of these assumptions is that the restriction of ∇ to tensor fields (i.e., generalized tensor fields with no latin indices) yields a torsion-free connection on M. However, ∇, as defined, can operate on *any* generalized tensor field. In particular, it is possible to have several distinct derivative operators on S whose restriction to tensor fields coincide. The algebraic properties of S and the defining properties of ∇ imply that the action of ∇ on elements of S^α (or S_α) and S^a (or S_a) determines its action on all of S.

Fix a derivative operator ∇ and consider the operator $\nabla_{[\alpha}\nabla_{\beta]}$. We have, for any C^∞ function f and any element t^a of S^a,

$$\begin{aligned}\nabla_{[\alpha}\nabla_{\beta]}(ft^a) &= f\nabla_{[\alpha}\nabla_{\beta]}t^a + t^a\nabla_{[\alpha}\nabla_{\beta]}f + (\nabla_{[\alpha}f)(\nabla_{\beta]}t^a) + (\nabla_{[\beta}f)(\nabla_{\alpha]}t^a)\\ &= f\nabla_{[\alpha}\nabla_{\beta]}t^a\end{aligned}$$

where, we have used conditions (ii) and (iii) in the definition of ∇. Thus, the operator $\nabla_{[\alpha}\nabla_{\beta]}$ maps S^a into $S_{\alpha\beta}{}^a$ and the map is (functionally) linear. Hence [by the requirement (vi) on the algebraic structure of S], there exists an element $F_{\alpha\beta}{}^a{}_b$ of $S_{\alpha\beta}{}^a{}_b$ such that

$$2\nabla_{[\alpha}\nabla_{\beta]}t^a = F_{\alpha\beta}{}^a{}_b t^b \tag{1}$$

for all t^a in S^a. An identical calculation shows that there exists a tensor field $R_{\alpha\beta\gamma}{}^{\delta}$ on M, such that, for all covector fields t_γ,

$$2\nabla_{[\alpha}\nabla_{\beta]}t_\gamma = R_{\alpha\beta\gamma}{}^{\delta}t_\delta \tag{2}$$

Combining (1) and (2) and using the Leibnitz rule, we have

$$\nabla_{[\alpha}\nabla_{\beta]}t_\gamma{}^a = \tfrac{1}{2}\,(R_{\alpha\beta\gamma}{}^{\delta}\delta^a{}_b + F_{\alpha\beta}{}^a{}_b\delta_\gamma{}^\delta)\,t_\delta{}^b \tag{3}$$

for all $t_\gamma{}^a$ in $S_\gamma{}^a$. The expression in the bracket is the curvature tensor of ∇ on S. If we restrict ourselves to tensor fields alone, the curvature reduces to the Riemann tensor, while if ∇ is an extension to S of a flat connection on M, the curvature expression yields the familiar "Yang-Mills" field.

Next, we derive certain properties of the curvature tensor. By the very definition of curvature, we have

$$F_{\alpha\beta}{}^a{}_b = F_{[\alpha\beta]}{}^a{}_b \qquad \text{and} \qquad R_{\alpha\beta\gamma}{}^\delta = R_{[\alpha\beta]\gamma}{}^\delta \tag{4}$$

Next, since any vector field V_μ can be written as a sum $f^{\boldsymbol{\alpha}}\nabla_\mu g_{\boldsymbol{\alpha}}$ for some functions $f^{\boldsymbol{\alpha}}$ and $g_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha}$ runs from 1 to dimension of M), we have

$$\nabla_{[\alpha}\nabla_\beta V_{\mu]} = \nabla_{[\alpha}\nabla_\beta f^{\boldsymbol{\alpha}}\nabla_{\mu]}g_{\boldsymbol{\alpha}} = 0$$

whence

$$R_{[\alpha\beta\mu]}{}^\nu = 0 \tag{5}$$

This is the first Bianchi identity on the Riemann tensor. Using this, we have, for any t^a in S^a,

$$\begin{aligned}\nabla_{[\alpha}(F_{\beta\gamma]}{}^a{}_b t^b) &= 2\nabla_{[\alpha}\nabla_\beta\nabla_{\gamma]}T^a \\ &= R_{[\alpha\beta\gamma]}{}^\delta\nabla_\delta T^a + F_{[\alpha\beta}{}^a{}_{|b|}\nabla_{\gamma]}t^b\end{aligned}$$

whence, by (5), we obtain

$$\nabla_{[\alpha}F_{\beta\gamma]}{}^a{}_b = 0 \tag{6}$$

Similarly, one has, for the Riemann tensor,

$$\begin{aligned}\nabla_{[\alpha}(R_{\beta\gamma]\mu}{}^\nu V_\nu) &= 2\nabla_{[\alpha}\nabla_\beta\nabla_{\gamma]}V_\mu \\ &= R_{[\alpha\beta\gamma]}{}^\delta\nabla_\delta V_\mu + R_{[\alpha\beta|\mu|}{}^\delta\nabla_{\gamma]}V_\delta\end{aligned}$$

whence, from (5), we obtain the second Bianchi identity

$$\nabla_{[\alpha}R_{\beta\gamma]\delta}{}^\mu = 0 \tag{7}$$

This establishes the basic properties of the curvature fields.

