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# Poincaré-Invariant Gravitational Field and Equations of Motion of Two Pointlike Objects: The Postlinear Approximation of General Relativity

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## *Abstract*

Using a fast-motion approximation method we obtain the second-order gravitational field and equations of motion for two pointlike objects in algebraically closed form. A regularization procedure is used which is shown to guarantee the consistency of the approximation scheme. The equations of motion are then transformed within the framework of relativistic predictive mechanics into a system of ordinary differential equations.

## §(1): *Introduction*

We consider in this paper the problem of obtaining the gravitational field and the equations of motion for two gravitationally interacting bodies in the post-linear approximation of general relativity. The results of this paper are the first

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step of a program aiming at studying the dynamics of gravitationally bound systems up to a precision sufficient to include secular effects due to gravitational radiation.

This work is motivated by the fact that the Einstein "quadrupole formula" (which links the energy loss at infinity due to gravitational radiation to the quadrupole moment of the system) has been proved convincingly only for systems interacting via nongravitational forces. Moreover, the quadrupole formula does not tell us anything about the effect of this energy loss on the motion of the system. These problems have been extensively considered in the literature, but have not yet been solved satisfactorily. Their resolution has, however, become an urgent necessity in order to be able to compare theory and observation in systems like the binary pulsar PSR 1913 + 16 where damping is probably due to gravitational radiation [1].

Since an exact solution of the two-body problem in general relativity is at present beyond reach, past analyses have used approximation techniques which may be classified into two main categories. In the "slow-motion" approach,[2] the velocities are assumed from the outset to be small compared to the velocity of light (this presupposes the use of some coordinate conditions corresponding to a frame of reference where the system is nearly at rest); although it has been possible to push this approximation scheme quite far [to order $(v/c)^5$] [3], it suffers from a serious drawback: since no convincing matching of the near-zone and far-zone fields has been yet performed, the simultaneous determination of the equations of motion and of the gravitational field far from the sources is not possible. In the "fast-motion" approach [4-6] one uses an expansion in powers of the strength of the gravitational field without making any assumptions about the magnitude of the velocities in the system. This approach also has the technically interesting feature of being Poincaré invariant because a formally Poincaré-invariant coordinate condition is used, and is, in principle, better suited to determining the gravitational field throughout space-time. In this paper we use a "fast-motion" approximation.

The interacting objects must be described by an appropriately chosen stress-energy tensor. However, since most often nothing is known observationally about the internal structure of the objects and since one is usually not interested in a detailed knowledge of that structure, the objects are frequently assumed to be pointlike, that is, characterized by their masses alone. Two techniques can be employed to deal with this assumption. The first considers extended bodies and formally lets their dimensions tend to zero at the end of the calculations; in such a technique the choice of the stress-energy tensor is arbitrary and is often artificially restricted for the sake of technical simplicity [7]. The second technique introduces the pointlike character of the sources from the very beginning by

[2] For a review of this approach and references to the pioneering works of Einstein, Fock, and Papapetrou, see Reference 2.

making an appropriate choice of a stress-energy tensor involving "delta functions"; although the introduction of delta distributions in a nonlinear theory certainly leads to difficulties, we chose this technique because it seems more straightforward and susceptible to yield quickly the significant physical results. It must be emphasized that in both techniques the assumption of pointlike sources leads to divergences.

In the line of approach we have chosen (fast-motion approximation with a stress-energy tensor involving delta functions), the first attempt to go beyond the linear approximation was that of Bertotti and Plebanski [5]; however these authors limited themselves to a discussion of the general features of the algorithm and ignored the problem of the divergencies. Later Havas and Goldberg [6] calculated the "Abraham-like" self-action terms, which form a small part of the full postlinear approximation. (For a critical review of the literature up to 1976, see [8]). Recently Rosenblum [9] claimed to have handled the full postlinear approximation and published a result which contradicts Einstein's quadrupole formula. Unfortunately his paper does not contain complete information on how the problem of divergencies has been addressed nor does it contain explicit results concerning the determination of the gravitational field and the full equations of motion. Westpfahl and Goller [10] give second-order equations of motion in an explicit form which agree with our results but no regularization technique is presented.

Our approach is the following. Using harmonic coordinates and a flat retarded propagator, the first-order metric is obtained straightforwardly by an unambiguous integration.[3] The first-order harmonicity condition (the zeroth-order equations of motion) requires the accelerations of the particles to be at least order one in $G$, the gravitational constant.

The postlinear stress-energy tensor constructed with the first-order metric is an undefined expression because it involves terms which are the product of a delta function $\delta$ and a function which is infinite on the support of $\delta$. This problem is handled by prescribing a definite regularization procedure which is based on a mean-value technique, and is chosen in order to ensure the consistency of the approximation scheme at order 2. The first-order equations of motion are then unambiguously obtained from the equation of conservation of the regularized second-order stress-energy tensor. They are shown to be equivalent to the regularized first-order geodesic equations.

The differential equations satisfied by the second-order metric are constructed with the first-order metric and its derivatives, taking account of the fact that the

[3] Using a flat retarded propagator at all steps (here at orders 1 and 2) introduces unknown errors (for an evaluation of these errors for a distribution of continuous matter see [11]). However this has the non-negligible advantage of leading to handleable calculations. The attitude adopted here is to use the flat retarded propagator, keeping in mind that it should be checked whether the obtained solution actually approximates the exact solution and satisfies the no-incoming-radiation condition.

accelerations of the particles are first-order in $G$. The second-order solution obtained using the flat retarded propagator diverges. A well-defined second-order metric is obtained, in algebraically closed form, by working out integration procedures which guarantee firstly that the second-order Schwarzschild metric is recovered in the one-body problem and secondly that the metric is a solution of the full Einstein equations (in other words the second-order harmonicity condition yields the first-order equations of motion as previously obtained). These integration procedures are shown to be consistent with the chosen regularization prescription.

The second-order equations of motion are obtained in algebraically closed form from the equation of conservation of the regularized third-order stress-energy tensor (constructed with the regularized second-order metric). They are shown to be equivalent to the regularized second-order geodesic equations.

The equations of motion we obtain are of hereditary character; they depend on the past history of the two particles. This complicated form leads both to technical problems—how can they be integrated?—and to problems of principle—how can important concepts like the total energy momentum of the system be defined? These problems are handled here without abandoning the manifest Poincaré invariance of the formalism by using the framework of relativistic predictive mechanics [12]; we derive explicitly the predictive Poincaré-invariant system associated with the second-order equations of motion.

## §(2): *Theoretical Framework*

*2.1. A Guideline to the Problem.* In this section the metric and its derivatives are assumed to be well behaved everywhere.

Einstein's equations for the metric $g_{\alpha\beta}(x^\rho)$ are written in harmonic coordinates:

$$2|g|\,S^{\alpha\beta} = 16\pi\mathfrak{T}^{\alpha\beta} \tag{1}$$

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = 0 \tag{2}$$

where

$$\mathfrak{g}^{\alpha\beta} = (-g)^{1/2} g^{\alpha\beta} \tag{3}$$

[$c = 1$; $\alpha, \beta = 0, 1, 2, 3$; $i, k = 1, 2, 3$; signature $(-+++)$; $g$ is the determinant of the matrix $g_{\alpha\beta}$]. $S^{\alpha\beta}$ is the Einstein tensor reduced by (2):

$$2|g|S^{\alpha\beta} = \mathfrak{g}^{\mu\nu}\partial^2_{\mu\nu}\mathfrak{g}^{\alpha\beta} + Q^{\alpha\beta} \tag{4}$$

where $Q^{\alpha\beta}$ is quadratic in the derivatives of $g^{\mu\nu}$ (cf. Appendix A for the explicit expression of $Q^{\alpha\beta}$).

$$\mathfrak{T}^{\alpha\beta} = G|g|T^{\alpha\beta} \tag{5}$$

where $T^{\alpha\beta}$ is the stress-energy tensor.

If the metric were well behaved everywhere, $T^{\alpha\beta}$ would be chosen as

$$T^{\alpha\beta}(x^\rho) = \sum m \int_{-\infty}^{+\infty} ds\, \delta_4 [x - z(s)]\, u^\alpha u^\beta (g g_{\mu\nu} u^\mu u^\nu)^{-1/2} \tag{6}$$

The equation of the world line $L$ is parametrized by $s$: $x^\alpha = z^\alpha(s)$; $u^\alpha = dz^\alpha/ds$ is the tangent to $L$ at point $z$; the sum $\Sigma$ is taken over the two particles of mass $m$ (world line $L$) and $m'$ (world line $L'$); $\delta_4(x)$ is the four-dimensional Dirac distribution, normalized by

$$\int \delta_4(x)\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = 1 \tag{7}$$

The choice (6) for the stress-energy tensor is deduced from the following considerations [6]:

(i) It depends only on the metric and the world lines of the particles.

(ii) It is symmetric.

(iii) It is independent of the parameter $s$ chosen to parametrize the world lines.

(iv) It is conservative if and only if the world lines are geodesics of the metric.

Choosing a Minkowskian parametrization of, e.g., the world line $L$,

$$ds^2 = -\eta_{\alpha\beta} dz^\alpha\, dz^\beta \tag{8}$$

$$(u \cdot u) \equiv \eta_{\alpha\beta} u^\alpha u^\beta = -1 \tag{9}$$

$$(u \cdot \dot{u}) \equiv \eta_{\alpha\beta} u^\alpha \dot{u}^\beta = 0 \tag{10}$$

where $\dot{u}^\alpha = du^\alpha/ds$, the geodesic equation reads for $L$:

$$\dot{u}^\alpha = -u^\mu u^\nu [\Gamma^\alpha_{\mu\nu}(z) + u^\alpha u_\rho \Gamma^\rho_{\mu\nu}(z)] \tag{11}$$

where

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\rho} (\partial_\beta g_{\rho\gamma} + \partial_\gamma g_{\rho\beta} - \partial_\rho g_{\beta\gamma}) \tag{12}$$

Equation (11) together with the similar equation for $L'$ is equivalent to the equation of conservation of $T^{\alpha\beta}$:

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{13}$$

or equivalently,

$$\nabla_\beta \mathfrak{T}^{\alpha\beta} = \partial_\beta \mathfrak{T}^{\alpha\beta} + \Gamma^\alpha_{\mu\beta} \mathfrak{T}^{\beta\mu} - \Gamma^\beta_{\mu\beta} \mathfrak{T}^{\mu\alpha} = 0 \tag{14}$$

$\nabla_\alpha$ denoting the covariant derivative, $\Gamma^\alpha_{\beta\gamma}$ being the Christoffel symbols (12). Were the metric well behaved the problem would be the following: Obtain $\mathfrak{g}^{\alpha\beta}$

as a functional of the unconstrained world lines $L, L'$ by solving equation (1) with $\mathfrak{T}^{\alpha\beta}$ given by (5) and (6). Then, find the equations of motion restricting the world lines by requiring either (i) that the solution of (1) is a solution of the full Einstein equations, that is, the harmonicity condition (2) is satisfied, (ii) that the equation of conservation for $T^{\alpha\beta}$ [equation (13) or (14)] is satisfied, or (iii) that the geodesic equation (11) and the similar equation for $L'$ are satisfied. The equivalence of (i)-(iii) is a check of the assumption (6).

Now, since we are dealing with pointlike particles and, at the same time, we are going to perform formal expansions in powers of $G$, the metric actually diverges on the world lines; the expression (6) for $T^{\alpha\beta}$ is thus meaningless and the equivalence between the above three ways of obtaining the equations of motion (when given a sense by means of a regularization procedure) is not guaranteed. Therefore this formal framework cannot be considered as anything more than a guideline to be followed as closely as possible.

*2.2. The Approximation Scheme.* Let $h^{\alpha\beta}$ denote the deviation from $\eta^{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$:

$$h^{\alpha\beta} \equiv g^{\alpha\beta} - \eta^{\alpha\beta} \tag{15}$$

and expand (4) in powers of $h^{\mu\nu}$ (see [5]):

$$2|g|S^{\alpha\beta} = \Box h^{\alpha\beta} - N^{\alpha\beta} + \mathcal{O}(h^3) \tag{16}$$

$$\begin{aligned} N^{\alpha\beta} = {} & -h^{\mu\nu}\partial^2_{\mu\nu}h^{\alpha\beta} + \tfrac{1}{8}\eta^{\alpha\beta}\partial_\mu h \partial^\mu h - \tfrac{1}{4}\partial^\alpha h \partial^\beta h - \tfrac{1}{4}\eta^{\alpha\beta}\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} \\ & + \tfrac{1}{2}\eta^{\alpha\beta}\partial_\rho h_{\mu\nu}\partial^\mu h^{\rho\nu} + \tfrac{1}{2}\partial^\alpha h^{\mu\nu}\partial^\beta h_{\mu\nu} - \partial^\alpha h_{\mu\nu}\partial^\mu h^{\beta\nu} \\ & - \partial^\beta h_{\mu\nu}\partial^\mu h^{\alpha\nu} + \partial_\mu h^{\alpha\nu}\partial^\mu h^\beta_\nu + \partial_\mu h^{\alpha\nu}\partial_\nu h^{\beta\mu} \end{aligned} \tag{17}$$

where $\Box = \eta^{\alpha\beta}\partial^2_{\alpha\beta}$, $h = \eta_{\alpha\beta}h^{\alpha\beta}$, and all indices are henceforth moved with the Minkowski metric.

Then replace (5) and (6) by a regularized expansion in powers of $h^{\mu\nu}$:

$$\mathfrak{T}^{\alpha\beta} = G|g|T^{\alpha\beta} = \sum Gm \int ds\,\delta_4(x - z)u^\alpha u^\beta[1 + \underset{(1)}{\bar{t}} + \underset{(2)}{\bar{t}} + \mathcal{O}(h^3)] \tag{18}$$

$$\overline{\underset{(1)}{t}(z)} = \tfrac{1}{4}\overline{h(z)} - \tfrac{1}{2}u^\mu u^\nu \overline{h_{\mu\nu}(z)} \tag{19}$$

$$\underset{(2)}{\bar{t}} = \tfrac{1}{32}\overline{h^2} - \tfrac{1}{8}\overline{h_{\mu\nu}h^{\mu\nu}} - \tfrac{1}{8}u^\mu u^\nu \overline{hh_{\mu\nu}} + \tfrac{1}{2}u^\mu u^\rho \overline{h_{\mu\nu}h^\nu_\rho} + \tfrac{3}{8}u^\mu u^\nu u^\rho u^\sigma \overline{h_{\mu\nu}h_{\rho\sigma}} \tag{20}$$

where the bar denotes a regularization procedure which eliminates the divergences in $\underset{(1)}{t}(z)$, $\underset{(2)}{t}(z)$ and which is defined in Section 2.3.

Finally we rewrite the equations of motion (2), (14), and (11) as[4]

$$\partial_\alpha h^{\alpha\beta} = 0 \tag{21}$$

$$\sum Gm \int ds\delta_4(x - z) \left[\frac{d}{ds}\,[u^\beta(1 + \underset{(1)}{\overline{t}} + \underset{(2)}{\overline{t}}\,)] + \underset{(2)}{\overline{\Gamma^\beta_{\mu\alpha}}}u^\alpha u^\mu - \underset{(2)}{\overline{\Gamma^\alpha_{\mu\alpha}}}u^\mu u^\beta \right.$$

$$\left. + (\underset{(1)}{\overline{\Gamma^\beta_{\mu\alpha}}}u^\alpha u^\mu - \underset{(1)}{\overline{\Gamma^\alpha_{\mu\alpha}}}u^\mu u^\beta)(1 + \underset{(1)}{\overline{t}}) + \mathcal{O}(h^3)\right] = 0 \tag{22}$$

$$\dot{u}^\alpha_a = -u^\mu_a u^\nu_a [\underset{(1)}{\overline{\Gamma^\alpha_{\mu\nu}(z_a)}} + \underset{(2)}{\overline{\Gamma^\alpha_{\mu\nu}(z_a)}} + u^\alpha_a u^a_\rho(\underset{(1)}{\overline{\Gamma^\rho_{\mu\nu}(z_a)}} + \underset{(2)}{\overline{\Gamma^\rho_{\mu\nu}(z_a)}})] + \mathcal{O}(h^3) \tag{23}$$

where

$$2\underset{(1)}{\overline{\Gamma^\alpha_{\beta\nu}(z)}} = -\overline{\partial_\beta h^\alpha_\gamma} - \overline{\partial_\gamma h^\alpha_\beta} + \overline{\partial^\alpha h_{\beta\gamma}} + \tfrac{1}{2}(\delta^\alpha_\gamma\overline{\partial_\beta h} + \delta^\alpha_\beta\overline{\partial_\gamma h} - \eta_{\beta\gamma}\overline{\partial^\alpha h}) \tag{24}$$

and where $\underset{(2)}{\overline{\Gamma^\alpha_{\beta\gamma}(z)}}$ is a sum of terms of the form $\overline{h_{\alpha\beta}\partial_\gamma h_{\mu\nu}}$ and is given in Appendix A. The equivalence between (21), (22) and the set of two equations (23) is not guaranteed and will have to be shown explicitly as a test of the consistency of the approximation scheme.

Our problem is to determine the gravitational potentials $h^{\alpha\beta} = h^{\alpha\beta}(x^\rho; L_a)$ together with the equations of motion $\dot{u}^\alpha_a = \Gamma^\alpha_a(z^\sigma_a, u^\rho_a; L_b)$; $h^{\alpha\beta}$ is a function of $x^\rho$ and a functional of the world lines $L_a$; $\dot{u}^\alpha_a$ is a function of $z^\sigma_a$, $u^\rho_a$, the tangent to $L_a$ at $z^\sigma_a$, and a functional of the world lines $L_b$; the equations of motion then restrict the class of possible world lines.

