*General Relativity and Gravitation, Vol. 22, No. 2, 1990*

# Some Exact Solutions in String Cosmology

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*Received March 17, 1989*

Some Bianchi-type string-cosmological models are presented here. The physical implications of the models are briefly discussed

## 1. INTRODUCTION

Gauge theories with spontaneous symmetry-breaking in elementary-particle physics have given rise to an intensive study of cosmic strings. It appears that after the big bang the universe may have experienced a number of phase transitions [1]. These phase transitions can produce vacuum domain structures such as domain walls, strings and monopoles [2]. Cosmic strings have excited considerable interest, as they may act as gravitational lenses [3] and may give rise to density perturbations leading to the formation of galaxies [4].

Letelier [5] has initiated the study of a new model of a cloud formed by massive strings in the context of general relativity. This model has been used as a source for Bianchi type I and Kantowski-Sachs cosmologies. The possibility that during the evolution of the universe the strings disappear leaving only particles has been examined.

In this paper we study the Letelier model in the context of Bianchi types II, $VI_0$, VIII and IX. We reproduce Einstein equations coupled to a cloud of strings in Section 2, derive the solutions in Section 3 and conclude with a brief discussion of the solutions in Section 4.

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0001-7701/90/0200-0123$06.00/0 

## 2. EINSTEIN EQUATIONS COUPLED TO A CLOUD OF STRINGS

The Einstein equations for a cloud of strings [5] are

$$R_{\mu\nu} - (1/2)g_{\mu\nu}R = -(\rho u_\mu u_\nu - \lambda x_\mu x_\nu) \tag{1}$$

Here $\rho$ is the rest energy density of the cloud of strings with particles attached to them. $\rho \equiv \rho_p + \lambda$, $\rho_p$ being the rest energy density of particles and $\lambda$ the tension density of the cloud of strings. As pointed out by Letelier, $\lambda$ may be positive or negative. The vector $u^\mu$ describes the cloud four-velocity and $x^\mu$ represents a direction of anisotropy, i.e. the direction of strings. We have

$$u^\mu u_\mu = -x^\mu x_\mu = 1 \quad \text{and} \quad u^\mu x_\mu = 0 \tag{2}$$

The contracted Bianchi identity for (1) is equivalent to

$$\nabla_\mu(\rho u^\mu) - \lambda x'^\nu u_\nu = 0 \tag{3a}$$

$$\nabla_\mu(\lambda x^\mu) - \rho \dot{u}^\nu x_\nu = 0 \tag{3b}$$

$$H^\mu_\nu(\rho \dot{u}^\nu - \lambda x'^\nu) = 0 \tag{3c}$$

where

$$x'^\nu = x^\mu \nabla_\mu(x^\nu) \quad \text{and} \quad \dot{u}^\nu = u^\mu \nabla_\mu(u^\nu) \tag{4}$$

Eqs. (3) are the evolutions for the cloud of strings. These equations are also the integrability conditions for (1).

## 3. FIELD EQUATIONS AND THEIR EXACT SOLUTIONS

(i) The locally rotationally symmetric (LRS) metric for the spatially homogeneous Bianchi-type II cosmological model is

$$ds^2 = dt^2 - (Sdx + Szdy)^2 - (Rdy)^2 - (Rdz)^2 \tag{5}$$

where $R$ and $S$ are functions of $t$ only.

From eqs. (5), (1) and (2) we write

$$u^\mu = u_\mu = (1, 0, 0, 0) \tag{6}$$

and $x^\mu$ must be taken along any of the three directions $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z$. Without loss of generality let us choose $x^\mu$ parallel to $\partial/\partial x$ so that

$$x^\mu = (0, S^{-1}, 0, 0) \tag{7}$$

With the help of eqs. (5)–(7) the Bianchi identities (3) can be reduced to a single equation

$$\dot{\rho} + (\rho - \lambda)\frac{\dot{S}}{S} + 2\rho\frac{\dot{R}}{R} = 0 \tag{8}$$

where a dot denotes derivative with respect to $t$.