Let us now analyze the issue of uniqueness of derivative operators on S. Let ∇ and ∇' be two derivative operators. A straightforward calculation shows that, for all functions f on M and for all t^a in S^a,

$$(\nabla_\alpha - \nabla'_\alpha)(f t^a) = f(\nabla_\alpha - \nabla'_\alpha)\, t^a$$

Hence there exists a generalized tensor field $\mathfrak{A}_\alpha{}^a{}_b$ such that

$$(\nabla_\alpha - \nabla'_\alpha)\, t^a = \mathfrak{A}_\alpha{}^a{}_b\, t^b \tag{8}$$

Similarly, it follows that

$$(\nabla_\alpha - \nabla'_\alpha)\, t_\beta = C_{\alpha\beta}{}^\gamma t_\gamma \tag{9}$$

for some tensor field $C_{\alpha\beta}{}^\gamma$ satisfying $C_{\alpha\beta}{}^\gamma = C_{(\alpha\beta)}{}^\gamma$. Thus, the derivative operators form an affine space; the difference between any two can be completely characterized by a pair $(\mathfrak{A}_\alpha{}^a{}_b, C_{\alpha\beta}{}^\gamma)$ of generalized tensor fields (where $C_{\alpha\beta}{}^\gamma$ is symmetric in α and β).

In the rest of the paper, for simplicity, we shall restrict ourselves to those derivative operators which are extensions to S of a fixed flat connection (i.e., a connection with zero Riemann tensor) on M.

Denote by C the collection of these derivative operators. Then, any two elements, ∇ and ∇' of C, are related via (8) by some generalized tensor field $\mathfrak{A}_\alpha{}^a{}_b$. Thus C is again an affine space.

To conclude this section, let us note certain properties of derivative operators $\mathring{\nabla}$ in C with vanishing $F_{\alpha\beta}{}^a{}_b$.

Lemma 3.1. There is a natural one-to-one correspondence between elements $\mathring{\nabla}$ of C with zero curvature and equivalence classes of bases $\{e^a{}_{\mathbf{a}}\}$ in S^a where two bases are considered as equivalent if they are related by an element of $GL(N, R)$. The pair $(\mathring{\nabla}, \{e^a{}_{\mathbf{a}}\})$ satisfies $\mathring{\nabla}_\alpha e^a{}_{\mathbf{a}} = 0$ for all $\mathbf{a}$ in $(1, \ldots, N)$.

Proof. Fix a basis $e^a{}_{\mathbf{a}}$ in S^a. Then, by the defining properties of derivative operators and property (vi) of S it follows that the action of a derivative operator on S^a suffices to determine it completely. (Recall that, by assumption, the action on ordinary tensor fields has been prespecified). Set $\mathring{\nabla}_\alpha e^a{}_{\mathbf{a}} = 0$. Then, since $0 = \mathring{\nabla}_{[\alpha}\mathring{\nabla}_{\beta]} e^a{}_{\mathbf{a}} = F_{\alpha\beta}{}^a{}_b\, e^b{}_{\mathbf{a}}$ for all basis vectors, the derivative operator $\mathring{\nabla}$ has zero curvature. Let us suppose that there exists, in addition, a basis $\hat{e}^a{}_{\mathbf{a}}$ such that $\mathring{\nabla}_\alpha \hat{e}^a{}_{\mathbf{a}} = 0$. Let $\hat{e}^a{}_{\mathbf{a}} = \Lambda^{\mathbf{b}}{}_{\mathbf{a}} e^a{}_{\mathbf{b}}$. Then $\mathring{\nabla}_\alpha \Lambda^{\mathbf{b}}{}_{\mathbf{a}} = 0$, whence for each value of $\mathbf{a}$ and $\mathbf{b}$, $\Lambda^{\mathbf{b}}{}_{\mathbf{a}}$ is a constant. Thus, $\hat{e}^a{}_{\mathbf{a}}$ and $e^a{}_{\mathbf{a}}$ are related by an element of the global gauge group $GL(N, R)$.

Let us now fix a derivative operator $\mathring{\nabla}'$ with zero curvature. Then, for k^a in S^a we have

$$\begin{aligned}
0 = \mathring{\nabla}'_{[\alpha}\mathring{\nabla}'_{\beta]} k^a &= \mathring{\nabla}_{[\alpha}\mathring{\nabla}'_{\beta]} k^a + \mathfrak{A}_{[\alpha}{}^a{}_{|b|}\mathring{\nabla}'_{\beta]} k^b \\
&= \mathring{\nabla}_{[\alpha}\mathring{\nabla}_{\beta]} k^a + \mathring{\nabla}_{[\alpha}(\mathfrak{A}_{\beta]}{}^a{}_b k^b) + \mathfrak{A}_{[\alpha}{}^a{}_{|b|}\mathring{\nabla}_{\beta]} k^b + \mathfrak{A}_{[\alpha}{}^a{}_{|b|}\mathfrak{A}_{\beta]}{}^b{}_c k^c \\
&= (\mathring{\nabla}_{[\alpha}\mathfrak{A}_{\beta]}{}^a{}_c + \mathfrak{A}_{[\alpha}{}^a{}_{|b|}\mathfrak{A}_{\beta]}{}^b{}_c)\, k^c
\end{aligned}$$

whence the field $\mathfrak{A}_\alpha{}^a{}_b$ relating $\mathring{\nabla}$ and $\mathring{\nabla}'$ must satisfy

$$\mathring{\nabla}_{[\alpha}\mathfrak{A}_{\beta]}{}^a{}_c + \mathfrak{A}_{[\alpha}{}^a{}_{|b|}\mathfrak{A}_{\beta]}{}^b{}_c = 0$$