We now assume that the solution can be expanded in power series of the gravitational constant $G$ [physically, this means an expansion in the dimensionless number $Gm/c^2d$ which is usually very small ($d$ being the characteristic length of the problem)]:

$$h^{\alpha\beta} \equiv g^{\alpha\beta} - \eta^{\alpha\beta} = G\underset{(1)}{h^{\alpha\beta}} + G^2\underset{(2)}{h^{\alpha\beta}} + \mathcal{O}(G^3) \tag{25}$$

$$\dot{u}^\alpha \equiv \Gamma^\alpha(z^\rho) = \underset{(0)}{\Gamma^\alpha} + G\underset{(1)}{\Gamma^\alpha} + G^2\underset{(2)}{\Gamma^\alpha} + \mathcal{O}(G^3) \tag{26}$$

(and a similar expansion for $\dot{u}'^\alpha$); $\underset{(1)}{h^{\alpha\beta}}$ and $\underset{(1)}{\Gamma^\alpha}$ are linear in the masses $m$ and $m'$; $\underset{(2)}{h^{\alpha\beta}}$ and $\underset{(2)}{\Gamma^\alpha}$ contain terms proportional to $m^2$, $m'^2$, and $mm'$. We expect

$$\underset{(0)}{\Gamma^\alpha} = \underset{(0)}{\Gamma'^\alpha} = 0 \tag{27}$$

[4]When it is convenient we shall use the index $a$ to label the particles ($a, b, \ldots = 1, 2$; $L_1 = L$; $L_2 = L'$).

when the gravitational interaction is shut off ($G = 0$), the world lines must reduce to straight lines in Minkowski space-time. [Equation (27) will be demonstrated in Section 3.]

Now when transforming the previous expansions for $S^{\alpha\beta}$, $\mathfrak{T}^{\alpha\beta}$, $\Gamma^{\alpha}_{\beta\gamma}$ in powers of $h^{\mu\nu}$ [see (16), (18), and (24)] into expansions in powers of $G$ care must be taken with derivatives of $h^{\mu\nu}$ in view of the assumed expansion (26) of the equations of motion. For instance, it will be shown in Section 3 that the functional dependence of $\underset{(1)}{h}^{\alpha\beta}$ on the world lines $L_a$ reduces to a dependence on the retarded points $z^{\alpha}_{aR}$ (associated with $x$ on $L_a$) and on the retarded tangent vectors $u^{\alpha}_{aR} = \dot{z}^{\alpha}_{aR}$. Since the derivatives of $u^{\alpha}_a$ are proportional to $\dot{u}^{\alpha}_a$, that is, at least order 1 in $G$ according to (26) and (27), the derivative of $h^{\alpha\beta}$ can be written as

$$\partial_\gamma h^{\alpha\beta} = G \underset{(0)}{\partial_\gamma} \underset{(1)}{h}^{\alpha\beta} + [G \underset{(1)}{\partial_\gamma} \underset{(1)}{h}^{\alpha\beta} + G^2 \underset{(0)}{\partial_\gamma} \underset{(2)}{h}^{\alpha\beta}] + \mathcal{O}(G^3) \tag{28}$$

where $\underset{(0)}{\partial_\gamma}$ means that the derivative is taken "as if" the trajectories were straight lines and where $\underset{(1)}{\partial_\gamma} \underset{(1)}{h}^{\alpha\beta}$ takes care of the terms proportional to $\dot{u}^\sigma = G \underset{(1)}{\Gamma}^\sigma + \mathcal{O}(G^2)$ and is order one in $G$ (see Appendix A for examples of $\underset{(0)}{\partial}$ and $\underset{(1)}{\partial}$).

We shall also have to consider partial functional derivatives with respect to $L$ (or $L'$). They are defined by

$$D_\alpha f(x^\rho; L; L') = \lim_{h \to 0} \frac{1}{h} [f(x^\rho; L + \delta_\alpha L; L') - f(x^\rho; L; L')] \tag{29}$$

where $L + \delta_\alpha L$ is the parallel displaced world line with parametric equations

$$x^\sigma = z^\sigma(s) + h\delta^\sigma_\alpha \tag{30}$$

Note that

$$\partial_\alpha + D_\alpha + D'_\alpha \equiv 0 \tag{31}$$

The functional derivatives $D_\alpha$ and $D'_\alpha$ can be decomposed, as in (28) in $\underset{(0)}{D_\alpha} + \underset{(1)}{D_\alpha} + \cdots$.

The field equations (1) are expanded step by step in powers of $G$ and are integrated at each step using the flat retarded propagator $D(x)$ such that

$$\Box D(x) = -4\pi\delta_4(x) \tag{32}$$

$$D(x) = 2\delta(x^2)\theta(x^0) = \delta(x^0 - |\mathbf{x}|)/|\mathbf{x}| \tag{33}$$

where $|\mathbf{x}| = (x_i x^i)^{1/2}$ and $\theta$ is the Heaviside step function.

The integration of (1) at first order will be straightforward. On the other hand, at second order, some retarded integrals will diverge and will have to be regularized. A remarkable result is that this regularization is not arbitrary but is in fact imposed by the structure of the theory, plus the demand that the second-order Schwarzschild metric be recovered in the one-body problem, as will be shown in detail in Section 5. Moreover, this regularization of the divergent retarded integrals restricts in turn the regularization of divergent field quantities on the world lines.

*2.3. Regularization Procedure.* Let $x^\alpha$ be in the hyperplane orthogonal to $L$ at $z^\alpha$ such that

$$x^\alpha - z^\alpha = \epsilon n^\alpha, \qquad (n \cdot n) = 1, \qquad (n \cdot u) = 0 \tag{34}$$

where $u^\alpha$ is the tangent to $L$ at $z^\alpha$. The function functionals $f(x; L_a)$ we shall have to deal with in the postlinear approximation can be expanded in a Laurent series in $\epsilon$:

$$f(x^\alpha; L; L') = \sum_{m=-s}^{\infty} \epsilon^m f_{[m]}(n^\alpha) \tag{35}$$

Inspired by the regularization procedures commonly used to deal with divergent quantities in classical electrodynamics, one is tempted to regularize $f(z)$ by taking the mean value of (35) over $n^\alpha$.[5] In this case $\epsilon$ (and $\epsilon'$) would enter the formalism as additional parameters which may or may not be eliminated by renormalization at the end of the calculations. When necessary we shall refer to this regularization procedure as W-regularization ("wrong" regularization). Indeed when applied to the present problem W-regularization leads to inconsistent results because the nonlinearity of the theory together with the fact that the equations of motions are not independent of the field equations but rather can be deduced from them, imply that the integration of the field equations and the regularization of the source cannot be considered separately. The regularization procedure we shall use throughout this paper replaces $f(z^\alpha)$ by

$$\overline{f(z^\alpha)} = \langle f_{[0]}(n^\alpha) \rangle \equiv \frac{1}{4\pi} \int_{S_2} d\Omega\, f_{[0]}(n^\alpha) \tag{36}$$

$\Omega$ being the measure on the unit 2-sphere in the 3-plane orthogonal to $u^\alpha$.

The regularization procedure (36) need not be complemented by a renormalization procedure since it does not introduce any additional parameters.

Finally, we note that a completely different approach based on a regularization using the Riesz potentials yields the same final results [14].

[5] For a critical review of the regularization procedures used in classical electrodynamics together with a consistent use of the mean-value techniques see [13].

## §(3): *The Linear Gravitational Field*

The first-order Einstein equations relaxed by the harmonicity condition are

$$\Box \underset{(1)}{h}^{\alpha\beta} = 16\pi \sum m \int ds \delta_4(x - z) u^\alpha u^\beta \tag{37}$$

The integration of (37) by means of the flat retarded propagator (33) is straightforward:

$$\underset{(1)}{h}^{\alpha\beta} = -4 \sum m \int ds\, D(x - z) u^\alpha u^\beta$$
$$= -4m \left(\frac{u^\alpha u^\beta}{r}\right)_R - 4m' \left(\frac{u'^\alpha u'^\beta}{r'}\right)_R \tag{38}$$

where if $z_R$ is the retarded point on $L$ associated with $x$ $[(x - z_R)^2 = 0]$, then $r_R = -(x^\alpha - z_R^\alpha) u_{\alpha R}$, $u_R^\alpha$ being the tangent to $L$ at $z_R^\alpha$. [See Appendix C, equation (C6).]

The solution (38) of equation (37) will be a solution of the full Einstein equations if the harmonicity condition is satisfied at order 1.

From (38) we have

$$\partial_\beta \underset{(1)}{h}^{\alpha\beta} = -4m \left(\frac{\dot{u}^\alpha}{r}\right)_R - 4m' \left(\frac{\dot{u}'^\alpha}{r'}\right)_R \tag{39}$$

(cf. Appendix A for calculation of derivatives of retarded quantities).

Because of the assumed expansion (26) of the equations of motion, (39) reads

$$\partial_\beta G \underset{(1)}{h}^{\alpha\beta} = -4Gm(\underset{(0)}{\Gamma}^\alpha/r)_R - 4Gm'(\underset{(0)}{\Gamma}'^\alpha/r')_R + \mathcal{O}(G^2) \tag{40}$$

Now the first-order harmonicity condition is

$$\partial_\alpha G \underset{(1)}{h}^{\alpha\beta} = \mathcal{O}(G^2) \tag{41}$$

Therefore we must have

$$\underset{(0)}{\Gamma}^\alpha = \underset{(0)}{\Gamma}'^\alpha = 0 \tag{42}$$

as already anticipated [equation (27)].

Equations (42) are the zeroth-order equations of motion. They have sometimes been considered puzzling because they were ambiguously written as $\dot{u}^\alpha = \dot{u}'^\alpha = 0$ and interpreted as meaning that the field equations had to be solved with a source moving on a straight line.

Equation (42) actually restricts the class of possible world lines only in the sense that they have to be solutions of a system of equations which reduces to (42) when the gravitational interaction is shut off ($G = 0$); this does not mean that the trajectories have to be close to straight lines when $G \neq 0$ but only that the right-hand sides of the equations of motion have to be at least first order in $G$.

The linear gravitational potentials, solution of the linearized Einstein equations, are therefore

$$\mathfrak{g}^{\alpha\beta} = \eta^{\alpha\beta} - 4Gm\left(\frac{u^\alpha u^\beta}{r}\right)_R - 4Gm'\left(\frac{u'^\alpha u'^\beta}{r'}\right)_R + \mathcal{O}(G^2) \tag{43}$$

$$\dot{u}_a^\alpha = \mathcal{O}(G) \tag{44}$$

## §(4): *Conservation of the Second-Order Stress-Energy Tensor; The First-Order Equations of Motion*

The nonregularized second-order stress-energy tensor $\mathfrak{T}^{\alpha\beta}$ constructed with the first-order metric [equation (43)] reads [cf. (18) and (19)]

$$\mathfrak{T}^{\alpha\beta}(x) = \sum \int ds\delta_4(x - z)u^\alpha u^\beta \Bigg\{Gm + G^2 m^2 \left[\frac{1 + 2(u_R \cdot u)^2}{r_R}\right] + G^2 mm' \left[\frac{1 + 2(u'_R \cdot u)^2}{r'_R}\right]\Bigg\} + \mathcal{O}(G^3) \tag{45}$$

The second term is meaningless because the functions which multiply $\delta_4 [x - z(s)]$ are infinite on the world lines $x = z(s)$. Therefore, strictly speaking, the approximation method breaks down here, and the formalism does not even allow a derivation of the first nontrivial term of the equations of motion.

Replacing the meaningless expression (45) by its regularization (36), a straightforward calculation leads to

$$\overline{\mathfrak{T}^{\alpha\beta}(x)} = \sum Gm \int ds\delta_4(x - z)u^\alpha u^\beta \left(1 + Gm' \frac{1 + 2\omega^2}{\rho}\right) + \mathcal{O}(G^3) \tag{46}$$

where if $\hat{z}'$ is the retarded point on $L'$ associated with $z$, then $\omega = (u \cdot \hat{u}')$, $\rho = -(z^\alpha - \hat{z}'^\alpha)\hat{u}'_\alpha$, $\hat{u}'^\alpha$ being the tangent to $L'$ at $\hat{z}'$. (cf. Appendix B for detailed expansions in Laurent series of retarded quantities).

The regularization procedure amounts here to ignoring the contribution from the self-field:

$$\overline{\partial_\gamma \underset{(1)}{h}{}^{\alpha\beta}(z)} = 4m'\hat{u}'^\alpha \hat{u}'^\beta \nu_\gamma/\rho^2 + \mathcal{O}(G) \tag{47}$$

where

$$\nu^\alpha = -\hat{u}'^\alpha + (z^\alpha - \hat{z}'^\alpha)/\rho, \quad (\nu \cdot \nu) = 1, \quad (\nu \cdot \hat{u}') = 0 \tag{48}$$

Since

$$\frac{d}{ds}\rho = (u \cdot v) + \mathcal{O}(G) \tag{49}$$

the equation of conservation of the regularized second-order stress-energy tensor [equation (22)] is obtained straightforwardly:

$$\sum Gm \int ds\delta_4(x - z)[\dot{u}^\alpha - G\underset{(1)}{\Gamma^\alpha}(z)] = \mathcal{O}(G^3) \tag{50}$$

where

$$G\underset{(1)}{\Gamma^\alpha}(z) = Gm'\left[(1 - 2\omega^2)\frac{v^\alpha + (v \cdot u)u^\alpha}{\rho^2} + 4\omega(v \cdot u)\frac{v^\alpha}{\rho^2}\right] \tag{51}$$

with

$$v^\alpha = \hat{u}'^\alpha + \omega u^\alpha, \qquad (v \cdot u) = 0 \tag{52}$$

Therefore the first-order equations of motion as obtained from the conservation of $\overline{\mathfrak{T}^{\alpha\beta}}$ are

$$\dot{u}_a^\alpha = G\underset{(1)}{\Gamma_a^\alpha}(z_a) + \mathcal{O}(G^2) \tag{53}$$

with $\underset{(1)}{\Gamma^\alpha}(z)$ given by (51). Had we used the first-order geodesic equations (23) together with the same regularization procedure (47), we would have obtained the same equations (53).[6] It should be noted that if we had used the W-regularization procedure mentioned in Section 2.3, we would have obtained a different regularized second-order $\mathfrak{T}^{\alpha\beta}$ (the W-$\mathfrak{T}^{\alpha\beta}$ contains terms proportional to $1/\epsilon$ and $1/\epsilon'$); the first-order equations of motion would nevertheless have been the same. At first order then, the inconsistency of the W-regularization procedure is not evident.

## §(5): *The Postlinear Gravitational Field*

The second-order Einstein equations relaxed by the harmonicity condition read

$$\Box G^2 \underset{(2)}{h^{\alpha\beta}} = 16\pi \sum Gm \int ds\delta_4(x - z)u^\alpha u^\beta \underset{(1)}{\overline{t}} + G^2 \underset{(2)}{N^{\alpha\beta}} \tag{54}$$

where, as shown in equation (46), $\underset{(1)}{\overline{t}} = [Gm'(1 + 2\omega^2)]/\rho$, and where care must

[6] Since $(d/ds)\overline{(1/r_R)} = u \cdot \overline{\partial(1/r_R)}$ (= 0).

be taken with derivatives to calculate $\underset{(2)}{N}^{\alpha\beta}$. $N^{\alpha\beta}$ as given by equation (17) is a linear function of $\partial h \times \partial h$ and $h \times \partial^2 h$; therefore according to Section 2.2 it is sufficient to replace $h$ by $G\underset{(1)}{h}$, $\partial h$ by $G\underset{(0)}{\partial}\underset{(1)}{h}$, and $\partial^2 h$ by $G\underset{(0)}{\partial^2}\underset{(1)}{h}$, thereby leaving out third-order terms proportional to $\dot{u}$. It is clear that this does not mean that we are replacing $L$ and $L'$ by straight lines but simply that up to second-order in $G$ we do not have to consider those terms which would appear at third order.