As a consequence of Einstein's equations $\rho$ and $\lambda$ are also functions of $t$ only.

The Einstein equations (1) are equivalent to

$$2\frac{\dot{R}}{R}\frac{\dot{S}}{S} + \frac{\dot{R}^2}{R^2} - \frac{1}{4}\frac{S^2}{R^4} = \rho \tag{9}$$

$$2\frac{\ddot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{3}{4}\frac{S^2}{R^4} = \lambda \tag{10}$$

$$\frac{\ddot{S}}{S} + \frac{\ddot{R}}{R} + \frac{\dot{S}}{S}\frac{\dot{R}}{R} + \frac{1}{4}\frac{S^2}{R^4} = 0 \tag{11}$$

Thus we have three equations for four unknowns $R$, $S$, $\lambda$ and $\rho$. A solution for (9)–(11) is given by

$$R = t^{9/14} \tag{12}$$

$$S = t^{2/7} \tag{13}$$

$$\rho = (26/49)t^{-2} \tag{14}$$

$$\lambda = -(39/49)t^{-2} \tag{15}$$

and

$$\rho_p = (65/49)t^{-2} \tag{16}$$

The solutions (12)–(15) identically satisfy (8).

(ii) The LRS metric for Bianchi type $\mathrm{VI}_0$ cosmological model is

$$ds^2 = dt^2 - Adx^2 - Be^{-2mx}dy^2 - Ce^{2mx}dz^2 \tag{17}$$

where $A$, $B$, $C$ are functions of $t$ only and $m$ is a nonzero constant.

As in case (i) we take

$$u^\mu = u_\mu = (1, 0, 0, 0) \tag{18}$$

and

$$x^\mu = (0, A^{-1/2}, 0, 0) \tag{19}$$

The Einstein equations (1) and Bianchi identity (3) are equivalent to

$$\frac{1}{4}(\frac{\dot{A}}{A}\frac{\dot{B}}{B} + \frac{\dot{B}}{B}\frac{\dot{C}}{C} + \frac{\dot{C}}{C}\frac{\dot{A}}{A}) - \frac{m^2}{A} = \rho \tag{20}$$

$$\frac{1}{2}(\frac{\ddot{B}}{B}\frac{\ddot{C}}{C}) - \frac{1}{4}(\frac{\dot{B}^2}{B^2} + \frac{\dot{C}^2}{C^2} - \frac{\dot{B}}{B}\frac{\dot{C}}{C}) + \frac{m^2}{A} = \lambda \tag{21}$$

$$\frac{1}{2}(\frac{\ddot{A}}{A} + \frac{\ddot{C}}{C}) - \frac{1}{4}(\frac{\dot{A}^2}{A^2} + \frac{\dot{C}^2}{C^2} - \frac{\dot{C}}{C}\frac{\dot{A}}{A}) - \frac{m^2}{A} = 0 \tag{22}$$

$$\frac{1}{2}(\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B}) - \frac{1}{4}(\frac{\dot{A}^2}{A^2} + \frac{\dot{B}^2}{B^2} - \frac{\dot{A}}{A}\frac{\dot{B}}{B}) - \frac{m^2}{A} = 0 \tag{23}$$

$$\frac{\dot{B}}{B} - \frac{\dot{C}}{C} = 0 \tag{24}$$

and

$$\dot{\rho} + \frac{1}{2}(\rho - \lambda)\frac{\dot{A}}{A} + \frac{1}{2}\rho(\frac{\dot{B}}{B} + \frac{\dot{C}}{C}) = 0 \tag{25}$$

Eq. (24) gives

$$C = KB \tag{26}$$

where $K$ is a constant of integration. Eq. (26) shows that eqs. (22) and (23) are not independent.

Thus we have four equations for five unknowns, $A$, $B$, $C$,$\rho$ and $\lambda$ and hence we can choose one of the unknowns to be arbitrary.