Transvecting this equation with $e^{\mathbf{a}}{}_a e^c{}_{\mathbf{c}}$ and using the fact that $\mathring{\nabla}_\alpha e^a{}_{\mathbf{a}}$ vanishes, we obtain

$$\mathring{\nabla}_{[\alpha}\mathfrak{A}_{\beta]}{}^{\mathbf{a}}{}_{\mathbf{c}} + \mathfrak{A}_{[\alpha}{}^{\mathbf{a}}{}_{|\mathbf{b}|}\mathfrak{A}_{\beta]}{}^{\mathbf{b}}{}_{\mathbf{c}} = 0$$

which is a partial differential equation involving ordinary vector fields on M. This equation can always be solved on M:

$$\mathfrak{A}_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}} = (\Lambda^{-1})^{\mathbf{a}}{}_{\mathbf{c}}\nabla_\alpha\Lambda^{\mathbf{c}}{}_{\mathbf{b}}$$

where $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}$, an element of $GL(N, R)_{\text{loc}}$, is determined up to a global gauge transformation.[7] Set $e'^a{}_{\mathbf{a}} = (\Lambda^{-1})^{\mathbf{b}}{}_{\mathbf{a}}e^a{}_{\mathbf{b}}$. Then, one has

$$\begin{aligned}
\mathring{\nabla}'_\alpha e'^a{}_{\mathbf{a}} &= \mathring{\nabla}'_\alpha[(\Lambda^{-1})^{\mathbf{b}}{}_{\mathbf{a}}e^a{}_{\mathbf{b}}] \\
&= -(\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{a}}\mathfrak{A}_\alpha{}^{\mathbf{b}}{}_{\mathbf{c}}e^a{}_{\mathbf{b}} + (\Lambda^{-1})^{\mathbf{b}}{}_{\mathbf{a}}\mathring{\nabla}'_\alpha e^a{}_{\mathbf{b}} \\
&= (\Lambda^{-1})^{\mathbf{b}}{}_{\mathbf{a}}\mathring{\nabla}_\alpha e^a{}_{\mathbf{b}} = 0
\end{aligned}$$

Hence $\mathring{\nabla}'$ determines an equivalence class of bases $\{e'^a{}_{\mathbf{a}}\}$ via $\mathring{\nabla}'_\alpha e'^a{}_{\mathbf{a}} = 0$. ∎

Using this Lemma and Theorem 2, we can now establish the following.

Theorem 3. Let S and S' be two systems of generalized tensor fields on M with N internal degrees of freedom. Let $\mathring{\nabla}$ and $\mathring{\nabla}'$ be derivative operators with zero curvature on S and S', respectively, such that their action coincides on tensor fields on M. Then, $(S, \mathring{\nabla})$ and $(S', \mathring{\nabla}')$ are isomorphic and $GL(N, R)$ acts simply and transitively on the space of isomorphisms.

Proof. Fix any basis $e^a{}_{\mathbf{a}}$ in S^a and $e'^a{}_{\mathbf{a}}$ in S'^a such that $\mathring{\nabla}_\alpha e^a{}_{\mathbf{a}} = 0$ and $\mathring{\nabla}'_\alpha e'^a{}_{\mathbf{a}} = 0$ and consider the unique isomorphism Ψ from S to S' which maps $e^a{}_{\mathbf{a}}$ to $e'^a{}_{\mathbf{a}}$. Then, clearly,

$$\Psi(\mathring{\nabla}_\alpha T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q}) = \mathring{\nabla}'_\alpha\Psi(T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q})$$

for all $T^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_q}$ in S; Ψ maps $\mathring{\nabla}$ to $\mathring{\nabla}'$. Thus, $(S, \mathring{\nabla})$ and $(S', \mathring{\nabla}')$ are isomorphic. Next, consider any other isomorphism $\hat{\Psi}$. Then, $0 = \hat{\Psi}(\mathring{\nabla}_\alpha e^a{}_{\mathbf{a}}) = \mathring{\nabla}'_\alpha\hat{\Psi}(e^a{}_{\mathbf{a}})$. Hence, by Lemma 3.1, $\hat{\Psi}(e^a{}_{\mathbf{a}})$ is related to $e'^a{}_{\mathbf{a}}$ by an element $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}$ of $GL(N, R)$; $\hat{\Psi}(e^a{}_{\mathbf{a}}) = \Lambda^{\mathbf{b}}{}_{\mathbf{a}}e'^a{}_{\mathbf{b}} = \Lambda^{\mathbf{b}}{}_{\mathbf{a}}\Psi(e^a{}_{\mathbf{a}})$. Thus, $GL(N, R)$ acts transitively on the space of isomorphisms. Since the only element $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}$ of $GL(N, R)$ satisfying $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}e'^b{}_{\mathbf{a}} = e'^b{}_{\mathbf{b}}$ is the identity, the action of $GL(N, R)$ on the space of isomorphisms is simple. ∎