We shall separate the solution of (54) as

$$\underset{(2)}{h}^{\alpha\beta} = h_T^{\alpha\beta} + h_N^{\alpha\beta} \tag{55}$$

where $h_T^{\alpha\beta}$ is generated by the first term on the right-hand side of (54) (see below for its computation) and where

$$G^2 h_N^{\alpha\beta}(x) = -\frac{1}{4\pi}\int d^4y\, D(x-y)\, G^2 \underset{(2)}{N}^{\alpha\beta}(y^\sigma, y_{aR}^\sigma, \dot{y}_{aR}^\sigma) \tag{56}$$

where the functional dependence of $\underset{(2)}{N}^{\alpha\beta}(y)$ on the lines $L_a$ reduces to a dependence on the retarded positions and velocities associated with $y$. The calculation of $h_N^{\alpha\beta}$ can be further simplified by remarking that we only need to know $G^2 h_N^{\alpha\beta}$ up to second order in $G$. Since the integration in equation (56) concerns only the parts of the world lines $L_a$ below the retarded point $z_{aR}$ associated with $x$, we can replace under the integral sign $\underset{(2)}{N}^{\alpha\beta}(y, y_{aR}, \dot{y}_{aR})$ by its lowest-order value, $\underset{(2)}{N}^{\alpha\beta}(y, y_{aR}^{(0)}, \dot{y}_{aR}^{(0)})$, where $y_{aR}^{(0)}$ and $\dot{y}_{aR}^{(0)}$ are calculated "as if" the lines $L_a$ were straight below $z_{aR}, \dot{z}_{aR}$. [Note that these fictitious straight lines used to evaluate (56) depend on the point $x$]. More precisely we define

$$\dot{y}_{aR}^{\alpha(0)} = u_{aR}^\alpha \equiv \dot{z}_{aR}^\alpha \tag{57}$$

$$y_{aR}^{\alpha(0)} = z_{aR}^\alpha - u_{aR}^\alpha\left(\{[(k\cdot u_{aR}) - r_{aR}]^2 + 2k\cdot(x - z_{aR})\}^{1/2} + (k\cdot u_{aR}) - r_{aR}\right) \tag{58}$$

where

$$k^\alpha = y^\alpha - x^\alpha \tag{59}$$

Since the curvature of the lines are $\mathcal{O}(G)$ we can write (formally)

$$G^2 \underset{(2)}{N}^{\alpha\beta}(y, y_{aR}, \dot{y}_{aR}) = G^2 \underset{(2)}{N}^{\alpha\beta}(y, y_{aR}^{(0)}, \dot{y}_{aR}^{(0)}) + \mathcal{O}(G^3) \tag{60}$$

so that

$$G^2 h_N^{\alpha\beta}(x) = G^2 h_{N(0)}^{\alpha\beta}(x, z_{aR}, \dot{z}_{aR}) + \mathcal{O}(G^3) \tag{61}$$

where by definition

$$h^{\alpha\beta}_{N(0)}(x, z_{aR}, \dot{z}_{aR}) = -\frac{1}{4\pi}\int d^4y\, D(x-y) \underset{(2)}{N}^{\alpha\beta}(y, y^{(0)}_{aR}, \dot{y}^{(0)}_{aR}) \tag{62}$$

Finally, it can be checked explicitly that $h^{\alpha\beta}_{N(0)}$ is a zero-order solution of equation (54):

$$\underset{(0)}{\Box}\, h^{\alpha\beta}_{N(0)}(x, z_{aR}, \dot{z}_{aR}) = \underset{(2)}{N}^{\alpha\beta}(x, z_{aR}, \dot{z}_{aR}) \tag{63}$$

with $\underset{(0)}{\Box} = \underset{(0)}{\partial} \cdot \underset{(0)}{\partial}$ (cf. Section 2.2 for the definition of $\underset{(0)}{\partial}$ ).

Once again we insist on the fact that replacing $y_R$ by $y^{(0)}_R$ does not mean that we are considering $L$ and $L'$ as straight lines but only that calculating the integral (54), taking into account the curvature of $L_a$ below $z_{aR}$, would introduce higher-order corrections. However, once the integral has been so calculated, $h^{\alpha\beta}_{N(0)}(x, z_{aR}, \dot{z}_{aR})$ is considered as a functional of the actual (curved) lines $L_a$. Were we to push the approximation scheme to third-order, this integration procedure might need revising.

In the following we shall decompose $h^{\alpha\beta}_N$ as

$$h^{\alpha\beta}_N = h^{\alpha\beta}_S + h^{\alpha\beta}_X \tag{64}$$

where $h^{\alpha\beta}_S$ contains the self-terms ($m^2$ and $m'^2$) and where $h^{\alpha\beta}_X$ contains the cross-terms ($mm'$).

*5.1. The "Self-Terms" $h^{\alpha\beta}_S$.* The equation satisfied by $h^{\alpha\beta}_S$ is

$$\Box\, G^2 h^{\alpha\beta}_S = \sum G^2 m^2 \left[\frac{4n^\alpha n^\beta - 2(\eta^{\alpha\beta} + 8u^\alpha u^\beta)}{r^4}\right]_R + \mathcal{O}(G^3) \tag{65}$$

where $n^\alpha_R = -u^\alpha_R + (x^\alpha - z^\alpha_R)/r_R$.

The retarded integral of (65) obtained by means of the flat retarded propagator is

$$G^2 h^{\alpha\beta}_{S\,\mathrm{div}} = -\frac{1}{4\pi}\sum G^2 m^2 \int \frac{d^3k}{|\mathbf{k}|}\left[\frac{4N^\alpha N^\beta - 2(\eta^{\alpha\beta} + 8u^\alpha_R u^\beta_R)}{R^4}\right] + \mathcal{O}(G^3) \tag{66}$$

where $k^0 = -|\mathbf{k}|$ and

$$R = [r^2 + (k \cdot u)^2 + 2r(k \cdot n)]^{1/2}_R \tag{67}$$

$$RN^\alpha = [rn^\alpha + k^\alpha + u^\alpha(k \cdot u)]_R \tag{68}$$

The expression (66) diverges and must be regularized. We first calculate the integrals (66) removing a small ball of radius $\epsilon$ ($\epsilon'$) centered at $R = 0$ ($R' = 0$); performing the integration in the frame where $u^0_{aR} = 1, n^3_{aR} = 1$, we obtain

$$G^2 h^{\alpha\beta}_{S\,\text{div}} = \sum m^2 G^2 \left[ - \frac{(7u^\alpha u^\beta + n^\alpha n^\beta)}{r^2} + \frac{\alpha\eta^{\alpha\beta} + \beta u^\alpha u^\beta}{r\epsilon} \right]_R + \mathcal{O}(G^3) \quad (69)$$

with:

$$\alpha = \tfrac{2}{3}, \qquad \beta = 14 + (\tfrac{2}{3}) \quad (70)$$

Now we demand that the metric reduce in the one-body problem to the Schwarzschild metric. In harmonic coordinates it reads [16]

$$\mathfrak{g}^{\alpha\beta}_{\text{Schw.}} = \eta^{\alpha\beta} - 4Gm\left(\frac{u^\alpha u^\beta}{r}\right)_R - G^2 m^2 \left(\frac{7u^\alpha u^\beta + n^\alpha n^\beta}{r^2}\right)_R + \mathcal{O}(G^3) \quad (71)$$

Therefore the extra terms in $(1/r_R\epsilon)$ and $(1/r'_R\epsilon')$ in (69) must be discarded. This amounts to taking the Hadamard *partie finie* [17] of the divergent integral (66). Thus we have

$$G^2 h^{\alpha\beta}_S = -\sum G^2 m^2 \left(\frac{7u^\alpha u^\beta + n^\alpha n^\beta}{r^2}\right)_R + \mathcal{O}(G^3) \quad (72)$$

This regularization of (66), imposed by the requirement $\mathfrak{g}^{\alpha\beta}_S = \mathfrak{g}^{\alpha\beta}_{\text{Schw.}}$, requires in turn the regularization of quantities which diverge on the world lines, and justifies the regularization procedure (36) we have chosen. Let us consider the conditions under which (72) is indeed a solution of (65) and compute the d'Alembertian of (72).

Using the Leibniz rule for the derivative of a product, we can write

$$-\Box\left(\frac{7u^\alpha u^\beta + n^\alpha n^\beta}{r^2}\right)_R = -2\left(\frac{7u^\alpha u^\beta + n^\alpha n^\beta}{r}\right)_R \Box \frac{1}{r_R} + \left[\frac{4n^\alpha n^\beta - 2(\eta^{\alpha\beta} + 8u^\alpha u^\beta)}{r^4}\right]_R + \mathcal{O}(G) \quad (73)$$

The first term of the right-hand side of (73) is meaningless unless regularized. Rewriting it as [cf. Appendix C, equation (C5)]

$$-2\left(\frac{7u^\alpha u^\beta + n^\alpha n^\beta}{r}\right)_R \Box \frac{1}{r_R} = 8\pi \int ds\,\delta_4(x - z)\left[\frac{7u^\alpha u^\beta + n^\alpha n^\beta}{r}\right]_R \quad (74)$$

the regularization procedure used throughout the paper leads to

$$\overline{-2\left(\frac{7u^\alpha u^\beta + n^\alpha n^\beta}{r}\right)_R \Box \frac{1}{r_R}} = 0 \quad (75)$$

thus ensuring that (72) is indeed a solution of (65).

Had we used the W-regularization procedure mentioned in Section 2.3, then (69) and not (72) would have been the consistent solution of (65) [since $\langle -2(7u^\alpha u^\beta + n^\alpha n^\beta)_R / r_R \rangle = -(\alpha\eta^{\alpha\beta} + \beta u^\alpha u^\beta)_R / r_R\epsilon$]; therefore the W-regularization procedure must be eliminated at this stage, since it does not allow to recover the Schwarzschild metric in the one-body problem.

We emphasize here that this consistency link between the integration of (65) and the regularization procedure cannot be overlooked if one wants to be sure that the results do not depend on how the Einstein equations are written down. Indeed any inconsistency at this point would have shown up as a failure of the Leibniz rule used in (73), so that the equivalence between solving $2|g|S^{\alpha\beta}(g^{\mu\nu}) = 16\pi|g|T^{\alpha\beta}$ and, e.g., $R_{\alpha\beta}(g_{\mu\nu}) = 8\pi(T_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}T)$, where $R_{\alpha\beta}$ is the Ricci tensor, would not have been guaranteed.

It can be noted that the regularization procedure used to give sense to (66) amounts to solving (65) in the framework of the theory of pseudofunctions [15]: to the right-hand side of (65) one can associate a pseudofunction whose retarded integral is the pseudofunction associated with (72).

Moreover, one can iterate Einstein's equations beyond second order in the one-body problem ($m' = 0$). The same regularization of the divergent integrals (Hadamard's *parties finies*) used at all orders leads unambiguously to a series in powers of $(Gm/r)$. This series is convergent when $Gm/r < 1$ and its summation leads to the exact Schwarzschild solution outside the horizon. This fact shows, at least in the one-body problem, that the use of $\delta$ distributions is consistent with Einstein's theory, provided that a suitable regularization procedure is used.

*5.2. The "Stress-Energy" Terms $h_T^{\alpha\beta}$.* Using the regularized second-order stress-energy tensor defined in Section 4, the equations satisfied by $h_T^{\alpha\beta}$ are

$$\Box h_T^{\alpha\beta} = 16\pi \sum m \int ds\delta_4(x - z) m' \frac{1 + 2\omega^2}{\rho} u^\alpha u^\beta \tag{76}$$

They are straightforwardly integrated by means of the propagator (33):

$$h_T^{\alpha\beta} = -4mm' \left[(1 + 2\omega^2) \frac{u^\alpha u^\beta}{r\rho}\right]_R - 4mm' \left[(1 + 2\omega'^2) \frac{u'^\alpha u'^\beta}{r'\rho'}\right]_R \tag{77}$$

where $\omega_R = u_R \cdot \hat{u}'_R$, $\rho_R = -(z_R - \hat{z}'_R) \cdot \hat{u}'_R$, $\hat{z}'_R$ being the retarded point on $L'$ associated with $z_R$, $\hat{u}'_R$ being the tangent vector to $L'$ at $\hat{z}'_R$.

Equation (77) is the exact solution of (76); since we are looking for a second-order solution, we can write

$$G^2 h_T^{\alpha\beta} = -4G^2 mm' \left[(1 + 2w^2) \frac{u^\alpha u^\beta}{r\rho_{(0)}}\right]_R - 4G^2 mm' \left[(1 + 2w^2) \frac{u'^\alpha u'^\beta}{r'\rho'_{(0)}}\right]_R + \mathcal{O}(G^3) \tag{78}$$

where $w_R = u_R \cdot u'_R$; $\rho_{(0)R} = -(z_R - \hat{z}'_{(0)R}) \cdot u'_R$, $\hat{z}'_{(0)R}$ being the retarded point associated with $z_R$ on the tangent $u'_R$ to $L'$ at $z'_R$ (cf. Figure 1 in Appendix A).

*5.3. The "Cross-Terms" $h_X^{\alpha\beta}$.* The equation satisfied by $h_X^{\alpha\beta}$ is

$$\Box G^2 h_X^{\alpha\beta} = G^2 mm' \Big\{ -16u^\alpha u^\beta \frac{1}{r'} u'^\mu u'^\nu \partial^2_{\mu\nu} \frac{1}{r} - 16u'^\alpha u'^\beta \frac{1}{r} u^\mu u^\nu \partial^2_{\mu\nu} \frac{1}{r'}$$
$$+ 16(u^\alpha u'^\beta + u'^\alpha u^\beta)\left[u'^\mu \partial_\mu \frac{1}{r} u^\nu \partial_\nu \frac{1}{r'} + w\partial^\mu \frac{1}{r} \partial_\mu \frac{1}{r'}\right]$$
$$- 16wu^\alpha u'^\mu \partial_\mu \frac{1}{r} \partial^\beta \frac{1}{r'} - 16wu'^\alpha u^\mu \partial_\mu \frac{1}{r'} \partial^\beta \frac{1}{r}$$
$$- 16wu^\beta u'^\mu \partial_\mu \frac{1}{r} \partial^\alpha \frac{1}{r'} - 16wu'^\beta u^\mu \partial_\mu \frac{1}{r'} \partial^\alpha \frac{1}{r}$$
$$+ 4(2w^2 - 1)\left(\partial^\alpha \frac{1}{r} \partial^\beta \frac{1}{r'} + \partial^\beta \frac{1}{r} \partial^\alpha \frac{1}{r'}\right)$$
$$+ 4\eta^{\alpha\beta}\left[4wu'^\mu \partial_\mu \frac{1}{r} u^\nu \partial_\nu \frac{1}{r'} - (2w^2 - 1)\partial^\mu \frac{1}{r} \partial_\mu \frac{1}{r'}\right]\Big\}_R + \mathcal{O}(G^3) \tag{79a}$$

(79a) can be written as

$$\Box G^2 h_X^{\alpha\beta} = G^2 mm' \left[(M^{\alpha\beta}_{\rho\sigma} D^\sigma + N^{\alpha\beta}_{\rho\sigma} D'^\sigma)\left(\frac{1}{r'} D^\rho \frac{1}{r}\right)\right.$$
$$\left. + (M'^{\alpha\beta}_{\rho\sigma} D'^\sigma + N'^{\alpha\beta}_{\rho\sigma} D^\sigma)\left(\frac{1}{r} D'^\rho \frac{1}{r'}\right)\right]_R + \mathcal{O}(G^3) \tag{79b}$$

where

$$(M^{\alpha\beta}_{\rho\sigma})_R = -16(u^\alpha u^\beta u'_\rho u'_\sigma)_R \tag{80}$$

$$(N^{\alpha\beta}_{\rho\sigma})_R = \{8(2u^\alpha u'^\beta + w\eta^{\alpha\beta})u'_\rho u_\sigma + 2[8wu^\alpha u'^\beta - (2w^2 - 1)\eta^{\alpha\beta}]\eta_{\rho\sigma}$$
$$- 16wu'^\alpha u_\sigma \delta^\beta_\rho - 16wu'^\beta u_\sigma \delta^\alpha_\rho + 4(2w^2 - 1)\delta^\alpha_\rho \delta^\beta_\sigma\}_R \tag{81}$$

and where the functional derivatives $D^\sigma$ and $D'^\sigma$ were defined by (29).

Let us now consider firstly the zero-order retarded solutions of

$$\underset{(0)}{\Box} P^\alpha = \left(\frac{1}{r'} \underset{(0)}{D^\alpha} \frac{1}{r}\right)_R = \left(\frac{n^\alpha}{r'r^2}\right)_R \tag{82}$$

$$P^\alpha = -\frac{1}{4\pi} \int \frac{d^3k}{|\mathbf{k}|} \frac{N^\alpha}{R'R^2} \tag{83}$$

(and similarly $P'^\alpha$ obtained by exchanging the roles of $L$ and $L'$); secondly let us consider the zero-order solution of

$$\underset{(0)}{\Box} G^{\alpha\beta} = \underset{(0)}{D^\beta}\left(\frac{n^\alpha}{r'r^2}\right)_R = \left[\frac{3n^\alpha n^\beta - (\eta^{\alpha\beta} + u^\alpha u^\beta)}{r'r^3}\right]_R \tag{84}$$

$$G^{\alpha\beta} = -\frac{\text{VP}}{4\pi}\int\frac{d^3k}{|\mathbf{k}|}\,\frac{3N^\alpha N^\beta - (\eta^{\alpha\beta} + u_R^\alpha u_R^\beta)}{R'R^3} \tag{85}$$

where VP denotes the Cauchy *valeur principale* [15] and where $R, N^\alpha$, are defined by (67) and (68).

Now because of the pole in the integrand of (85) at $R = 0$, $\underset{(0)}{D^\beta P^\alpha} \neq G^{\alpha\beta}$. When calculating $\underset{(0)}{D^\beta P^\alpha}$ by differentiating the integral which has as a domain of integration the complement of the ball of radius $\epsilon$ centered at $R = 0$, and letting $\epsilon \to 0$, we obtain

$$\underset{(0)}{D^\beta P^\alpha} = G^{\alpha\beta} + \frac{1}{3}\left[\frac{\eta^{\alpha\beta} + u^\alpha u^\beta}{r\rho_{(0)}}\right]_R \tag{86}$$

On the other hand, the zero-order retarded solution of

$$\underset{(0)}{\Box}\, F^{\alpha\beta} = \underset{(0)}{D'^\beta}\left(\frac{n^\alpha}{r' r^2}\right)_R = \left(\frac{n^\alpha n'^\beta}{r'^2 r^2}\right)_R \tag{87}$$

is

$$F^{\alpha\beta} = -\frac{1}{4\pi}\int\frac{d^3k}{|\mathbf{k}|}\,\frac{N^\alpha N'^\beta}{R^2 R'^2} \tag{88}$$

$F^{\alpha\beta}$ converges and we have

$$F^{\alpha\beta} = \underset{(0)}{D'^\beta P^\alpha} \tag{89}$$

and therefore

$$\underset{(0)}{D'^\beta P^\alpha} = \underset{(0)}{D^\alpha P'^\beta} \tag{90}$$

There are therefore two nonequivalent ways of obtaining the retarded solution of (79):

$$G^2 h_{\mathrm{X}w}^{\alpha\beta} = G^2 mm'\{M_{R\rho\sigma}^{\alpha\beta} G^{\rho\sigma} + N_{R\rho\sigma}^{\alpha\beta} F^{\rho\sigma} + M_{R\rho\sigma}'^{\alpha\beta} G'^{\rho\sigma} + N_{R\rho\sigma}'^{\alpha\beta} F'^{\rho\sigma}\} + \mathcal{O}(G^3) \tag{91}$$

$$G^2 h_{\mathrm{X}}^{\alpha\beta} = G^2 mm'\{M_{R\rho\sigma}^{\alpha\beta} D^\sigma P^\rho + N_{R\rho\sigma}^{\alpha\beta} D'^\sigma P^\rho + M_{R\rho\sigma}'^{\alpha\beta} D'^\sigma P'^\rho + N_{R\rho\sigma}'^{\alpha\beta} D^\sigma P'^\rho\} + \mathcal{O}(G^3) \tag{92}$$

However, here too, a definite choice is imposed by the theory: (92) is the correct solution, since it ensures that the second-order harmonicity condition is satisfied, as demonstrated below.