Let us choose

$$A = t^2 \tag{27}$$

Then the field equations (20)–(24) give

$$B = K_1 t^{2m}(c_1 + c_2 t^{-2m})^2 \tag{28}$$

$$C = K_2 t^{2m}(c_1 + c_2 t^{-2m})^2 \tag{29}$$

$$\rho = \frac{2m}{t^2} - \frac{c_1 t^{2m} - c_2}{c_1 t^{2m} + c_2} + \frac{m^2}{t^2}(\frac{c_1 t^{2m} - c_2}{c_1 t^{2m} + c_2})^2 - \frac{m^2}{t^2} \tag{30}$$

and

$$\lambda = \frac{2m[(2m-1){c_1}^2 t^{2m-2} + (2m+1){c_2}^2 t^{-(2m+2)}]}{t^{-2m}(c_1 t^{2m} + c_2)^2} - \frac{m^2}{t^2}(\frac{c_1 t^{2m} - c_2}{c_1 t^{2m} + c_2})^2 + \frac{m^2}{t^2} \tag{31}$$

$$\rho_p = \frac{4m}{t^2}\left[\frac{c_1 t^{2m} - c_2}{c_1 t^{2m} + c_2} - m\right] \tag{32}$$

The solutions (27)–(31) satisfy the Bianchi identity (25) identically.
(iii) The LRS metric for Bianchi type VIII ($\delta = -1$) and Bianchi type IX ($\delta = +1$) is

$$ds^2 = dt^2 - (Sdx - Shdz)^2 - (Rdy)^2 - (Rfdz)^2 \tag{33}$$

where $S$ and $R$ are functions of $t$ only and

$$f(y) = \begin{bmatrix} \sin y \\ \sinh y \end{bmatrix}, \quad h(y) = \begin{bmatrix} \cos y \\ -\cosh y \end{bmatrix} \quad \text{for } \delta = \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

Taking $u^\mu$ and $x^\mu$ as in (6) and (7) respectively, the field equations and Bianchi identity are equivalent to

$$2\frac{\dot{S}}{S}\frac{\dot{R}}{R} + \frac{1}{R^2}(\dot{R}^2 + \delta) - \frac{1}{4}\frac{S^2}{R^4} = \rho \tag{34}$$

$$2\frac{\ddot{R}}{R} + \frac{1}{R^2}(\dot{R}^2 + \delta) - \frac{3}{4}\frac{S^2}{R^4} = \lambda \tag{35}$$

$$\frac{\ddot{S}}{S} + \frac{\ddot{R}}{R} + \frac{\dot{S}}{S}\frac{\dot{R}}{R} + \frac{1}{4}\frac{S^2}{R^4} = 0 \tag{36}$$

and

$$\dot{\rho} + (\rho - \lambda)\frac{\dot{S}}{S} + 2\rho\frac{\dot{R}}{R} = 0 \tag{37}$$

A solution for eqs. (34)–(36) is given by

$$R = t^{9/14} \tag{38}$$

$$S = t^{2/7} \tag{39}$$

$$\rho = (26/49)t^{-2} + \delta t^{-(9/7)} \tag{40}$$

$$\lambda = -(39/49)t^{-2} + \delta t^{-(9/7)} \tag{41}$$

$$\rho_p = (65/49)t^{-2} \tag{42}$$

These solutions (38)–(41) satisfy the Bianchi identity (37) identically.

## 4. DISCUSSION

For the Bianchi type II solution, we have from (15) and (16)

$$\frac{\rho_p}{|\lambda|} = \frac{5}{3} \tag{43}$$

Thus, throughout the whole process of evolution, the universe is dominated by massive strings. For the Bianchi type $VI_0$ solution we have from (31) and (32) as shown in the following table (Table 1).