[7]That is, $\Lambda'^{\mathbf{a}}{}_{\mathbf{b}} = k^{\mathbf{a}}{}_{\mathbf{c}}\Lambda^{\mathbf{c}}{}_{\mathbf{b}}$ solves the equation on $\mathfrak{A}_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}}$ iff $k^{\mathbf{a}}{}_{\mathbf{c}}$ is an element of $GL(N, R)$. To establish the result, fix a point p and set $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}(x) = P\exp\int_p^x \mathfrak{A}_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}}\,dS^\alpha$, where P is the path-ordering symbol. $\Lambda^{\mathbf{a}}{}_{\mathbf{b}}(x)$, so defined, is path independent because $F_{\alpha\beta}{}^{\mathbf{a}}{}_{\mathbf{b}} = 0$. It satisfies $\mathfrak{A}_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}} = (\Lambda^{-1})^{\mathbf{a}}{}_{\mathbf{c}}\nabla_\alpha\Lambda^{\mathbf{c}}{}_{\mathbf{b}}$ by very construction. We thank V. P. Nair for suggesting this argument.

Remarks. (i) Theorem 3 shows that the introduction of a derivative operator with zero curvature greatly reduces the freedom in the construction of systems of generalized tensor fields; the remaining freedom is trivial since it involves only global gauge transformations. This fact underlies the particle physicists' formulation of gauge theories although the chosen curvature-free derivative operator is not always explicitly exhibited.

(ii) The derivative operators with zero curvature are often referred to as "classical vacua." Lemma 3.1 shows that each classical vacuum is left invariant by the global gauge group $GL(N, R)$ and that the quotient, $GL(N, R)_{\text{loc}}/GL(N, R)$, of the local gauge group by the global one acts simply and transitively on the collection of classical vacua. Classification of vacua by topological quantum numbers rests on this fact.

(iii) In Section 2, we saw that the introduction of preferred generalized tensor fields reduces the gauge group. For example, if we introduce a (positive definite) metric field g_{ab}, the local gauge group reduces from $GL(N, R)_{\text{loc}}$ to $O(N)_{\text{loc}}$: if, in the statement of Theorem 2, we consider only those systems of generalized tensor fields which are equipped with such a metric, then we must also replace $GL(N, R)_{\text{loc}}$ by $O(N)_{\text{loc}}$ in that statement. How does the introduction of g_{ab} affect calculus? It is now natural to restrict ourselves to derivative operators ∇ in C which satisfy $\nabla_\alpha g_{ab} = 0$. Given any two derivative operators, ∇ and ∇' satisfying this condition, we have

$$0 = (\nabla'_\alpha - \nabla_\alpha)\, g_{ab} = \mathfrak{A}_\alpha{}^c{}_a g_{cb} + \mathfrak{A}_\alpha{}^c{}_b g_{ac} = 2\mathfrak{A}_{\alpha(ab)}.$$

(Internal indices can now be raised and lowered by g_{ab}.) Hence, the components $\mathfrak{A}_{\alpha\mathbf{ab}}$ of $\mathfrak{A}_{\alpha ab}$ in any basis which is orthonormal with respect to g_{ab} satisfy $\mathfrak{A}_{\alpha(\mathbf{ab})} = 0$; $\mathfrak{A}_{\alpha\mathbf{ab}}$ is a 1-form which takes values in the Lie algebra of $O(N)_{\text{loc}}$. A corresponding result holds if one introduces, in place of g_{ab}, other generalized tensor fields with no space-time indices. More precisely, it is easy to establish the following result. Fix an N-dimensional, *vector space* V and denote by G the subgroup of $GL(N, R)$ which preserves a specified list

$$\{T^{a_1 \dots a_m}{}_{b_1 \dots b_n}, U^{a_1 \dots a_p}{}_{b_1 \dots b_q} \dots\}$$

of tensors on V. (The reason behind the use of latin indices here will be clear from what follows.) Then, if we introduce on S, *generalized tensor fields*

$$\{t^{a_1 \dots a_m}{}_{b_1 \dots b_n}, u^{a_1 \dots a_p}{}_{b_1 \dots b_q}, \dots\}$$

which have the same algebraic properties as

$$\{T^{a_1 \dots a_m}{}_{b_1 \dots b_n}, U^{a_1 \dots a_p}{}_{b_1 \dots b_q} \dots\},$$

respectively, then, the difference between any two derivative operators which annihilate these tensor fields can be represented by a 1-form $\mathfrak{A}_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}}$ which takes values in the Lie algebra of G_{loc}.

§(4): *Trivialization*

The space of derivative operators ∇ in C has a natural affine structure. In many circumstances–e.g., in the variational calculations, quantum field theory, etc.–it is convenient to reduce this affine space to a vector space by choosing an operator $\mathring{\nabla}$ with zero curvature as an origin and labeling any element ∇ of C by the generalized tensor field $A_\alpha{}^a{}_b$ defined by $(\nabla_\alpha - \mathring{\nabla}_\alpha)\, k^a = A_\alpha{}^a{}_b k^b$ for all k^a in S^a. (From now on, we shall use the symbol $A_\alpha{}^a{}_b$ in place of $\mathfrak{A}_\alpha{}^a{}_b$ when one of the connections, i.e., the "origin," has zero curvature.) However, by Lemma 3.1, the choice of such an origin is *not* a natural one: By selecting a particular $\mathring{\nabla}$, one introduces an auxiliary structure which is not available naturally. One must therefore investigate how the resulting framework changes in response to a change in the choice of origin. It is via these transformation properties that gauge makes its appearance in the "intrinsic," basic-independent framework.