It should be noted that an integration of (79) within the framework of the theory of distributions would have led unambiguously to (92). Associating with

$(1/r')_R \underset{(0)}{D^\alpha}(1/r)_R$ the corresponding distribution, the zero-order retarded solution of (82) is the distribution $P^\alpha$. Since distributions are indefinitely differentiable, the retarded integral of the distribution $(1/r')_R \underset{(0)}{D^{\alpha\beta}}(1/r)_R$ is the distribution $\underset{(0)}{D^\beta P^\alpha}$ (and not $G^{\alpha\beta}$). Thus (92) is obtained as the zero-order retarded integral of (78).

Finally the explicit integration of (83) can be carried out, so that $P^\alpha$ is known in terms of elementary transcendental functions. (Cf. Appendix C.)

*5.4. Second-Order Harmonicity Condition.* The metric we obtained [equations (72), (77), and (92)] will be a solution of the full Einstein equations if the second-order harmonicity condition is satisfied.

Gathering the results, the metric reads

$$\begin{aligned}
\mathfrak{g}^{\alpha\beta} &= \eta^{\alpha\beta} - 4Gm(u^\alpha u^\beta/r)_R - 4Gm'(u'^\alpha u'^\beta/r')_R - G^2m^2[(7u^\alpha u^\beta + n^\alpha n^\beta)/r^2]_R \\
&\quad - G^2m'^2[(7u'^\alpha u'^\beta + n'^\alpha n'^\beta)/r'^2]_R - 4G^2mm'\{[(1+2\omega^2)u^\alpha u^\beta]/r\rho\}_R \\
&\quad - 4G^2mm'\{[(1+2\omega'^2)u'^\alpha u'^\beta/r'\rho'\}_R + G^2mm'\{-16u^\alpha u^\beta u'_\rho u'_\sigma D^\rho P^\sigma \\
&\quad - 16u'^\alpha u'^\beta u_\rho u_\sigma D'^\rho P'^\sigma + [8(2u^\alpha u'^\beta + w\eta^{\alpha\beta})u'_\rho u_\sigma \\
&\quad + (16wu^\alpha u'^\beta - 2\eta^{\alpha\beta}(2w^2-1))\eta_{\rho\sigma} - 16wu'^\alpha u_\sigma \delta^\beta_\rho - 16wu'^\beta u_\sigma \delta^\alpha_\rho \\
&\quad + 4(2w^2-1)\delta^\alpha_\rho \delta^\beta_\sigma]D'^\sigma P^\rho + [8(2u'^\alpha u^\beta + w\eta^{\alpha\beta})u_\rho u'_\sigma \\
&\quad + (16wu'^\alpha u^\beta - 2\eta^{\alpha\beta}(2w^2-1))\eta_{\rho\sigma} - 16wu^\alpha u'_\sigma \delta^\beta_\rho - 16wu^\beta u'_\sigma \delta^\alpha_\rho \\
&\quad + 4(2w^2-1)\delta^\alpha_\rho \delta^\beta_\sigma]D^\sigma P'^\rho\}_R + \mathcal{O}(G^3)
\end{aligned} \tag{93}$$

where the explicit expression of $P^\alpha$ is given in Appendix C.

Now

$$\partial_\beta G \underset{(1)}{h^{\alpha\beta}} = -4Gm(\dot{u}^\alpha/r)_R - 4Gm'(\dot{u}'^\alpha/r')_R \tag{94}$$

$$\partial_\beta G^2 h_S^{\alpha\beta} = \mathcal{O}(G^3) \tag{95}$$

$$\begin{aligned}
\partial_\beta G^2 h_T^{\alpha\beta} &= 4G^2mm'\left[\frac{(1+2\omega^2)u^\alpha(v\cdot u)}{r\rho^2}\right]_R \\
&\quad + 4G^2mm'\left[\frac{(1+2\omega'^2)u'^\alpha(v'\cdot u')}{r'\rho'^2}\right]_R + \mathcal{O}(G^3)
\end{aligned} \tag{96}$$

$$\begin{aligned}
\partial_\beta G^2 h_X^{\alpha\beta} &= G^2mm'\left[16\omega u'^\alpha \frac{(v\cdot u)}{r\rho^2} - 4(2\omega^2-1)\frac{v^\alpha}{r\rho^2}\right]_R \\
&\quad + G^2mm'\left[16\omega' u^\alpha \frac{(v'\cdot u')}{r'\rho'^2} - 4(2\omega'^2-1)\frac{v'^\alpha}{r'\rho'^2}\right]_R + \mathcal{O}(G^3)
\end{aligned} \tag{97}$$

where

$$v_R^\alpha = -\hat{u}_R'^\alpha + (z_R^\alpha - \hat{z}_R'^\alpha)/\rho_R$$

To compute (97) the following property of $P^\alpha$ was used

$$\underset{(0)}{D_\alpha} P^\alpha = \left[\frac{1}{r\rho_{(0)}}\right]_R \tag{98}$$

as can be deduced from (86) and (85). Equation (98) was then rewritten as

$$D_\alpha P^\alpha = [1/r\rho]_R + \mathcal{O}(G) \tag{99}$$

We also replaced $w_R$ by either $\omega_R$ or $\omega_R'$ since

$$w_R = \omega_R + \mathcal{O}(G) = \omega_R' + \mathcal{O}(G) \tag{100}$$

The second-order harmonicity condition then reads

$$\partial_\beta \mathfrak{g}^{\alpha\beta} = \frac{-4Gm}{r_R}\{\dot{u}^\alpha - G\underset{(1)}{\Gamma^\alpha}\}_R - \frac{4Gm'}{r_R'}\{\dot{u}'^\alpha - G\underset{(1)}{\Gamma'^\alpha}\}_R = \mathcal{O}(G^3) \tag{101}$$

which implies

$$\dot{u}_R^\alpha = G\underset{(1)}{\Gamma_R^\alpha} + \mathcal{O}(G^2), \qquad \dot{u}_R'^\alpha = G\underset{(1)}{\Gamma_R'^\alpha} + \mathcal{O}(G^2) \tag{102}$$

where

$$G\underset{(1)}{\Gamma_R^\alpha} = Gm'\left\{(1 - 2\omega^2)\frac{[v^\alpha + (v\cdot u)u^\alpha]}{\rho^2} + 4\omega(v\cdot u)\frac{v^\alpha}{\rho^2}\right\}_R \tag{103}$$

with

$$v_R^\alpha = \hat{u}_R'^\alpha + \omega_R u_R^\alpha \tag{104}$$

Equations (103) and (104) are nothing but the first-order equations of motion (52) written for $z_R$.

We have therefore shown that the first-order equations of motion can be obtained either from the conservation equation of the regularized second-order stress-energy tensor, or from the first-order geodesic equation, or from the second-order harmonicity condition, thus ensuring the consistency of the approximation scheme at this order.

## §(6): *Conservation of the Third-Order Stress-Energy Tensor; The Second-Order Equations of Motion*

The equation of conservation of the regularized third-order stress-energy tensor is obtained from (22) where $h^{\alpha\beta}$ is replaced by $G\underset{(1)}{h^{\alpha\beta}} + G^2\underset{(2)}{h^{\alpha\beta}}$. Some elementary algebra yields

$$\overline{\nabla_\beta \overline{\mathfrak{T}^{\alpha\beta}}} = \sum Gm \int ds \delta_4(x-z)\{\dot{u}^\alpha + u^\mu u^\nu[\overline{\underset{(1)}{\Gamma}{}^\alpha_{\mu\nu}} + \overline{\underset{(2)}{\Gamma}{}^\alpha_{\mu\nu}} + u^\alpha u_\rho(\overline{\underset{(1)}{\Gamma}{}^\rho_{\mu\nu}} + \overline{\underset{(2)}{\Gamma}{}^\rho_{\mu\nu}})]\} + \mathcal{O}(G^3) \quad (105)$$

[that is, $\overline{\nabla_\beta \overline{\mathfrak{T}^{\alpha\beta}}} = 0$ is equivalent to the second-order regularized geodesic equations (23)], provided that the three following conditions are satisfied:

$$\overline{\underset{(1)}{h}{}^{\alpha\beta}\underset{(1)}{h}{}^{\mu\nu}} = \overline{\underset{(1)}{h}{}^{\alpha\beta}} \times \overline{\underset{(1)}{h}{}^{\mu\nu}} + \mathcal{O}(G) \quad (106)$$

$$\overline{\underset{(1)}{h}{}^{\alpha\beta}\underset{(0)}{\partial^\lambda}\underset{(1)}{h}{}^{\mu\nu}} = \overline{\underset{(1)}{h}{}^{\alpha\beta}} \times \overline{\underset{(0)}{\partial^\lambda}\underset{(1)}{h}{}^{\mu\nu}} + \mathcal{O}(G) \quad (107)$$

$$\frac{d}{ds}(\overline{\underset{(1)}{h}{}^{\alpha\beta} + G\underset{(2)}{h}{}^{\alpha\beta}}) = u^\lambda(\overline{\partial_\lambda \underset{(1)}{h}{}^{\alpha\beta} + G\partial_\lambda \underset{(2)}{h}{}^{\alpha\beta}}) + \mathcal{O}(G^2)$$

$$= u^\lambda(\overline{\underset{(0)}{\partial_\lambda}\underset{(1)}{h}{}^{\alpha\beta} + \underset{(1)}{\partial_\lambda}\underset{(1)}{h}{}^{\alpha\beta} + G\underset{(0)}{\partial_\lambda}\underset{(2)}{h}{}^{\alpha\beta}}) + \mathcal{O}(G^2) \quad (108)$$

Now since

$$\underset{(0)}{\partial^\lambda}\underset{(1)}{h}{}^{\mu\nu} = 4m\left(\frac{u^\mu u^\nu n^\lambda}{r^2}\right)_R + 4m'\left(\frac{u'^\mu u'^\nu n'^\lambda}{r'^2}\right)_R \quad (109)$$

(cf. Appendix A for the calculation of derivatives of retarded quantities), and since

$$\overline{(1/r_R^n)} = \overline{(n^\gamma/r^n)_R} = \overline{(1/rr')_R} = \overline{(n'^\gamma/rr'^2)_R} = \overline{(n^\gamma/r^2r')_R} = \mathcal{O}(G) \quad (110)$$

(cf. Appendix B for developments in Laurent series and the regularization of retarded quantities), the conditions (106) and (107) are clearly fulfilled. The importance of the last condition (108) for the consistency of any regularization scheme has been stressed in [13]; extending the methods in [13] it can be seen easily that (108) is fulfilled when using the regularization procedure we have chosen.

Therefore the conservation of the regularized third-order stress-energy tensor is equivalent to the regularized second-order geodesic equations. We shall not deduce here the second-order equations of motion from the third-order harmonicity condition since we do not calculate the third-order metric. However, the equivalence between the third-order harmonicity condition and the regularized second-order geodesic equations can be shown explicitly by using a procedure of analytic continuation based on a generalization of the Riesz potentials [14].

The second-order geodesic equation is obtained from (23) by replacing $h^{\alpha\beta}$ by $G\underset{(1)}{h}{}^{\alpha\beta} + G^2\underset{(2)}{h}{}^{\alpha\beta}$ given by (94). We shall separate it as

$$\dot{u}^\alpha = G\underset{(1)}{\Gamma}{}^\alpha + G^2(\Gamma^\alpha_{\bullet} + \Gamma^\alpha_\cap + \Gamma^\alpha_T + \Gamma^\alpha_S + \Gamma^\alpha_X) + \mathcal{O}(G^3) \quad (111)$$

Introducing the quantities

$$\underset{(1)}{\Gamma^\alpha}(x) = \frac{m}{r_R^2}\langle n_R^\alpha[1 - 2(u_R \cdot u)^2] + (n_R \cdot u)\{[1 + 2(u_R \cdot u)^2]u^\alpha + 4(u_R \cdot u)u_R^\alpha\}\rangle + \cdots \tag{112}$$

$$G\underset{\bullet}{\Gamma^\alpha}(x) = m\left[\frac{n_R^\alpha + (n_R \cdot u)u^\alpha}{r_R}\right]\{[1 - 2(u_R \cdot u)^2](n_R \cdot \dot{u}_R) - 4(\dot{u}_R \cdot u)(u_R \cdot u)\} + 4m\left[\frac{\dot{u}_R^\alpha + (\dot{u}_R \cdot u)u^\alpha}{r_R}\right](u_R \cdot u)[(n_R \cdot u) + (u_R \cdot u)] + m\left[\frac{u_R^\alpha + (u_R \cdot u)u^\alpha}{r_R}\right][(n \cdot \dot{u})_R + 4(n_R \cdot u)(u_R \cdot u)(n_R \cdot \dot{u}_R) + 4(n_R \cdot u)(\dot{u}_R \cdot u) + 2(u_R \cdot u)^2(n_R \cdot \dot{u}_R)] + \cdots \tag{113}$$

$$\Gamma^\alpha_{\cap}(x) = 2m^2\left[\frac{n_R^\alpha + (n_R \cdot u)u^\alpha}{r_R^3}\right][2(u_R \cdot u)^2 - 1] - 4mm'\frac{(n'_R \cdot u)}{r_R r'^2_R}\{2[u_R^\alpha + (u_R \cdot u)u^\alpha][(u_R \cdot u) + 2(u'_R \cdot u)(u' \cdot u)] + 2[u'^\alpha_R + (u'_R \cdot u)u^\alpha](u'_R \cdot u)\} + 2mm'\frac{[2(u'_R \cdot u)^2 - 1]}{r_R r'^2_R} \cdot \{n'^\alpha_R + (n'_R \cdot u)u^\alpha + 2(n'_R \cdot u_R)[u_R^\alpha + (u_R \cdot u)u^\alpha]\} + \cdots \tag{114}$$

$$\Gamma^\alpha_T(x) = mm'\left(\frac{1 + 2w^2}{r\rho_{(0)}}\right)_R\left\{\left[\frac{n_R^\alpha + (n_R \cdot u)u^\alpha}{r_R} + \frac{v^\alpha_{(0)R} + (v_{(0)R} \cdot u)u^\alpha}{\rho_{(0)R}}\right] \cdot [1 - 2(u_R \cdot u)^2] + 4(u_R \cdot u)[u_R^\alpha + (u_R \cdot u)u^\alpha] \cdot \left[\frac{(n_R \cdot u)}{r_R} + \frac{(v_{(0)R} \cdot u)}{\rho_{(0)R}}\right]\right\} + \cdots \tag{115}$$

$$\Gamma^\alpha_S(x) = \frac{2m^2}{r_R{}^3}\{[n_R^\alpha + (n_R \cdot u)u^\alpha][1 + (n_R \cdot u)^2] - (n_R \cdot u)(u_R \cdot u)[u_R^\alpha + (u_R \cdot u)u^\alpha]\} + \cdots \tag{116}$$

$$\frac{\Gamma^\alpha_X(x)}{mm'} = 4(u'^\mu u'^\nu \partial^\alpha D_\mu P_\nu + u^\alpha u^\rho u'^\mu u'^\nu \partial_\rho D_\mu P_\nu) - 16\omega v^\alpha u^\rho u^\mu u^\nu \partial_\rho D'_\mu P'_\nu + 4(2\omega^2 - 1)(u^\mu u^\nu \partial^\alpha D'_\mu P'_\nu + u^\alpha u^\rho u^\mu u^\nu \partial_\rho D'_\mu P'_\nu) - 16v^\alpha u^\rho u^\mu u'^\nu \partial_\rho D_\nu P'_\mu + 16\omega(u^\rho u'^\mu \partial_\rho D_\mu P'^\alpha + u^\alpha u^\rho u'^\mu u^\nu \partial_\rho D_\mu P'_\nu) - 4(2\omega^2 + 1)u^\rho u^\mu \partial_\rho D^\alpha P'_\mu - 16\omega v^\alpha u^\rho \partial_\rho D_\mu P'^\mu + 8\omega^2(\partial^\alpha D_\mu P'^\mu + u^\alpha u^\rho \partial_\rho D_\mu P'^\mu) \tag{117}$$

$\underset{(1)}{\Gamma^\alpha}, \Gamma^\alpha_{\bullet}$, etc. are the regularized values of $\underset{(1)}{\Gamma^\alpha}(x), \Gamma^\alpha_{\bullet}(x)$ etc., when $x$ tends to the point $z$ on the world-line $L$. In (112)-(116) the dots $(\cdots)$ mean the exchange of the particles ($m \to m', u_R \to u'_R, n_R \to n'_R, r_R \to r'_R$, etc.). Equations (112)-(114) yield the contributions to the second-order geodesic equation due to the linear field $G\underset{(1)}{h^{\alpha\beta}}$. Equation (112) yields the first-order equations of motion. When calculating (113) from equation (23) all the terms proportional to $\dot{u}^2$ were discarded since they are second order in $G$; the vector $\dot{u}^\mu_R$ entering (113) is a known functional of the world lines given by the first-order equations of motion (102) and (103); equation (113) contains the term $(11\ddot{u}^\alpha/3)$ first obtained by Havas [6]. Equation (114) is obtained from equation (A.28) in Appendix A by replacing $h^{\alpha\beta}$ by $\underset{(1)}{h^{\alpha\beta}}$. Equation (115) is the contribution of the stress-energy terms $h^{\alpha\beta}_T$; it is obtained from (23)-(24) by replacing $h^{\alpha\beta}$ by $h^{\alpha\beta}_T$. Similarly (116) is the contribution from the self-terms $h^{\alpha\beta}_S$. Finally (117) is the contribution from the cross-terms $h^{\alpha\beta}_X$.