**Table 1**

| At | | $m < (1/2)$ | $(1/2) < m < 1$ | $m > 1$ |
|---|---|---|---|---|
| $t \to 0$ | $\lambda \sim$ | $\frac{2m(1+2m)}{t^2}$ | $\frac{2m(2m+1)}{t^2}$ | $\frac{2m(2m+1)}{t^2}$ |
| | $\rho_p \sim$ | $-\frac{4m(1+m)}{t^2}$ | $-\frac{4m(1+m)}{t^2}$ | $-\frac{4m(1+m)}{t^2}$ |
| $t \to \infty$ | $\lambda \sim$ | $-\frac{2m(1-2m)}{t^2}$ | $\frac{2m(2m-1)}{t^2}$ | $\frac{2m(2m-1)}{t^2}$ |
| | $\rho_p \sim$ | $\frac{4m(1-m)}{t^2}$ | $\frac{4m(1-m)}{t^2}$ | $-\frac{4m(m-1)}{t^2}$ |

Strings exist throughout the evolution of the universe for values of $m$ as shown in Table 1. But for all these values of $m$, particles have negative mass and hence are unobservable in the early era of the universe. In the later phase of the universe, however, particles appear with positive mass only for $m < 1$.

Finally, for the Bianchi types VIII ($\delta = -1$) and IX ($\delta = +1$) from (41) and (42)

$$\text{at } t \to 0 \frac{\rho_p}{|\lambda|} \to \frac{5}{3} \tag{44}$$

$$\text{and at } t \to \infty \frac{\rho_p}{|\lambda|} \to \frac{65}{49} t^{-(5/7)} \tag{45}$$

(44) shows that in the early era the universe is dominated by massive strings, but, according to (45), in the later phase the strings dominate over the particles.

It would be in order to point out a peculiar feature of all four solutions presented in this paper. Strings exist throughout the process of evolution of the universe. On the other hand, matter (with positive mass) either exists all through the process of evolution or appears only in the later phase of evolution. It is, however, to be remembered that all these are classical solutions. Their character is likely to be different in the context of quantum gravity, particularly in the early era.

We conclude with a comment on the case $\rho = \lambda$ (geometric strings). The Bianchi type $VI_0$ metric (Section 3.ii) admits of a physical solution

$$\begin{aligned} A &= t^2 \\ B &= kt^2 \\ C &= t^2 \\ \lambda &= \rho = (3 - m^2)/t^2 \end{aligned} \tag{46}$$

where $k$ is a constant and $m = \pm 1$.

For Bianchi types VIII ($\delta = -1$) and IX ($\delta = +1$) (Section 3.iii) in the case $\rho = \lambda$ we have from (34) and (35)

$$\dot{R}\frac{\dot{S}}{S} - \ddot{R} + \frac{1}{4}\frac{S^2}{R^3} = 0 \tag{47}$$

Again from (36) and (47)

$$\frac{\ddot{S}}{S} + 2\frac{\ddot{R}}{R} = 0 \tag{48}$$

$R$ and $S$ may be determined from (47) and (48). Unfortunately we could not solve these equations. We may, however, make one or two general remarks assuming that a physical solution exists. For $\lambda = \rho$ we find from (37) that

$$\lambda = \rho = \text{constant}/R^2 \tag{49}$$

Eqs. (47)–(49) also follow from Bianchi type II ($\delta = 0$) (Section 3.i). Thus $R$ and $S$ will be of the same form for all the three Bianchi types II, VIII and IX. (48) shows that either $\ddot{S}$ or $\ddot{R}$ must be negative. (47), however, points to the fact that $\ddot{R}$ must be positive assuming that both $R$ and $S$ are increasing functions of $t$ (expanding universe). Since $\lambda(= \rho)$ depends on $\delta$ (which may be $-1$, 0 or $+1$), the constant occurring in (49) must be different for different values of $\delta$.

## ACKNOWLEDGEMENTS

The authors express their profound gratitude to the Government of Assam, Dispur, for all facilities provided at Cotton College, Gauhati, to carry out the investigations reported in this paper.

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