Fix an operator $\mathring{\nabla}$ with zero curvature and coordinatize C by generalized tensor fields $A_\alpha{}^a{}_b$: to each point, there corresponds a unique $A_\alpha{}^a{}_b$, and, conversely, every generalized tensor field $A_\alpha{}^a{}_b$ labels a point of C. One can therefore express any property of elements ∇ of C in terms of $A_\alpha{}^a{}_b$. Associated with every ∇ in C, we have a curvature field $F_{\alpha\beta}{}^a{}_b$: $\nabla_{[\alpha}\nabla_{\beta]}\, k^a = F_{\alpha\beta}{}^a{}_b k^b$ for all k^a in S^a. We can express this curvature in terms of $A_\alpha{}^a{}_b$ as follows: Since

$$\begin{aligned}\tfrac{1}{2} F_{\alpha\beta}{}^a{}_b k^b &= \nabla_{[\alpha}\nabla_{\beta]}\, k^a = \mathring{\nabla}_{[\alpha}\nabla_{\beta]}\, k^a + A_{[\alpha}{}^a{}_{|b|}\nabla_{\beta]}\, k^b \\ &= \mathring{\nabla}_{[\alpha}\mathring{\nabla}_{\beta]}\, k^a + \mathring{\nabla}_{[\alpha}(A_{\beta]}{}^a{}_b k^b) + A_{[\alpha}{}^a{}_{|b|}\mathring{\nabla}_{\beta]}\, k^b + A_{[\alpha}{}^a{}_{|b|}A_{\beta]}{}^b{}_c k^c \\ &= (\mathring{\nabla}_{[\alpha}A_{\beta]}{}^a{}_c + A_{[\alpha}{}^a{}_{|b|}A_{\beta]}{}^b{}_c)\, k^c\end{aligned}$$

for all k^c, we have

$$\tfrac{1}{2} F_{\alpha\beta}{}^a{}_c = \mathring{\nabla}_{[\alpha}A_{\beta]}{}^a{}_c + A_{[\alpha}{}^a{}_{|b|}A_{\beta]}{}^b{}_c \tag{10}$$

an expression familiar from the Yang-Mills theory. One can now express Bianchi identities (and, Yang-Mills source-free equation, $\nabla^\alpha F_{\alpha\beta}{}^a{}_b = 0$) in terms of $\mathring{\nabla}$ and $A_\alpha{}^a{}_b$.

Let $\mathring{\nabla}'$ be another derivative operator with zero curvature. By Lemma 3.1, there exists a generalized tensor field $\Lambda^a{}_b$ such that

$$(\mathring{\nabla}'_\alpha - \mathring{\nabla}_\alpha)\, k^a = ((\Lambda^{-1})^a{}_b \nabla_\alpha \Lambda^b{}_c)\, k^c,$$

where $(\Lambda^{-1})^a{}_b$ and $\Lambda^a{}_b$ are related by $(\Lambda^{-1})^a{}_c\Lambda^c{}_b = \delta^a{}_b$. Let us choose $\mathring{\nabla}'$ as the origin in C and label elements ∇ of C by $A'_\alpha{}^a{}_b$, defined by

$$(\nabla_\alpha - \mathring{\nabla}'_\alpha)\, k^a = A'_\alpha{}^a{}_b k^b.$$

Then, we have

$$\begin{aligned}A'_\alpha{}^a{}_b k^b &= (\nabla_\alpha - \mathring{\nabla}'_\alpha)\, k^a = (\nabla_\alpha - \mathring{\nabla}_\alpha)\, k^a - (\mathring{\nabla}'_\alpha - \mathring{\nabla}_\alpha)\, k^a \\ &= A_\alpha{}^a{}_b k^b - ((\Lambda^{-1})^a{}_c \nabla_\alpha \Lambda^c{}_b)\, k^b\end{aligned}$$

whence

$$A'_\alpha{}^a{}_b = A_\alpha{}^a{}_b - (\Lambda^{-1})^a{}_c \nabla_\alpha \Lambda^c{}_b \tag{11}$$

is the transformation law. Since the curvature field $F_{\alpha\beta}{}^a{}_b$ of ∇ depends only on ∇ and not on the choice of origin, we have

$$F'_{\alpha\beta}{}^a{}_b = F_{\alpha\beta}{}^a{}_b \tag{12}$$

a result which may also be verified by a direct substitution in (10).