Equations (112)-(117) are then expanded in Laurent series around the point $z, x$ being in the hyperplane orthogonal to $L$ at $z$. The resulting expansions are then regularized by taking the mean value of the term of the expansion which is independent of $\epsilon = [(x - z)^2]^{1/2}$. A rather long but straightforward calculation outlined in Appendices B and C then yields

$$\underset{(1)}{\Gamma^\alpha} = m'\rho^{-2}[(1 - 2\omega^2)A^\alpha - (1 + 2\omega^2 + 4\omega A)v^\alpha] \tag{118}$$

$$\begin{aligned}\frac{\Gamma^\alpha_{\bullet}}{mm'} = \frac{A^\alpha}{\rho^3}\Bigg[&\frac{-4\omega^4 + 12\omega^2 - 1}{A^3} + \frac{-4\omega^5 + 12\omega^3 - \omega}{A^2} + \frac{4\omega^4 - 1}{A} \\ &- 11\omega + 22\omega^3 + 11A(2\omega^2 - 1)\Bigg] + \frac{v^\alpha}{\rho^3}\Bigg[\frac{-8\omega^3 + 12\omega}{A^2} \\ &+ \frac{-8\omega^4 + 28\omega^2}{A} + 26\omega + \frac{4 \times 17}{3}\omega^3 + 11A(1 + 6\omega^2 + 4\omega A)\Bigg]\end{aligned} \tag{119}$$

$$\Gamma^\alpha_\cap = \frac{2m'^2}{\rho^3}(2\omega^2 - 1)(A^\alpha - v^\alpha) \tag{120}$$

$$\begin{aligned}\frac{\Gamma^\alpha_T}{mm'} = &\frac{A^\alpha}{\rho^3}\left[\frac{4\omega^4 - 1}{A^3} + \frac{4\omega^5 - \omega}{A^2} - \frac{4\omega^4 - 1}{A}\right] \\ &+ \frac{v^\alpha}{\rho^3}\left[\frac{8\omega^3 + 4\omega}{A^2} + \frac{4\omega^4 - 1}{A} - 4\omega - 8\omega^3\right]\end{aligned} \tag{121}$$

$$\Gamma^\alpha_S = \frac{2m'^2}{\rho^3}[A^\alpha(1 + \omega^2 + 2A\omega + A^2) - v^\alpha(1 + A\omega + A^2)] \tag{122}$$

$$\frac{\Gamma^{\alpha}_{\times}}{mm'} = -4\left[\frac{d}{ds}\underset{(1)}{\Gamma^{\alpha}}\right]\frac{\ln A}{m'} - \frac{A^{\alpha}}{\rho^3}\left[\frac{2}{A^5} + \frac{5\omega}{A^4} + \frac{(22\omega^2 - 7)}{A^3}\right.$$
$$+ \frac{4\omega}{A^2}(5\omega^2 - 2) + \frac{8(1 - 2\omega^2 - \omega^4)}{A}$$
$$\left. + 17\omega(1 - 2\omega^2) + 5A(1 - 2\omega^2)\right] - \frac{v^{\alpha}}{\rho^3}\left[\frac{47}{3A^3} + \frac{48\omega}{A^2}\right.$$
$$\left. + \frac{4(\omega^2 - 2)}{A} - 62\omega - \frac{4 \times 31}{3}\omega^3 - A(5 + 78\omega^2) - 20\omega A^2\right] \tag{123}$$

where if $\hat{z}'$ is the retarded point on $L'$ associated with $z$, $\hat{u}'$, the tangent to $L'$ at $z'$, then

$$\rho = -(z - \hat{z}') \cdot \hat{u}' \tag{124}$$

$$\omega = (u \cdot \hat{u}') \tag{125}$$

$$v^{\alpha} = \hat{u}'^{\alpha} + \omega u^{\alpha} \tag{126}$$

$$A^{\alpha} = u^{\alpha}[(z - \hat{z}') \cdot u]/\rho + (z^{\alpha} - \hat{z}'^{\alpha})/\rho, \quad (A \cdot u) = 0 \tag{127}$$

$$A = (A \cdot A)^{1/2} = -[(z - \hat{z}') \cdot u]/\rho \tag{128}$$

Gathering the results we obtain

$$\dot{u}^{\alpha} = G\underset{(1)}{\Gamma^{\alpha}} + G^2\underset{(2)}{\Gamma^{\alpha}} + \mathcal{O}(G^3) \tag{129}$$

where $\underset{(1)}{\Gamma^{\alpha}}$ is given by (51) or (118) and where

$$\underset{(2)}{\Gamma^{\alpha}} = \rho^{-3}[m'^2 H^{\alpha}_{S} + mm'(aA^{\alpha} + bv^{\alpha})] - 4md[\underset{(1)}{\Gamma^{\alpha}}\ln A]/ds \tag{130}$$

with:

$$H^{\alpha}_{S} = 2A^{\alpha}(3\omega^2 + 2A\omega + A^2) - 2v^{\alpha}(2\omega^2 + A\omega + A^2) \tag{131}$$

$$a = -2A^{-5} - 5\omega A^{-4} + 5(1 - 2\omega^2)A^{-3} + 2\omega(3 - 4\omega^2)A^{-2}$$
$$+ 4(2\omega^4 + 2\omega^2 - 1)A^{-1} + 20\omega(2\omega^2 - 1) + 12(2\omega^2 - 1)A \tag{132}$$

$$b = -(47/3)A^{-3} - 32\omega A^{-2} + (3 + 16\omega^2 - 4\omega^4)A^{-1} + 20\omega(2\omega^2 + 3)$$
$$+ 4(3 + 26\omega^2)A + 48\omega A^2 \tag{133}$$

The equations of motion (130)-(133) are equivalent to those obtained in [10].

### §(7): *Predictive Poincaré-Invariant System Associated with the Second-Order Equations of Motion*

The second-order equations of motion obtained in Section 6 are of hereditary character. The acceleration of each particle is a functional of the past history of the two particles. Indeed equations (129) can be symbolically written as

$$\begin{aligned} \frac{dz^\alpha}{ds} &= u^\alpha, \qquad \frac{du^\alpha}{ds} = W^\alpha(z^\beta, u^\beta, \hat{z}'^\beta, \hat{u}'^\beta) + \mathcal{O}(G^3) \\ \frac{dz'^\alpha}{ds'} &= u'^\alpha, \qquad \frac{du'^\alpha}{ds'} = W'^\alpha(z'^\beta, u'^\beta, \hat{z}^\beta, \hat{u}^\beta) + \mathcal{O}(G^3) \end{aligned} \tag{134}$$

where $W^\alpha$ denotes $G\underset{(1)}{\Gamma^\alpha} + G^2 \underset{(2)}{\Gamma^\alpha}$ [the differentiation in the last term of (130) having been worked out "as if" the lines were straight].

$s(s')$ is the Minkowskian proper time along $L$ $(L')$ and $\hat{z}' \in L'$ $(\hat{z} \in L)$ is the retarded point associated with $z \in L$ $(z' \in L')$. Equations (134) exhibit a hereditary character, since the right-hand side involves configurations of the particles (4-positions and 4-velocities) which are connected by light cones and which are not therefore generic. This kind of equation is known in the literature as differential-delay systems or retarded-functional differential systems. Unfortunately, there is at present no theorem of existence and uniqueness of solution for those systems, except in very particular cases which do not include the present one.

In order to transform equations (134) into an ordinary system of second-order differential equations, we shall use here the hypotheses and the framework of predictive relativistic mechanics [12]. In this theory it is assumed that the evolution of an isolated system of interacting particles is described by an ordinary differential system of the type

$$\begin{aligned} \frac{dz^\alpha(\lambda)}{d\lambda} &= u^\alpha, \qquad \frac{du^\alpha(\lambda)}{d\lambda} = \xi^\alpha(z^\beta, u^\beta, z'^\beta, u'^\beta) \\ \frac{dz'^\alpha(\lambda)}{d\lambda} &= u'^\alpha, \qquad \frac{du'^\alpha(\lambda)}{d\lambda} = \xi'^\alpha(z^\beta, u^\beta, z'^\beta, u'^\beta) \end{aligned} \tag{135}$$

or

$$\frac{dz_a^\alpha(\lambda)}{d\lambda} = u_a^\alpha, \qquad \frac{du_a^\alpha(\lambda)}{d\lambda} = \xi_a^\alpha(z_b^\beta, u_c^\gamma)$$

where, for compactness, the Latin indices labeling the particles have been reintroduced: $a, b, a', \ldots = 1, 2; z_1 \equiv z, z_2 \equiv z'$; with the conventions: $a \neq a'$ and no summation on repeated Latin indices. The function $\xi_a^\alpha$ must satisfy the following requirements

(i) Be invariant, under the transformations of the Poincaré group.

(ii) Satisfy the orthogonality condition

$$\xi_a^\alpha u_{a\alpha} = 0 \tag{136}$$

so that the unitarity (in general, the constancy of the modulus) of the 4-velocities $u_a^\alpha$ is a consequence of the corresponding condition imposed on the initial conditions. In other words the common parameter $\lambda$ measures the Minkowskian propertime along both lines $z(\lambda), z'(\lambda)$.

(iii) Be solutions of the following nonlinear system of partial differential equations [18]:

$$u_{a'}^\rho \frac{\partial \xi_a^\alpha}{\partial z_{a'}^\rho} + \xi_{a'}^\rho \frac{\partial \xi_a^\alpha}{\partial u_{a'}^\rho} = 0 \tag{137}$$

Equation (137) can be obtained from the requirement that the solution of (135), i.e., a pair of world lines, is unchanged when the initial data $z_a(0), u_a(0)$ are arbitrarily shifted along the trajectories. This condition together with (136) ensures that the general solution of (135) depends on 12 essential parameters and not on 16 as one could expect (case of two particles). Thus the system (135) can be changed, in each frame of reference, into another one of Newtonian type, (i.e., using $t = z^0 = z'^0$ as parameter $\lambda$ for both particles), preserving the invariance under the Poincaré group [19].

According to the above considerations, we construct a Poincaré-invariant predictive system of the type (135) such that the functions $\xi_a^\alpha$ coincide with the right-hand side of (134) when the configurations of the particles are connected by light cones. (This requirement plays the role of a boundary condition.) In order to do so we assume that the functions $\xi_a^\alpha$ can be expanded in a power series of the gravitational constant $G$ as

$$\xi_a^\alpha = G \underset{(1)}{\xi_a^\alpha} + G^2 \underset{(2)}{\xi_a^\alpha} + \mathcal{O}(G^3) \tag{138}$$

so that the proposed program can be carried out order by order.

Substituting the expansion (138) into equation (137) and equating terms of the same order we obtain the following equations:

$$u_{a'}^\rho \frac{\partial \underset{(1)}{\xi_a^\alpha}}{\partial z_{a'}^\rho} = 0 \tag{139a}$$

$$u_{a'}^\rho \frac{\partial \underset{(2)}{\xi_a^\alpha}}{\partial z_{a'}^\rho} = -\underset{(1)}{\xi_{a'}^\rho} \frac{\partial \underset{(1)}{\xi_a^\alpha}}{\partial u_{a'}^\rho} \tag{139b}$$

and so on. Equations (139) give a recurrent method to compute the different terms of the expansion (138) once the first one is known, provided that we have a criterion for the selection of solutions [20]. An essentially equivalent method would consist in transforming the differential equation (137) into integro-functional equations [21] [12] which automatically include the boundary condition (134).

*7.1 First Order.* To obtain the first-order term $\underset{(1)}{\xi}{}^{\alpha}_{a}$ of the expansion (138) we impose the "boundary condition" (134) to the general solution of (139a) at first order. As is well known [20], we thus obtain a *unique* result, which can be deduced from $\underset{(1)}{\Gamma}{}^{\alpha}_{a}$, [equation (118)], by means of the following substitutions:

$$\begin{aligned} \hat{z}^{\alpha}_{a'} &\longrightarrow z^{\alpha}_{a'} - [(z_{aa'} \cdot u_{a'}) + r_a] u^{\alpha}_{a'} \\ \hat{u}^{\alpha}_{a'} &\longrightarrow u^{\alpha}_{a'} \end{aligned} \tag{140}$$

where the final configurations $(z_a, u_a, z_{a'}, u_{a'})$ are generic, and where the following notations have been used:

$$z^{\alpha}_{aa'} \equiv z^{\alpha}_{a} - z^{\alpha}_{a'} \tag{141a}$$

$$r_a \equiv +[z^2_{aa'} + (z_{aa'} \cdot u_{a'})^2]^{1/2} \tag{141b}$$

The final result (see Appendix D for detailed notations and a sketch of the derivation) is

$$\underset{(1)}{\xi}{}^{\alpha}_{a} = m_{a'} r_a^{-3} [(1 - 2k^2) h^{\alpha}_{aa'} + k(3 - 2k^2) \tau_a t^{\alpha}_{a'}] \tag{142}$$

with

$$\begin{aligned} k &\equiv -(u_a \cdot u_{a'}), \qquad \Lambda^2 \equiv k^2 - 1 \\ \tau_a &\equiv \Lambda^{-2} [(z_{aa'} \cdot u_a) - k(z_{aa'} \cdot u_{a'})] \end{aligned} \tag{143}$$

$$\begin{aligned} h^{\alpha}_{aa'} &\equiv z^{\alpha}_{aa'} - \tau_a u^{\alpha}_{a} + \tau_{a'} u^{\alpha}_{a'} \\ t^{\alpha}_{a} &\equiv u^{\alpha}_{a} - k u^{\alpha}_{a'} \end{aligned} \tag{144}$$

This result was first obtained by Portilla [22].

The vector $h^{\alpha}_{aa'}$ is orthogonal to the two velocities $u^{\alpha}_{a}$. Its length is the least distance between the straight lines constructed from the generic configurations $(z^{\alpha}_{a}, u^{\beta}_{b})$. The scalars $\tau_a$ are the parametric distance on the above straight lines, between $z^{\alpha}_{a}$ and the endpoints of their common perpendicular $h^{\alpha}_{aa'}$. Equation (141b) can be rewritten as

$$r_a \equiv \{h^2_{aa'} + \Lambda^2 \tau^2_a\}^{1/2} \tag{145}$$

Note that in order to obtain (142) we have used neither the condition of invariance under the Poincaré group nor the orthogonality condition (136). These requirements are automatically fulfilled as a consequence of the structure of $\underset{(1)}{\Gamma}{}^{\alpha}_{a}$.

*7.2. Second Order.* The second-order term in the expansion (138) is obtained from equation (139b), where $\underset{(1)}{\xi_a^\alpha}$ is given by (142). We shall write the general solution of (139b) as

$$\underset{(2)}{\xi_a^\alpha} = \underset{(2)}{\xi_a^{*\alpha}} + \underset{(2)}{\xi_a^{\#\alpha}} \tag{146}$$

where $\underset{(2)}{\xi_a^{*\alpha}}$ represents the general solution of the associated homogeneous equation, and $\underset{(2)}{\xi_a^{\#\alpha}}$ is a particular solution of (139b), which is chosen in order to vanish when the configurations of the particles are connected by retarded light cones. Thus, at second order, the "boundary condition" (134) selects a solution from the family $\underset{(2)}{\xi_a^{*\alpha}}$. This solution is obtained as in Section 7.1 by making the substitutions (140) in the expression (130) of $\underset{(2)}{\Gamma_a^\alpha}$. We obtain

$$\begin{aligned}\underset{(2)}{\xi_a^{*\alpha}} &= m_{a'}^2 r_a^{-3} H_{aS}^\alpha + m_a m_{a'} r_a^{-4}\{A_a h_{aa'}^\alpha - [k\tau_a A_a - r_a(A_a + B_a)]\, t_{a'}^\alpha\} \\ &\quad - 4m_a m_{a'} r_a^{-3} p_a^{-1}(1 - 2kr_a^{-1}p_a + r_a^{-2}p_a^2)\{(1 - 2k^2)h_{aa'}^\alpha \\ &\quad + k(3 - 2k^2)\tau_a t_{a'}^\alpha\} + 4m_a m_{a'} r_a^{-4}\ln(r_a^{-1}p_a) \\ &\quad \cdot\{3(1 - 2k^2)(k - r_a^{-1}p_a)h_{aa'}^\alpha + k[(5 - 2k^2)(k - r_a^{-1}p_a)\tau_a \\ &\quad + (1 - 2k^2)r_a - 4r_a(1 - 2kr_a^{-1}p_a + r_a^{-2}p_a^2)]\, t_{a'}^\alpha\}\end{aligned} \tag{147}$$

where

$$\begin{aligned}H_{aS}^\alpha &\equiv 2r_a^{-1}\{(3k^2 - 2kr_a^{-1}p_a + r_a^{-2}p_a^2)h_{aa'}^\alpha \\ &\quad - k\tau_a(1 + 2k^2 - 2kr_a^{-1}p_a + r_a^{-2}p_a^2)\, t_{a'}^\alpha\}\end{aligned} \tag{148a}$$

$$\begin{aligned}A_a &\equiv -2r_a^5 p_a^{-5} + 5kr_a^4 p_a^{-4} + 5(1 - 2k^2)r_a^3 p_a^{-3} - 2k(3 - 4k^2)r_a^2 p_a^{-2} \\ &\quad + 4(2k^4 + 2k^2 - 1)r_a p_a^{-1} - 20k(2k^2 - 1) + 12(2k^2 - 1)r_a^{-1}p_a\end{aligned} \tag{148b}$$

$$\begin{aligned}B_a &\equiv -\tfrac{47}{3} r_a^3 p_a^{-3} + 32kr_a^2 p_a^{-2} + (3 + 16k^2 - 4k^4)r_a p_a^{-1} - 20k(2k^2 + 3) \\ &\quad + 4(3 + 26k^2)r_a^{-1}p_a - 48kr_a^{-2}p_a^2\end{aligned} \tag{148c}$$

$$p_a \equiv kr_a - \Lambda^2\tau_a \tag{149}$$

The particular solution $\underset{(2)}{\xi_a^{\#\alpha}}$ is given by

$$\underset{(2)}{\xi_a^{\#\alpha}} = \int_0^{\hat\lambda_a} d\lambda\, R_{a'}(\lambda)\left(\underset{(1)}{\xi_{a'}^\rho}\, \frac{\partial \underset{(1)}{\xi_a^\alpha}}{\partial u_{a'}^\rho}\right) \tag{150}$$

where

$$\hat\lambda_a \equiv -(z_{aa'} \cdot u_{a'}) - r_a \tag{151}$$

and $R_a(\lambda)$ is an operator acting on any function $f$ of the argument $(z_b^\alpha, u_c^\beta)$ in the following way:

$$R_a(\lambda) f(z_a^\alpha, z_{a'}^\beta; u_a^\gamma, u_{a'}^\delta) = f(z_a^\alpha + \lambda u_a^\alpha, z_{a'}^\beta; u_a^\gamma, u_{a'}^\delta) \tag{152}$$

and similarly for $R_{a'}(\lambda)$, which "shifts" $z_{a'}$. The computation of the integral (150) using (142) is rather long but straightforward. Noting that

(i) The computation of the quantity in brackets in (150) is notably simplified when introducing the linear differential operators:

$$N_{a'} = h_{a'a}^\rho \frac{\partial}{\partial u_{a'}^\rho}, \qquad Q_a \equiv t_a^\rho \frac{\partial}{\partial u_{a'}^\rho} \tag{153}$$

whose action on the different scalars and vectors is

$$N_{a'}\{h_{aa'}^2, \tau_a, \tau_{a'}, k\} = \{-2h_{aa'}^2 \tau_{a'}, \Lambda^{-2} k h_{aa'}^2, \Lambda^{-2} h_{aa'}^2, 0\} \tag{154a}$$

$$Q_a\{h_{aa'}^2, \tau_a, \tau_{a'}, k\} = \{0, \tau_{a'}, k\tau_{a'}, -\Lambda^2\} \tag{154b}$$

$$N_{a'} r_a = -r_a^{-1} h_{aa'}^2 (\tau_{a'} - k\tau_a) \tag{154c}$$

$$Q_a r_a = \Lambda^2 r_a^{-1} \tau_a (\tau_{a'} - k\tau_a) \tag{154d}$$

$$N_{a'} h_{aa'}^\alpha = \Lambda^{-2} h_{aa'}^2 t_{a'}^\alpha - \tau_{a'} h_{aa'}^\alpha, \qquad Q_a h_{aa'}^\alpha = 0 \tag{154e}$$

$$N_{a'} t_{a'}^\alpha = -h_{aa'}^\alpha, \qquad Q_a t_{a'}^\alpha = -k t_{a'}^\alpha \tag{154f}$$

(ii) The operator $R_{a'}(\lambda)$ acts on the variables depending on $z_{a'}^\alpha$ as follows:

$$\begin{aligned} R_{a'}(\lambda)\tau_a &= \tau_a \\ R_{a'}(\lambda)\tau_{a'} &= \tau_{a'} + \lambda \\ R_{a'}(\lambda) h_{aa'}^\alpha &= h_{aa'}^\alpha \end{aligned} \tag{155}$$

and hence

$$\begin{aligned} R_{a'}(\lambda) r_a &= r_a \\ R_{a'}(\lambda) r_{a'} &= [h_{aa'}^2 + \Lambda^2(\tau_{a'} + \lambda)^2]^{1/2} \equiv r_{a'}(\lambda) \end{aligned} \tag{156}$$

We finally obtain

$$\begin{aligned} \underset{(2)}{\xi_a^{\#\alpha}} \equiv{}& m_a m_{a'} r_a^{-5} \{-k(2k^2 - 1)[3(2k^2 - 1)h_{aa'}^2 + (2k^2 - 3)r_a^2]\tau_a I_{a'} \\ &+ \langle (2k^2 - 1)^2 (3h_{aa'}^2 - r_a^2) + k^2 \Lambda^2 (2k^2 - 3)[3(2k^2 - 1)\tau_a^2 - 4r_a^2] \rangle J_{a'} \\ &- 3k\Lambda^2 (2k^2 - 1)(2k^2 - 3)\tau_a K_{a'}\} h_{aa'}^\alpha \\ &+ m_a m_{a'} r_a^{-5} \{(2k^2 - 1)\langle -3k^2 (2k^2 - 3)\tau_a^2 + \Lambda^{-2}[(2k^2 - 1) \\ &+ k^2 (2k^2 - 3)] r_a^2 \rangle h_{aa'}^2 I_{a'} + k[3(2k^2 - 1)(2k^2 - 3)(h_{aa'}^2 - \Lambda^2 r_a^2) \\ &+ k^2 (2k^2 - 3)^2 (3\Lambda^2 \tau_a^2 - r_a^2)]\tau_a J_{a'} + k^2 (2k^2 - 3)^2 (-3\Lambda^2 \tau_a^2 + r_a^2) K_{a'}\} t_{a'}^\alpha \end{aligned} \tag{157}$$

where

$$I_{a'} \equiv h_{aa'}^{-2}\left(\frac{k\tau_a - r_a}{kr_a - \Lambda^2\tau_a} - \frac{\tau_{a'}}{r_{a'}}\right) \tag{158a}$$

$$J_{a'} \equiv -\Lambda^{-2}[(kr_a - \Lambda^2\tau_a)^{-1} - r_{a'}^{-1}] \tag{158b}$$

$$K_{a'} \equiv -\Lambda^{-2} h_{aa'}^2 I_{a'} + \Lambda^{-3} \ln \frac{(\Lambda\tau_a + r_a)(k - \Lambda)}{\Lambda\tau_{a'} + r_{a'}} \tag{158c}$$

As at first order, the result is invariant under the Poincaré group and the condition (136) is automatically fulfilled.

The expansion (138) where $\underset{(1)}{\xi_a^\alpha}$ is given by (142) and where $\underset{(2)}{\xi_a^\alpha}$ is the sum of $\underset{(2)}{\xi_a^{*\alpha}}$ given by (147) and of $\underset{(2)}{\xi_a^{\#\alpha}}$ given by (157) solves the problem of writing the equations of motion in the form of an ordinary system of second-order differential equations.

By known methods already used at first order by Portilla [22], the equations (135) can be transformed in Newtonian-like equations of motion which reduce to the usual Einstein-Infeld-Hoffman (EIH) equations of motion [2], when one neglects all terms of order higher than or equal to $Gm(v/c)^3$ or $G^2m^2(v/c)$.

Moreover it is possible to construct by known methods [23, 24] a Hamiltonian formulation (invariant under the Poincaré group) for the dynamics of this two-particle system and thereby to define unambiguously the energy, linear momentum, and angular momentum of the system. The computation of these quantities was done at first order in [22].

## *Appendix A: $c = 1$; signature $-+++$; $\alpha = 0, 1, 2, 3$; $ds^2 = -\eta_{\alpha\beta}dz^\alpha dz^\beta$*

*A.1. Einstein's Equations in Terms of the Deviation from the Minkowski Metric.* The Einstein tensor $S^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}Rg^{\alpha\beta}$ reads in harmonic coordinates [16]

$$2|g|S^{\alpha\beta} = \mathfrak{g}^{\mu\nu}\partial_{\mu\nu}^2\mathfrak{g}^{\alpha\beta} - |g|(2\Pi^{\alpha,\mu\nu}\Pi^\beta_{\mu\nu} - y^\alpha y^\beta + g^{\alpha\beta}L) \tag{A1}$$

where

$$\mathfrak{g}^{\alpha\beta} = (-g)^{1/2}g^{\alpha\beta} \tag{A2}$$

is such that

$$\partial_\beta\mathfrak{g}^{\alpha\beta} = 0 \tag{A3}$$

and where

$$-2|g|\Pi^{\alpha,\mu\nu} = \mathfrak{g}^{\mu\rho}\partial_\rho\mathfrak{g}^{\alpha\nu} + \mathfrak{g}^{\nu\rho}\partial_\rho\mathfrak{g}^{\alpha\mu} - \mathfrak{g}^{\alpha\rho}\partial_\rho\mathfrak{g}^{\mu\nu} \tag{A4}$$

$$\Pi^{\beta}_{\mu\nu} = g_{\mu\rho} g_{\nu\sigma} \Pi^{\beta,\rho\sigma} \quad \text{(A5)}$$

$$y^{\alpha} = g^{\alpha\beta} y_{\beta}, \qquad y_{\beta} = \partial_{\beta} [\ln (-g)^{1/2}] \quad \text{(A6)}$$

$$L = -\frac{1}{2(-g)^{1/2}} \Pi^{\rho}_{\mu\nu} \partial_{\rho} \mathfrak{g}^{\mu\nu} + \frac{1}{2} y_{\rho} y^{\rho} \quad \text{(A7)}$$

$g$ being the determinant of the matrix $g_{\alpha\beta}$ (or equivalently of $\mathfrak{g}^{\alpha\beta}$).

Introducing the deviation from the Minkowski metric:

$$h^{\alpha\beta} = \mathfrak{g}^{\alpha\beta} - \eta^{\alpha\beta} \quad \text{(A8)}$$

we obtain the following expansions in powers of $h^{\mu\nu}$:

$$-g = 1 + h + \tfrac{1}{2}(h^2 - h_{\mu\nu} h^{\mu\nu}) + \tfrac{1}{6} h^3 + \tfrac{1}{3} h_{\mu\nu} h^{\nu\rho} h_{\rho}{}^{\mu} - \tfrac{1}{2} h h_{\mu\nu} h^{\mu\nu} + \mathcal{O}(h^4) \quad \text{(A9)}$$

$$g^{\alpha\beta} = \eta^{\alpha\beta} + h^{\alpha\beta} - \tfrac{1}{2}\eta^{\alpha\beta} h + \eta^{\alpha\beta}(\tfrac{1}{8}h^2 + \tfrac{1}{4} h_{\mu\nu} h^{\mu\nu}) - \tfrac{1}{2} h h^{\alpha\beta} + \tfrac{1}{8} h^2 h^{\alpha\beta}$$
$$+ \tfrac{1}{4} h^{\alpha\beta} h_{\mu\nu} h^{\mu\nu} - \eta^{\alpha\beta}(\tfrac{1}{48} h^3 + \tfrac{1}{6} h_{\mu\nu} h^{\nu\rho} h^{\mu}_{\rho} + \tfrac{1}{8} h h_{\mu\nu} h^{\mu\nu}) + \mathcal{O}(h^4) \quad \text{(A10)}$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} - h_{\alpha\beta} + \tfrac{1}{2}\eta_{\alpha\beta} h + \eta_{\alpha\beta}(\tfrac{1}{8}h^2 - \tfrac{1}{4} h_{\mu\nu} h^{\mu\nu}) - \tfrac{1}{2} h h_{\alpha\beta} + h_{\alpha\mu} h^{\mu}_{\beta}$$
$$+ \eta_{\alpha\beta}(\tfrac{1}{48} h^3 + \tfrac{1}{6} h_{\mu\nu} h^{\nu\rho} h^{\mu}_{\rho} - \tfrac{1}{8} h h_{\mu\nu} h^{\mu\nu}) - \tfrac{1}{8} h^2 h_{\alpha\beta} + \tfrac{1}{2} h h_{\alpha\mu} h^{\mu}_{\beta}$$
$$+ \tfrac{1}{4} h_{\alpha\beta} h_{\mu\nu} h^{\mu\nu} - h_{\alpha\mu} h^{\mu\rho} h_{\rho\beta} + \mathcal{O}(h^4) \quad \text{(A11)}$$

In equations (A.9)-(A.11) and henceforth, the indices are moved with the Minkowski metric $\eta^{\alpha\beta}$; $h = \eta_{\alpha\beta} h^{\alpha\beta}$. A similar expansion of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$ yields

$$\Gamma^{\alpha}_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\rho}(\partial_{\beta} g_{\rho\gamma} + \partial_{\gamma} g_{\rho\beta} - \partial_{\rho} g_{\beta\gamma})$$
$$= \underset{(1)}{\Gamma^{\alpha}_{\beta\gamma}} + \underset{(2)}{\Gamma^{\alpha}_{\beta\gamma}} + \mathcal{O}(h^3) \quad \text{(A12)}$$

with

$$2\underset{(1)}{\Gamma^{\alpha}_{\beta\gamma}} = \tfrac{1}{2}\delta^{\alpha}_{\gamma}\partial_{\beta} h + \tfrac{1}{2}\delta^{\alpha}_{\beta}\partial_{\gamma} h - \tfrac{1}{2}\eta_{\beta\gamma}\partial^{\alpha} h - \partial_{\gamma} h^{\alpha}_{\beta} - \partial_{\beta} h^{\alpha}_{\gamma} + \partial^{\alpha} h_{\beta\gamma} \quad \text{(A13)}$$

$$2\underset{(2)}{\Gamma^{\alpha}_{\beta\gamma}} = -\tfrac{1}{2}\delta^{\alpha}_{\gamma} h_{\mu\nu}\partial_{\beta} h^{\mu\nu} - \tfrac{1}{2}\delta^{\alpha}_{\beta} h_{\mu\nu}\partial_{\gamma} h^{\mu\nu} + \tfrac{1}{2}\eta_{\beta\gamma} h_{\mu\nu}\partial^{\alpha} h^{\mu\nu}$$
$$+ h_{\gamma\mu}\partial_{\beta} h^{\alpha\mu} + h_{\beta\mu}\partial_{\gamma} h^{\alpha\mu} - h_{\beta\mu}\partial^{\alpha} h^{\mu}_{\gamma} - h_{\gamma\mu}\partial^{\alpha} h^{\mu}_{\beta}$$
$$- \tfrac{1}{2}\eta_{\beta\gamma} h^{\mu\alpha}\partial_{\mu} h + \tfrac{1}{2} h_{\beta\gamma}\partial^{\alpha} h + h^{\alpha\mu}\partial_{\mu} h_{\beta\gamma} \quad \text{(A14)}$$

The expansion of (A.1) then is

$$2|g| S^{\alpha\beta} = \Box h^{\alpha\beta} - N^{\alpha\beta} - M^{\alpha\beta} + \mathcal{O}(h^4) \quad \text{(A15)}$$

where $\Box = \eta^{\alpha\beta}\partial^2_{\alpha\beta}$ and

$$N^{\alpha\beta} = -h^{\mu\nu}\partial^2_{\mu\nu} h^{\alpha\beta} + \tfrac{1}{8}\eta^{\alpha\beta}\partial_{\mu} h \partial^{\mu} h - \tfrac{1}{4}\partial^{\alpha} h \partial^{\beta} h - \tfrac{1}{4}\eta^{\alpha\beta}\partial_{\mu} h_{\nu\rho}\partial^{\mu} h^{\nu\rho}$$

$$+ \tfrac{1}{2}\eta^{\alpha\beta}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho} + \tfrac{1}{2}\partial^\alpha h^{\mu\nu}\partial^\beta h_{\mu\nu} - \partial^\alpha h^{\mu\nu}\partial_\mu h^\beta_\nu - \partial^\beta h^{\mu\nu}\partial_\mu h^\alpha_\nu + \partial_\nu h^{\alpha\mu}\partial^\nu h^\beta_\mu + \partial_\nu h^{\alpha\mu}\partial^\mu h^\beta_\nu \qquad \text{(A16)}$$

$$M^{\alpha\beta} = \tfrac{1}{8}\eta^{\alpha\beta}\partial_\mu h\partial_\nu h h^{\mu\nu} + \tfrac{1}{8}h^{\alpha\beta}\partial_\mu h\partial^\mu h - \tfrac{1}{4}\partial^\alpha h\partial_\mu h h^{\mu\beta} - \tfrac{1}{4}\partial^\beta h\partial_\mu h h^{\mu\alpha}$$
$$- \tfrac{1}{4}\eta^{\alpha\beta}\partial^\mu h\partial_\mu h_{\nu\rho}h^{\nu\rho} + \tfrac{1}{4}\partial^\alpha h\partial^\beta h_{\mu\nu}h^{\mu\nu} + \tfrac{1}{4}\partial^\beta h\partial^\alpha h_{\mu\nu}h^{\mu\nu}$$
$$+ \tfrac{1}{2}h^{\alpha\beta}\partial_\mu h^{\nu\rho}\partial_\nu h^\mu_\rho - \tfrac{1}{4}h^{\alpha\beta}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho} - \partial^\mu h^{\alpha\nu}h^{\beta\rho}\partial_\rho h_{\mu\nu}$$
$$- \partial^\mu h^{\beta\nu}h^{\alpha\rho}\partial_\rho h_{\mu\nu} - \partial^\nu h^{\alpha\mu}\partial_\nu h^{\beta\rho}h_{\mu\rho} + \partial^\nu h^{\alpha\mu}\partial^\beta h^\rho_\nu h_{\mu\rho}$$
$$+ \partial^\nu h^{\beta\mu}\partial^\alpha h^\rho_\nu h_{\mu\rho} + \tfrac{1}{2}\partial^\alpha h^{\mu\nu}h^{\beta\rho}\partial_\rho h_{\mu\nu} + \tfrac{1}{2}\partial^\beta h^{\mu\nu}h^{\alpha\rho}\partial_\rho h_{\mu\nu}$$
$$- \partial^\alpha h^{\mu\nu}\partial^\beta h^\rho_\mu h_{\rho\nu} + h^{\mu\rho}\partial_\rho h^{\alpha\nu}\partial_\mu h^\beta_\nu - \tfrac{1}{4}\eta^{\alpha\beta}\partial_\rho h^{\mu\nu}h^{\rho\sigma}\partial_\sigma h_{\mu\nu}$$
$$- \tfrac{1}{2}\eta^{\alpha\beta}\partial_\rho h^{\mu\nu}\partial_\mu h^{\rho\sigma}h_{\nu\sigma} + \tfrac{1}{2}\eta^{\alpha\beta}\partial^\rho h^{\mu\nu}\partial_\rho h^\sigma_\mu h_{\sigma\nu} \qquad \text{(A17)}$$