The transformation laws used in the particle physics literature differ from equations (11) and (12). This is because the framework there is not intrinsic but depends on the choice of basis. More precisely, one chooses, at the outset, a basis $\{e^a{}_{\mathbf{a}}\}$ in S^a and expresses generalized tensor fields in terms of their components; one trivializes generalized tensor fields $t^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{a_1\ldots a_p}{}_{b_1\ldots b_{q_0}}$ to ordinary tensor fields $t^{\alpha_1\ldots\alpha_m}{}_{\beta_1\ldots\beta_n}{}^{\mathbf{a}_1\ldots\mathbf{a}_p}{}_{\mathbf{b}_1\ldots\mathbf{b}_q}$. The derivative operator $\mathring{\nabla}$ which annihilates the choen basis $\{e^a{}_{\mathbf{a}}\}$ is then selected as the origin in C and derivative operators ∇ are labeled by the components $A_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}}$ of $A_\alpha{}^a{}_b$ in the basis $\{e^a{}_{\mathbf{a}}\}$: $A_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}} = A_\alpha{}^a{}_b e^{\mathbf{a}}{}_a e^b{}_{\mathbf{b}}$. Under the change of basis $\{e^a{}_{\mathbf{a}}\} \to \{e'^a{}_{\mathbf{a}}\}$, the label $A_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}}$ is transformed to $A'_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}}$ which, in view of equation (11), is given by

$$\begin{aligned}
A'_\alpha{}^{\mathbf{a}}{}_{\mathbf{b}} &\equiv A'_\alpha{}^a{}_b e'^b{}_{\mathbf{b}} e'^{\mathbf{a}}{}_a \\
&= A'_\alpha{}^a{}_b (\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}} e^b{}_{\mathbf{c}} \Lambda^{\mathbf{a}}{}_{\mathbf{d}} e^{\mathbf{d}}{}_a \\
&= (A_\alpha{}^a{}_b - (\Lambda^{-1})^a{}_c \mathring{\nabla}_\alpha \Lambda^c{}_b)(\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}} e^b{}_{\mathbf{c}} \Lambda^{\mathbf{a}}{}_{\mathbf{d}} e^{\mathbf{d}}{}_a \\
&= \Lambda^{\mathbf{a}}{}_{\mathbf{d}} A_\alpha{}^{\mathbf{d}}{}_{\mathbf{c}} (\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}} - (\Lambda^{-1})^{\mathbf{d}}{}_{\mathbf{m}} (\mathring{\nabla}_\alpha \Lambda^{\mathbf{m}}{}_{\mathbf{c}})(\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}} \Lambda^{\mathbf{a}}{}_{\mathbf{d}} \\
&= \Lambda^{\mathbf{a}}{}_{\mathbf{d}} A_\alpha{}^{\mathbf{d}}{}_{\mathbf{c}} (\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}} + \Lambda^{\mathbf{a}}{}_{\mathbf{c}} \mathring{\nabla}_\alpha (\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}}
\end{aligned} \tag{13}$$

Similarly, for the curvature field, we have

$$\begin{aligned}
F'_{\alpha\beta}{}^{\mathbf{a}}{}_{\mathbf{b}} &= F'_{\alpha\beta}{}^a{}_b e'^{\mathbf{a}}{}_a e'^b{}_{\mathbf{b}} \\
&= F_{\alpha\beta}{}^a{}_b (\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}} e'^b{}_{\mathbf{c}} (\Lambda)^{\mathbf{a}}{}_{\mathbf{d}} e^{\mathbf{d}}{}_a \\
&= \Lambda^{\mathbf{a}}{}_{\mathbf{d}} F_{\alpha\beta}{}^{\mathbf{d}}{}_{\mathbf{c}} (\Lambda^{-1})^{\mathbf{c}}{}_{\mathbf{b}}
\end{aligned} \tag{14}$$

Thus, the change in basis has a dual effect: components of generalized tensor fields transform according to the familiar law [equations (13) and (14)] *and* the origin in the space C of derivative operators is shifted.

To summarize, one can work at three levels. The first level is the most intrinsic one. One introduces generalized tensor fields abstractly (Section 2) without any reference to bases or any other auxiliary structure, defines derivative operators axiomatically (Section 3), and analyzes properties of curvature tensors. The fundamental variables are the derivative operators in C. Yang-Mills equations can be introduced directly on these variables and the subspace of C satisfying these equations can be given the structure of a symplectic manifold. At the second level, one introduces an origin $\overset{\circ}{\nabla}$ in C and labels elements ∇ of C by generalized tensor fields $A_\alpha{}^a{}_b$. Under a change of origin, these labels transform via equation (11). However, no basis is introduced and one deals, throughout, with generalized tensor fields, rather than with their components. It is because of this that the curvature $F_{\alpha\beta}{}^a{}_b$ remains invariant under the change of origin [equation (12)]. (Thus $F_{\alpha\beta}{}^a{}_b$ is thought of as an intrinsic object which is manifestly gauge invariant and not merely gauge covariant. That is, $F_{\alpha\beta}{}^a{}_b$ is treated in the same spirit as, say, the Riemann tensor $R_{\alpha\beta\gamma}{}^\delta$ is, in general relativity.) This level is the most convenient one in computations and in the problem of quantization. The last level is the least intrinsic one. Here, the entire framework is tied down to the choice of a basis. Thus the generalized tensor fields are trivialized to ordinary tensor fields and the emphasis is on transformation properties under the change of basis [equations (13) and (14)].

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Appendix: N = 1 Case

Generalized tensor fields with only one internal degree of freedom arise naturally in several different contexts in relativistic physics. Therefore, in this Appendix, we consider this case in some detail.