From (A.16) and (A.17) we deduce

$$\partial_\beta N^{\alpha\beta} = -\tfrac{1}{4}\partial^\alpha h\Box h + \tfrac{1}{2}\partial^\alpha h^{\mu\nu}\Box h_{\mu\nu} - \partial^\mu h^{\alpha\nu}\Box h_{\mu\nu} \qquad \text{(A18)}$$

$$\partial_\beta M^{\alpha\beta} = \tfrac{1}{4}\partial^\alpha hN - \tfrac{1}{2}\partial^\alpha h^{\mu\nu}N_{\mu\nu} + \partial^\mu h^{\alpha\nu}N_{\mu\nu} - \tfrac{1}{4}\partial_\mu h h^{\alpha\mu}\Box h$$
$$+ \tfrac{1}{4}\partial^\alpha h h^{\mu\nu}\Box h_{\mu\nu} + \tfrac{1}{4}\partial^\alpha h^{\mu\nu}h_{\mu\nu}\Box h + \partial^\nu h^{\alpha\mu}h_{\mu\rho}\Box h^\rho_\nu$$
$$+ \tfrac{1}{2}h^{\alpha\rho}\partial_\rho h_{\mu\nu}\Box h^{\mu\nu} - \partial^\alpha h^{\mu\nu}h_{\nu\rho}\Box h^\rho_\mu \qquad \text{(A19)}$$

where $N = \eta_{\alpha\beta}N^{\alpha\beta}$.

The stress-energy tensor density

$$\mathfrak{T}^{\alpha\beta} = G|g|T^{\alpha\beta} = \sum Gm\int ds\delta_4(x-z)u^\alpha u^\beta|g|^{1/2}|g_{\mu\nu}u^\mu u^\nu|^{-1/2} \qquad \text{(A20)}$$

can be formally expanded in powers of $h^{\mu\nu}$:

$$\mathfrak{T}^{\alpha\beta} = \sum Gm\int ds\delta_4(x-z)u^\alpha u^\beta[1 + \underset{(1)}{t} + \underset{(2)}{t} + \mathcal{O}(h^3)] \qquad \text{(A21)}$$

where

$$\underset{(1)}{t} = \tfrac{1}{4}h - \tfrac{1}{2}h_{\mu\nu}u^\mu u^\nu \qquad \text{(A22)}$$

$$\underset{(2)}{t} = \tfrac{1}{32}h^2 - \tfrac{1}{8}h_{\mu\nu}h^{\mu\nu} - \tfrac{1}{8}hh_{\mu\nu}u^\mu u^\nu + \tfrac{3}{8}(h_{\mu\nu}u^\mu u^\nu)^2 + \tfrac{1}{2}u^\mu u^\nu h_{\mu\rho}h^\rho_\nu \qquad \text{(A23)}$$

The covariant divergence of the tensor-density $\mathfrak{T}^{\alpha\beta}$ of weight + 2 is

$$\nabla_\beta\mathfrak{T}^{\alpha\beta} = \partial_\beta\mathfrak{T}^{\alpha\beta} + \Gamma^\alpha_{\beta\gamma}\mathfrak{T}^{\beta\gamma} - \Gamma^\beta_{\beta\gamma}\mathfrak{T}^{\alpha\gamma} \qquad \text{(A24)}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols (A12). Equation (A24) reads, when formally expanded in powers of $h^{\mu\nu}$,

$$\nabla_\beta \mathfrak{T}^{\alpha\beta} = \sum Gm \int ds\,\delta_4(x-z)\Bigg\{\frac{d}{ds}\,[u^\alpha(1+\underset{(1)}{t}+\underset{(2)}{t})] + \underset{(2)}{\Gamma^\alpha_{\beta\gamma}}u^\beta u^\gamma$$

$$- \underset{(2)}{\Gamma^\beta_{\beta\gamma}}u^\alpha u^\gamma + (\underset{(1)}{\Gamma^\alpha_{\beta\gamma}}u^\beta u^\gamma - \underset{(1)}{\Gamma^\beta_{\beta\gamma}}u^\alpha u^\gamma)(1+\underset{(1)}{t}) + \mathcal{O}(h^3)\Bigg\} \quad \text{(A25)}$$

where $\underset{(1)}{t}$ and $\underset{(2)}{t}$ are given by (A22) and (A23); $\underset{(1)}{\Gamma^\alpha_{\beta\gamma}}$ and $\underset{(2)}{\Gamma^\alpha_{\beta\gamma}}$ are given by (A13) and (A14).

Finally the formal expansion of the geodesic equations (which could be deduced from $\nabla_\beta \mathfrak{T}^{\alpha\beta} = 0$, were the metric well behaved) read

$$\dot{u}^\alpha = -u^\mu u^\nu[\underset{(1)}{\Gamma^\alpha_{\mu\nu}} + u^\alpha u_\rho \underset{(1)}{\Gamma^\rho_{\mu\nu}} + \underset{(2)}{\Gamma^\alpha_{\mu\nu}} + u^\alpha u_\rho \underset{(2)}{\Gamma^\rho_{\mu\nu}}] + \mathcal{O}(h^3) \quad \text{(A26)}$$

where

$$-u^\mu u^\nu[\underset{(1)}{\Gamma^\alpha_{\mu\nu}} + u^\alpha u_\rho \underset{(1)}{\Gamma^\rho_{\mu\nu}}] = u^\mu u^\nu \partial_\nu h^\alpha_\mu - \tfrac{1}{2}u^\mu u^\nu \partial^\alpha h_{\mu\nu} - \tfrac{1}{4}\partial^\alpha h$$

$$- u^\alpha(\tfrac{1}{4}u^\mu \partial_\mu h - \tfrac{1}{2}u^\mu u^\nu u^\rho \partial_\rho h_{\mu\nu}) \quad \text{(A27)}$$

$$-u^\mu u^\nu[\underset{(2)}{\Gamma^\alpha_{\mu\nu}} + u^\alpha u_\rho \underset{(2)}{\Gamma^\rho_{\mu\nu}}] = -\tfrac{1}{4}\partial_\mu h h^{\alpha\mu} - \tfrac{1}{4}\partial^\alpha h u^\mu u^\nu h_{\mu\nu} + \tfrac{1}{4}h_{\mu\nu}\partial^\alpha h^{\mu\nu}$$

$$+ u^\nu u^\mu \partial^\alpha h_{\mu\rho}h^\rho_\nu - \tfrac{1}{2}u^\mu u^\nu \partial_\rho h_{\mu\nu}h^{\alpha\rho}$$

$$- u^\mu u^\nu \partial_\mu h^\alpha_\rho h^\rho_\nu - u^\alpha[\tfrac{1}{4}u_\nu \partial_\mu h h^{\mu\nu}$$

$$+ \tfrac{1}{2}u^\mu u^\nu u_\rho \partial_\sigma h_{\mu\nu}h^{\rho\sigma} + \tfrac{1}{4}u^\nu u^\rho u^\mu \partial_\mu h h_{\nu\rho}$$

$$- \tfrac{1}{4}u^\rho \partial_\rho h^{\mu\nu}h_{\mu\nu}] \quad \text{(A28)}$$

*A.2. Notations and Derivatives of Retarded Quantities.* By definition

$$r_R = -(x-z_R)\cdot u_R, \qquad r'_R = -(x-z'_R)\cdot u'_R \quad \text{(A29)}$$

$$n^\alpha_R = -u^\alpha_R + (x^\alpha - z^\alpha_R)/r_R, \qquad n'^\alpha_R = -u'^\alpha_R + (x^\alpha - z'^\alpha_R)/r'_R \quad \text{(A30)}$$

$$\rho_R = -(z_R - \hat{z}'_R)\cdot \hat{u}'_R, \qquad \rho'_R = -(z'_R - \hat{z}_R)\cdot \hat{u}_R \quad \text{(A31)}$$

$$\nu^\alpha_R = -\hat{u}'^\alpha_R + (z^\alpha_R - \hat{z}'^\alpha_R)/\rho_R, \qquad \nu'^\alpha_R = -\hat{u}^\alpha_R + (z'^\alpha_R - \hat{z}^\alpha_R)/\rho'_R \quad \text{(A32)}$$

$$\rho_{(0)R} = -(z_R - \hat{z}'_{(0)R})\cdot u'_R, \qquad \rho'_{(0)R} = -(z'_R - \hat{z}_{(0)R})\cdot u_R \quad \text{(A33)}$$

$$\nu^\alpha_{(0)R} = -u'^\alpha_R + (z^\alpha_R - \hat{z}'^\alpha_{(0)R})/\rho_{(0)R}, \qquad \nu'^\alpha_{(0)R} = -u^\alpha_R + (z'^\alpha_R - \hat{z}^\alpha_{(0)R})/\rho'_{(0)R} \quad \text{(A34)}$$

$$w_R = u_R\cdot u'_R, \qquad \omega_R = u_R\cdot \hat{u}'_R, \qquad \omega'_R = u'_R\cdot \hat{u}_R \quad \text{(A35)}$$

$$0 \equiv (n\cdot u)_R = (n'\cdot u')_R = (\nu\cdot \hat{u}')_R = (\nu'\cdot \hat{u})_R = (\nu_{(0)}\cdot u')_R = (\nu'_{(0)}\cdot u)_R \quad \text{(A35)}'$$

$$1 \equiv n^2_R = n'^2_R = \nu^2_R = \nu'^2_R = \nu^2_{(0)R} = \nu'^2_{(0)R} \quad \text{(A35)}''$$

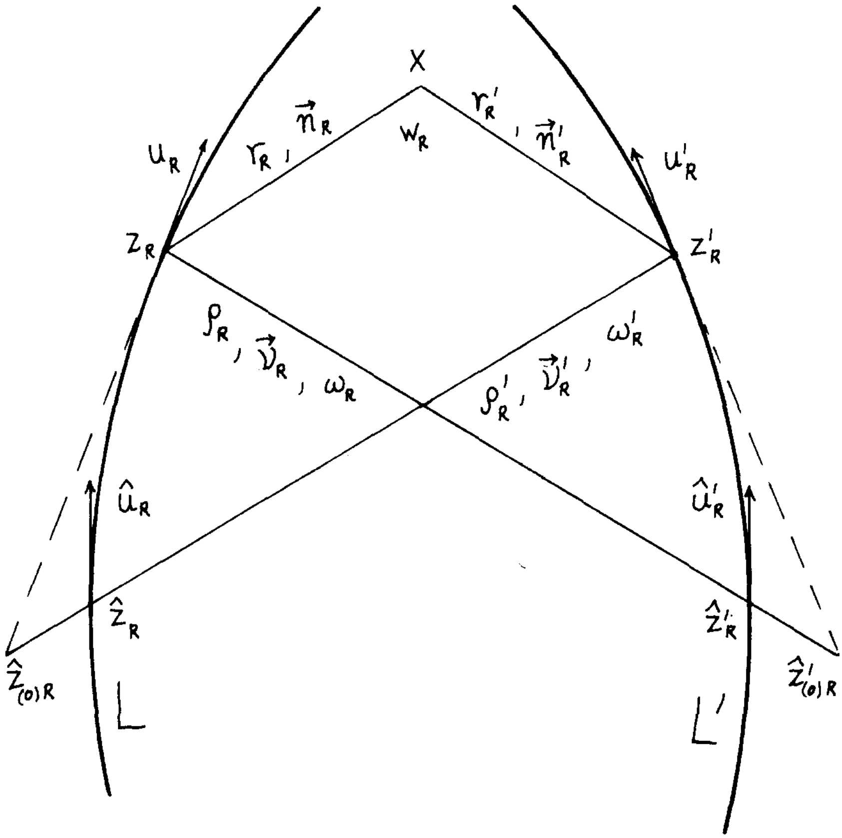


**Fig. 1.** $(x - z_R)^2 = (x - z'_R)^2 = (z_R - \hat{z}'_R)^2 = (z'_R - \hat{z}_R)^2 = (z_R - \hat{z}'_{(0)R})^2 = (z'_R - \hat{z}_{(0)R})^2 = 0.$

Since the curvature of the world lines is of order 1 in $G$, we have

$$\begin{aligned} \rho_{(0)R} &= \rho_R + \mathcal{O}(G), \quad \rho'_{(0)R} = \rho'_R + \mathcal{O}(G), \quad \nu'^{\alpha}_{(0)R} = \nu'^{\alpha}_R + \mathcal{O}(G) \\ \nu^{\alpha}_{(0)R} &= \nu^{\alpha}_R + \mathcal{O}(G), \quad \omega_R = w_R + \mathcal{O}(G), \quad \omega'_R = w_R + \mathcal{O}(G) \end{aligned} \tag{A36}$$

The derivatives of the retarded quantities are deduced from differentiating the identity

$$(x - z_R)^2 \equiv \{x - z[s_R(x)]\}^2 \equiv 0 \tag{A37}$$

and read

$$-\partial^{\alpha} s_R(x) = (x^{\alpha} - z^{\alpha}_R)/r_R \equiv n^{\alpha}_R + u^{\alpha}_R \tag{A38}$$

$$\partial^\alpha(x^\beta - z^\beta_R) = \eta^{\alpha\beta} + u^\beta_R(x^\alpha - z^\alpha_R)/r_R = -D^\alpha(x^\beta - z^\beta_R), \qquad D'^\alpha(x^\beta - z^\beta_R) = 0 \tag{A39}$$

$$\partial^\alpha u^\beta_R = -\dot{u}^\beta_R(n^\alpha_R + u^\alpha_R) = -D^\alpha u^\beta_R, \qquad D'^\alpha u^\beta_R = 0 \tag{A40}$$

$$\partial^\alpha r_R = n^\alpha_R + r_R(n^\alpha_R + u^\alpha_R)(n_R \cdot \dot{u}_R) = -D^\alpha r_R, \qquad D'^\alpha r_R = 0 \tag{A41}$$

and similar expressions for $\partial_\alpha(x^\beta - z'^\beta_R)$; $\partial_\alpha u'^\beta_R$, $\partial_\alpha r'_R$. $\dot{u}_R = du_R/ds$ and $D_\alpha$ is the partial functional derivative associated with parallel displacements of the line $L$; [cf. equation (29) in text]. Equations (A40) and (A41) can be rewritten as

$$\underset{(0)}{\partial^\alpha} u^\beta_R = -\underset{(0)}{D^\alpha} u^\beta_R = 0, \qquad \underset{(1)}{\partial^\alpha} u^\beta_R = -\underset{(1)}{D^\alpha} u^\beta_R = -\dot{u}^\beta_R(n^\alpha_R + u^\alpha_R) \tag{A42}$$

$$\underset{(0)}{\partial^\alpha} r_R = n^\alpha_R = -\underset{(0)}{D^\alpha} r_R, \qquad \underset{(1)}{\partial^\alpha} r_R = -\underset{(1)}{D^\alpha} r_R = r_R(n^\alpha_R + u^\alpha_R)(n_R \cdot \dot{u}_R) \tag{A43}$$

where $\underset{(0)}{\partial}$, $\underset{(0)}{D}$ means that the derivative is taken "as if" the world lines were straight lines and where $\underset{(1)}{\partial}$, $\underset{(1)}{D}$ are the first-order corrections to $\partial - \underset{(0)}{\partial}$, $D - \underset{(0)}{D}$ due to the curvature of the world lines; $\underset{(1)}{\partial} f$, $\underset{(1)}{D} f$ are proportional to $\dot{u}^\beta_R$ and are first order in $G$. Similarly we have

$$\underset{(0)}{\partial^\alpha} n^\beta_R = [(-n^\alpha n^\beta + \eta^{\alpha\beta} + u^\alpha u^\beta)/r]_R = -\underset{(0)}{D^\alpha} n^\beta_R, \qquad \underset{(0)}{D'^\alpha} n^\beta_R = 0 \tag{A44}$$

$$-\underset{(0)}{D'^\alpha} \rho_R = \nu^\alpha_R, \qquad -\underset{(0)}{D'^\alpha} \nu^\beta_R = [(-\nu^\alpha \nu^\beta + \eta^{\alpha\beta} + \hat{u}'^\alpha \hat{u}'^\beta)/\rho]_R \tag{A45}$$

$$-\underset{(0)}{D'^\alpha} \rho_{(0)R} = \nu^\alpha_{(0)R}, \qquad -\underset{(0)}{D'^\alpha} \nu^\beta_{(0)R} = [(-\nu^\alpha_{(0)} \nu^\beta_{(0)} + \eta^{\alpha\beta} + u'^\alpha u'^\beta)/\rho_{(0)}]_R \tag{A46}$$

Noting that $\rho_{(0)R}$ and $\nu^\alpha_{(0)R}$ can be expressed as a function of retarded quantities:

$$(\rho\nu^\alpha)_{(0)R} = \{r'n'^\alpha + r[n^\alpha + (n \cdot u')u'^\alpha + u^\alpha + wu'^\alpha]\}_R \tag{A47}$$

$$(\nu \cdot \nu)_{(0)R} = 1 \tag{A48}$$

$\underset{(0)}{D^\alpha} \rho_{(0)R}$ and $\underset{(0)}{D^\alpha} \nu^\beta_{(0)R}$ are easily calculated using (A43) and (A44).