Since $GL(1, R)$ [and $GL(1, \mathbb{C})$] is Abelian, applications of this framework will yield only Abelian gauge groups. Moreover, since the module S^a is one-dimensional, an important simplification occurs: $S^{a_1 \dots a_m}{}_{b_1 \dots b_n}$ is also one-dimensional for all m and n. Hence a generalized tensor field $t^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}{}^{a_1 \dots a_m}{}_{b_1 \dots b_n}$ can now be denoted simply by $\underset{(m-n)}{t}{}^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}$, where the integer below the stem letter is the difference between the number of contravariant and covariant latin indices. [Let us, for simplicity, suppress the tensor indices. Then, in a basis e^a in S^a, $t^{a_1 \dots a_m}{}_{b_1 \dots b_n}$ reduces to a function $t^{\mathrm{a}_1 \dots \mathrm{a}_m}{}_{\mathrm{b}_1 \dots \mathrm{b}_n}$ on M and the change $e^a \to \Lambda e^a$ induces the transformation $t^{\mathrm{a}_1 \dots \mathrm{a}_m}{}_{\mathrm{b}_1 \dots \mathrm{b}_n} \to (\Lambda)^{n-m} t^{\mathrm{a}_1 \dots \mathrm{a}_m}{}_{\mathrm{b}_1 \dots \mathrm{b}_n}$; only the difference $(m - n)$ matters.] Note, in particular, that generalized tensor fields $\underset{0}{t}{}^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}$ are *naturally* isomorphic with ordinary tensor fields $t^{\alpha_1 \dots \alpha_p}{}_{\beta_1 \dots \beta_q}$.

Consider a derivative operator ∇ in C. We have, for any $t^a \equiv \underset{1}{t}$ in S^a, $\nabla_{[\alpha}\nabla_{\beta]}\underset{1}{t} = F_{\alpha\beta}\underset{1}{t}$; the curvature is represented by a 2-form $F_{\alpha\beta}$ on M. More generally, we have $\nabla_{[\alpha}\nabla_{\beta]}\underset{n}{t} = nF_{\alpha\beta}\underset{n}{t}$. Thus, n may be interpreted as the "charge" of the field $\underset{n}{t}$. Bianchi identity implies that $F_{\alpha\beta}$ is closed. Given a $\mathring{\nabla}$ with zero curvature, one can introduce a vector potential A_α: $(\nabla_\alpha - \mathring{\nabla}_\alpha)\underset{1}{t} = A_\alpha \underset{1}{t}$. Thus, A_α satisfies $\nabla_{[\alpha}A_{\beta]} = F_{\alpha\beta}$ on M. The change of origin $\mathring{\nabla} \to \mathring{\nabla}'$ in C causes the familiar gauge transformation: Since $\mathring{\nabla}$ and $\mathring{\nabla}'$ are related by

$$(\mathring{\nabla}'_\alpha - \mathring{\nabla}_\alpha)\underset{1}{t} = (\Lambda^{-1}\nabla_\alpha\Lambda)\underset{1}{t} \equiv (\nabla_\alpha f)\underset{1}{t},$$

say, A_α is transformed to $A'_\alpha = A_\alpha - \nabla_\alpha f$. Thus, the main simplification in the calculus of generalized tensor field is that the curvature $F_{\alpha\beta}{}^a{}_b$ and the potential $A_\alpha{}^a{}_b$ are now ordinary tensor fields on M since the difference between the number of their contravariant and covariant internal indices vanishes.

We now consider some examples.

(1) *Tensor densities.* Let M be n dimensional and orientable. Then, any two nowhere vanishing n-forms, $\epsilon_{\alpha_1\ldots\alpha_n}$ and $\bar{\epsilon}_{\alpha_1\ldots\alpha_n}$ are related by

$$\bar{\epsilon}_{\alpha_1\ldots\alpha_n} = \Lambda\epsilon_{\alpha_1\ldots\alpha_n}$$

for a nowhere vanishing function Λ. A *tensor density* $\underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p}$ of weight n is an assignment to each n-form $\epsilon_{\alpha_1\ldots\alpha_n}$ of a tensor field $\underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_p}(\epsilon)$ such that $\underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}(\Lambda\epsilon) = \Lambda^n \underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}(\epsilon)$ for all nowhere vanishing functions Λ. (By introducing charts on M, it is easy to verify that this definition is equivalent to the familiar one involving transformation properties and Jacobians.) A derivative operator ∇ defined on tensor fields can be naturally extended to one on densities: Using the fact that $\nabla_\alpha\epsilon_{\alpha_1\ldots\alpha_n} = \lambda_\alpha\epsilon_{\alpha_1\ldots\alpha_n}$ holds for some vector field λ_α on M, one sets

$$(\nabla_\alpha \underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q})(\epsilon) = \nabla_\alpha(\underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}(\epsilon)) - n\lambda_\alpha \underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q}(\epsilon)$$

It is easy to verify that $(\nabla_\alpha \underset{n}{t}{}^{\alpha_1\ldots\alpha_p}{}_{\beta_1\ldots\beta_q})$ is again a tensor density of weight n. To show that tensor densities yield a system of generalized tensor fields with one internal degree of freedom, one can first verify, step by step, that the algebraic axioms of Section 2 are satisfied. Next, a simple calculation yields

$$\nabla_{[\alpha}\nabla_{\beta]}\underset{1}{t} = -(\nabla_{[\alpha}\lambda_{\beta]})\underset{1}{t}.$$

Thus, the curvature $F_{\alpha\beta}$ is given by $F_{\alpha\beta} = -\nabla_{[\alpha}\underset{1}{t}_{\beta]}$. (Note that, although the definition of λ_α involves a choice of $\epsilon_{\alpha_1\ldots\alpha_n}$, $F_{\alpha\beta}$ is independent of this choice since under the transformation $\epsilon_{\alpha_1\ldots\alpha_n} \to \Lambda\epsilon_{\alpha_1\ldots\alpha_n}$ one has $\lambda_\alpha \to \lambda_\alpha + \Lambda^{-1}\nabla_\alpha\Lambda$.) However, since the action of ∇ on ordinary tensor fields extends uniquely to that on generalized tensor fields in this example, the "internal" curvature $F_{\alpha\beta}$ is intertwined with the "manifold" curvature $R_{\alpha\beta\gamma}{}^\delta$ to a certain extent. For example, if $R_{\alpha\beta\gamma}{}^{\delta} = 0$, there exists an n-form $\overset{\circ}{\epsilon}_{\alpha_1 \ldots \alpha_n}$ such that $\nabla_\alpha \overset{\circ}{\epsilon}_{\alpha_1 \ldots \alpha_n} = 0$, whence $\nabla_\alpha \epsilon_{\alpha_1 \ldots \alpha_n} = (\Lambda^{-1} \nabla_\alpha \Lambda)\, \epsilon_{\alpha_1 \ldots \alpha_n}$ for any n-form, for some Λ, so that $F_{\alpha\beta} = 0$. This is a special feature of this example; it is not universal even in the $N = 1$ case. Finally, note that the choice of a specific n-form, $\bar{\epsilon}_{\alpha_1 \ldots \alpha_n}$, say, corresponds to the introduction of a basis $\underset{1}{e}$ in $\underset{1}{S} \equiv \underset{1}{S}{}^a$: e is the density of weight 1 which assigns to any n-form $\epsilon_{\alpha_1 \ldots \alpha_n}$ the function Λ defined by $\epsilon_{\alpha_1 \ldots \alpha_n} = \Lambda \bar{\epsilon}_{\alpha_1 \ldots \alpha_n}$.

(2) *Charged tensor fields.* Fix an electromagnetic field $F_{\alpha\beta}$ on the space-time manifold M and consider a matter field with a given spin and mass interacting with this fixed, background $F_{\alpha\beta}$. Such a matter field with charge n will be represented by a "*charged tensor field*" $\underset{n}{t}{}^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q}$ which assigns to each choice A_α of vector potential of $F_{\alpha\beta}$ a tensor field $t^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q}(A)$ such that $\underset{n}{t}{}^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q}(A + df) = (\exp inf)\, \underset{n}{t}{}^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q}(A)$. A derivative operator ∇ on ordinary tensor fields can now be extended to charged tensor fields as follows:

$$(\nabla_\alpha \underset{n}{t}{}^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q})(A) = \nabla_\alpha (\underset{n}{t}{}^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q}(A)) - in A_\alpha \underset{n}{t}{}^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q}(A)$$

It is easy to check that $(\nabla_\alpha \underset{n}{t}{}^{\alpha_1 \cdots \alpha_p}{}_{\beta_1 \ldots \beta_q})$ is again a charged tensor field with charge n. The structure of charged tensor fields is similar to that of densities except that the "internal" curvature $F_{\alpha\beta}$ is now detached from the space-time curvature $R_{\alpha\beta\gamma}{}^{\delta}$. It is easy to verify that the charged tensors provide an example of the system of complex-valued[8] generalized tensor fields with one internal degree of freedom. In addition, the system is now equipped with a Hermitean metric field $g_{a\bar{b}}$ which reduces the gauge group from $GL(1, \mathbb{C})$ to $U(1)$. Other examples include conformally weighted tensor fields, spin and boost weighted fields on null infinity, $\mathscr{I}$, and normal modes of perturbations on a nonstatic, stationary background.

Finally, we wish to point out a curious property of the $N = 1$ case which, to our knowledge, has not found an application in physics. In the framework above, n denoted the difference between contravariant and covariant latin indices and could therefore assume only integral values. However, having obtained the framework, we may let n, the "charge," assume any real value. (Indeed, tensor densities do assume nonintegral weights in physical applications.) one can generalize further and consider objects whose "charge" takes values in any Abelian group. This can be achieved by letting the curvature depend on the charge g in such a way that $F_{\alpha\beta}(g_1 + g_2) = F_{\alpha\beta}(g_1) + F_{\alpha\beta}(g_2)$. The charge group may either be chosen arbitrarily or intertwined with the gauge group by requiring, e.g., that it be dual to the gauge group in a suitable sense. The resulting framework, however, is tied down to the $N = 1$ contexts and Abelian charge groups.

[8]See Remark (3) at the end of Section 2.

References

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2. Penrose, R., and Rindler, W. *Spinors, Twistors and Space-time Structure* (to appear).