## *Appendix B*

*B.1. Notations.* By definition

$$x^\alpha - z^\alpha = \epsilon n^\alpha, \qquad (n \cdot u) = 0, \qquad (n \cdot n) = 1, \qquad \epsilon = [(x - z)^2]^{1/2} \tag{B1}$$

$$\rho = -(z - \hat{z}') \cdot \hat{u}', \qquad \nu^\alpha = -\hat{u}'^\alpha + (z^\alpha - \hat{z}'^\alpha)/\rho, \qquad \omega = u \cdot \hat{u}' \tag{B2}$$

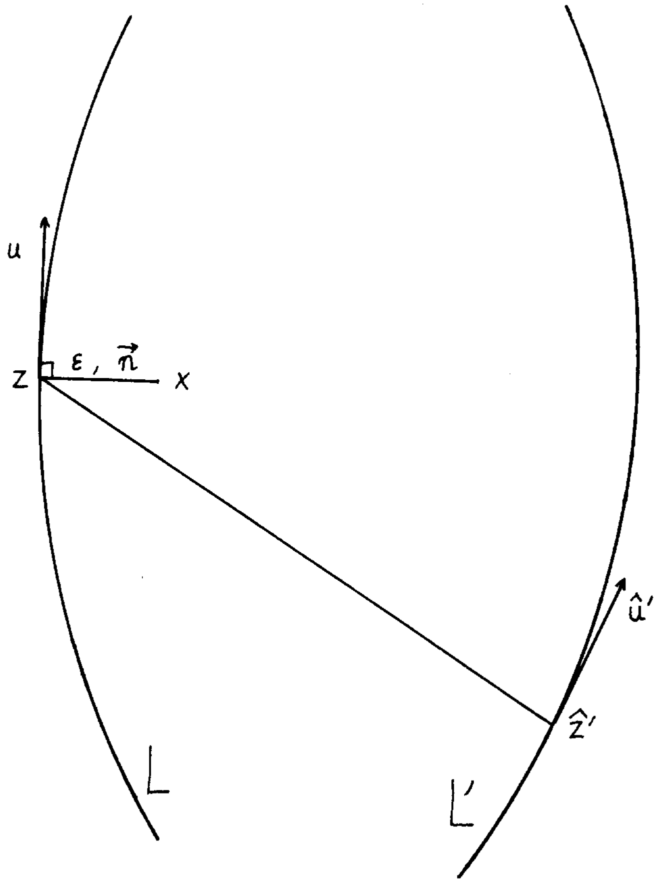


**Fig. 2.** $(x - z) \cdot u = 0$; $(z - \hat{z}')^2 = 0$.

It will be also convenient to introduce

$$v^\alpha = \hat{u}'^\alpha + \omega u^\alpha \tag{B3}$$

$$A^\alpha = \nu^\alpha + (\nu \cdot u) u^\alpha + v^\alpha = u^\alpha [(z - \hat{z}') \cdot u]/\rho + (z^\alpha - \hat{z}'^\alpha)/\rho \tag{B4}$$

$$(A \cdot u) = 0, \qquad A = (A \cdot A)^{1/2} = -[(z - \hat{z}') \cdot u]/\rho = -[\omega + (\nu \cdot u)] \tag{B5}$$

*B.2. Laurent Series Expansions of Retarded Quantities.* We have

$$n_R^\alpha = n^\alpha + \frac{\epsilon}{2}[\dot{u}^\alpha - (n \cdot \dot{u})(2u^\alpha + n^\alpha)]$$
$$+ \frac{\epsilon^2}{6}[(n \cdot \ddot{u})(3u^\alpha + 2n^\alpha) - 2\ddot{u}^\alpha] + \mathcal{O}(\epsilon^3) + \mathcal{O}(G^2) \tag{B6}$$

$$r_R = \epsilon + \frac{\epsilon^2}{2}(n \cdot \dot{u}) - \frac{\epsilon^3}{3}(n \cdot \ddot{u}) + \mathcal{O}(\epsilon^4) + \mathcal{O}(G^2) \tag{B7}$$

$$u_R^\alpha = u^\alpha - \epsilon \dot{u}^\alpha + \frac{\epsilon^2}{2}\ddot{u}^\alpha + \mathcal{O}(\epsilon^3) + \mathcal{O}(G^2) \tag{B8}$$

$$\dot{u}_R^\alpha = \dot{u}^\alpha - \epsilon \ddot{u}^\alpha + \mathcal{O}(\epsilon^2) + \mathcal{O}(G^2) \tag{B9}$$

$$n_R'^\alpha = \nu^\alpha + \frac{\epsilon}{\rho}[n^\alpha + (n \cdot \hat{u}')\hat{u}'^\alpha - (n \cdot \nu)\nu^\alpha] + \mathcal{O}(\epsilon^2) + \mathcal{O}(G) \tag{B10}$$

$$r_R' = \rho + \epsilon(n \cdot \nu) + \frac{\epsilon^2}{2\rho}[1 + (n \cdot \hat{u}')^2 - (n \cdot \nu)^2] + \mathcal{O}(\epsilon^3) + \mathcal{O}(G) \tag{B11}$$

$$u_R'^\alpha = \hat{u}'^\alpha + \mathcal{O}(\epsilon) + \mathcal{O}(G) \tag{B12}$$

$$\dot{u}_R'^\alpha = \hat{\dot{u}}'^\alpha + \mathcal{O}(\epsilon) + \mathcal{O}(G^2) \tag{B13}$$

$$\rho_{(0)R} = \rho - \epsilon(\nu \cdot u) - \frac{\epsilon^2}{2\rho}[1 + (\nu \cdot u)^2 - \omega^2] + \mathcal{O}(\epsilon^3) + \mathcal{O}(G) \tag{B14}$$

$$(\rho\nu^\alpha)_{(0)R} = \rho\nu^\alpha - \epsilon(u^\alpha + \omega\hat{u}'^\alpha) + \mathcal{O}(\epsilon^2) + \mathcal{O}(G) \tag{B15}$$

*B.3. Regularization of Retarded Quantities.* The regularization procedure used throughout the paper consists in taking the mean value of the term of the Laurent series expansion which does not depend on $\epsilon$. For a simple method of calculating them see [13].

$$\overline{\frac{S_R}{r_R}} = -\dot{S}(z) \tag{B16}$$

$$\overline{\partial_\alpha\left(\frac{S_R}{r_R}\right)} = \frac{S}{3}(\ddot{u}_\alpha - \dot{u}^2 u_\alpha) + \dot{S}\dot{u}_\alpha + \ddot{S}u_\alpha$$

$$= \frac{S\ddot{u}_\alpha}{3} + \dot{S}\dot{u}_\alpha + \ddot{S}u_\alpha + \mathcal{O}(G^2) \tag{B17}$$

where

$$S_R = S(z_R). \tag{B18}$$

$$\overline{(n^\alpha/r^n)_R} = \mathcal{O}(G) \tag{B19}$$

$$\overline{(n'^\alpha/rr'^2)_R} = \mathcal{O}(G) \tag{B20}$$

$$\overline{(n^\alpha/r'r^2)_R} = \mathcal{O}(G) \tag{B21}$$

$$\overline{(n^\alpha/r^2\rho_{(0)})_R} = \mathcal{O}(G) \tag{B22}$$

From (B6) and (B23) the contribution of $\underset{(1)}{h}^{\alpha\beta}$, $h_S^{\alpha\beta}$, and $h_T^{\alpha\beta}$ to the geodesic equation can be calculated [equations (118)-(122) in text].

## Appendix C

*C.1. Calculation of $P^\alpha$.* The equation satisfied by $P^\alpha$ is

$$\underset{(0)}{\Box} P^\alpha = -\frac{1}{r'_R}\partial^\alpha \frac{1}{r_R} = +\frac{1}{r'_R} D^\alpha \frac{1}{r_R} \tag{C1}$$

where the index (0) indicates that we are looking for a zero-order solution of (C1):

$$r_R = -(x - z_R)\cdot u_R, \qquad r'_R = -(x - z'_R)\cdot u'_R \tag{C2}$$

$z_R$ being the retarded point on $L$ associated with $x$, $u_R$ the tangent to $L$ at $z_R$.

$D(x)$ being the flat retarded propagator:

$$\Box D(x) = -4\pi\delta_4(x) \tag{C3}$$

$$D(x) = \delta(x^0 - |\mathbf{x}|)/|\mathbf{x}|, \qquad |\mathbf{x}| = (x_i x^i)^{1/2} \tag{C4}$$

we have

$$\Box(1/r_R) = -4\pi \int ds\, \delta_4(x - z) \tag{C5}$$

$$1/r_R = \int ds\, D(x - z), \qquad 1/r'_R = \int ds'\, D(x - z') \tag{C6}$$

so that the solution of (C1) is

$$P^\alpha = -\frac{1}{4\pi}\int d^4y\, D(x - y)\int ds'\, D(y - z')D^\alpha \int ds\, D(y - z) \tag{C7}$$

The integration on $d^4y$ is performed using cylindrical coordinates:

$$x - y = t, \qquad t = (t^0, t^1 = R\cos\varphi, t^2 = R\sin\varphi, t^3) \tag{C8}$$

in a frame such that

$$x - z' = \alpha'^0 e^0, \qquad x - z = \alpha^0 e^0 + \alpha^3 e^3 \tag{C9}$$

One thus obtains (cf. [5])

$$P^\alpha(x) = -\frac{1}{2}\int_{-\infty}^{z'_R} ds'\, D^\alpha \int_{-\infty}^{z_R} ds\, \theta[(z - z')^2]\{[(x - z)\cdot(x - z')]^2 - (x - z)^2(x - z')^2\}^{-1/2} \tag{C10}$$

where $\theta$ is the Heaviside step function. In lowest order we can integrate on straight lines tangent to $L$ and $L'$ at $z_R$ and $z'_R$. The integration of (C10) on $ds$ then is elementary and yields

$$P^{\alpha}(x, z_R, z'_R) = + \int_{\hat{z}'_{(0)R}}^{z'_R} ds' \, \frac{\beta^{\alpha}}{r_R \beta^2} \tag{C11}$$

where $\hat{z}'_{(0)R}$ is the retarded point on the tangent to $L'$ at $z'_R$ associated with $z_R$, and where

$$\beta^{\alpha}(s') = R'^{\alpha} - r_R n_R^{\alpha} - n_R^{\alpha}[(R' - r_R n_R)^2]^{1/2} \tag{C12}$$

$$R'^{\alpha}(s') = [x^{\alpha} - z'^{\alpha}(s')] + \{[x - z'(s')] \cdot u_R\} u_R^{\alpha} \tag{C13}$$

$$n_R^{\alpha} = -u_R^{\alpha} + (x^{\alpha} - z_R^{\alpha})/r_R \tag{C14}$$

Finally the integration on $ds'$, using three-dimensional vector notations after a projection $\Pi$ orthogonal to $u_R^{\alpha}$ such that

$$\mathbf{v} = \Pi(u'_R) \tag{C15}$$

$$\mathbf{l} = \Pi[z'(s') - z_R], \qquad l = |\mathbf{l}| \tag{C16}$$

$$w_R = (u_R \cdot u'_R) \tag{C17}$$

yields

$$2(\mathbf{v} \times \mathbf{n}_R)^2 r_R \mathbf{P} = \Bigg[ \frac{\mathbf{v} \times (\mathbf{v} \times \mathbf{n}_R)}{(w_R^2 - 1)^{1/2}} \ln \left[ l + \frac{\mathbf{l} \cdot \mathbf{v}}{(w_R^2 - 1)^{1/2}} \right] - \mathbf{n}_R \times (\mathbf{v} \times \mathbf{n}_R) \ln (l + \mathbf{l} \cdot \mathbf{n}_R) - 2(\mathbf{v} \times \mathbf{n}_R) \tan^{-1} c \Bigg]_{\hat{z}'_{(0)R}}^{z'_R} \tag{C18}$$

with

$$\mathbf{l} \cdot (\mathbf{v} \times \mathbf{n}_R) c = l[(w_R^2 - 1)^{1/2} + \mathbf{v} \cdot \mathbf{n}_R] + \mathbf{l} \cdot [\mathbf{v} + (w_R^2 - 1)^{1/2} \mathbf{n}_R] \tag{C19}$$

where $\times$ denotes the vector product.

From (C18) we deduce

$$2 r_R (P \cdot u'_R) = [\ln (l + \mathbf{l} \cdot \mathbf{n}_R)]_{\hat{z}'_{(0)R}}^{z'_R} \tag{C20}$$

*C.2 Laurent Series Expansions of $P^{\alpha}$ and $P'^{\alpha}$.* The Laurent series expansion of $P^{\alpha}$ around the point $z$ defined in Appendix B is

$$P_{\alpha} = \frac{1}{\epsilon} \left[ S^1_{\alpha\mu} \xi^{\mu} + \frac{1}{2} S^2_{\alpha(\mu\nu)} \xi^{\mu} \xi^{\nu} + \frac{1}{6} S^3_{\alpha(\mu\nu\rho)} \xi^{\mu} \xi^{\nu} \xi^{\rho} \right] + R^0_{\alpha} + R^1_{\alpha\mu} \xi^{\mu} + \frac{1}{2} R^2_{\alpha(\mu\nu)} \xi^{\mu} \xi^{\nu} + \mathcal{O}(\epsilon^3) \tag{C21}$$

where

$$\xi^{\alpha} = \epsilon n^{\alpha}, \qquad (\xi \cdot u) = 0 \tag{C22}$$

(cf. Appendix B for notation) and where

$$-2S^1_{\alpha\mu} = \Pi_{\alpha\mu}/\rho \tag{C23}$$

$$-2S^2_{\alpha(\mu\nu)} = [2\Pi_{\mu\nu}\tilde{\nu}_\alpha - \Pi_{\alpha\mu}\tilde{\nu}_\nu - \Pi_{\alpha\nu}\tilde{\nu}_\mu]/2\rho^2 \tag{C24}$$

$$-2S^3_{\alpha(\mu\nu\rho)} = \frac{1}{\rho^3}\left[\frac{v_\alpha}{3} v_{(\mu}\Pi_{\nu\rho)} - \tilde{\nu}_\alpha\tilde{\nu}_{(\mu}\Pi_{\nu\rho)} - \frac{1}{3}\Pi_{\alpha(\mu}v_\nu v_{\rho)} + \Pi_{\alpha(\mu}\tilde{\nu}_\nu\tilde{\nu}_{\rho)} + \frac{[3(\tilde{\nu})^2 - v^2 - 3]}{3}\Pi_{\alpha(\mu}\Pi_{\nu\rho)}\right] \tag{C25}$$

$$-2R^0_\alpha = -\frac{A_\alpha}{\rho A} \tag{C26}$$

$$-2R^1_{\alpha\mu} = \frac{1}{2\rho^2}\left\{-\frac{A_\alpha v_\mu}{A} - \frac{A_\mu v_\alpha}{A} + \left[A + \frac{(A\cdot v)}{A}\right]\frac{A_\alpha A_\mu}{A^2} - \left[A - \frac{(A\cdot v)}{A}\right]\Pi_{\alpha\mu}\right\} \tag{C27}$$

$$-2R^2_{\alpha(\mu\nu)} = \frac{1}{3\rho^3}\left\{-\frac{A_\alpha}{A}\left[a\Pi_{\mu\nu} + 2v_\mu v_\nu - \frac{bA_{(\mu}v_{\nu)}}{A} + \frac{cA_\mu A_\nu}{A^2}\right] + f\Pi_{\alpha(\mu}v_{\nu)} + v_\alpha\left[d\Pi_{\mu\nu} - \frac{2A_{(\mu}v_{\nu)}}{A} + \frac{bA_\mu A_\nu}{A^2}\right] - \frac{e\Pi_{\alpha(\mu}A_{\nu)}}{A}\right\} \tag{C28}$$

where

$$\begin{aligned} a &= -3 + v^2 - (A\cdot v)^2/A^2 - 3(A\cdot v)/A^2 - 3(A\cdot v) + 3A^2 \\ b &= 2(A\cdot v)/A + 3A, \qquad c = 3(A\cdot v)^2/A^2 - v^2 + 3(A\cdot v) + 3A^2 \\ d &= -3/A - 2(A\cdot v)/A + 3A, \qquad f = (A\cdot v)/A - 3A/2 \\ e &= \tfrac{1}{2}[-v^2 + (A\cdot v)^2/A^2 + 3(A\cdot v) - 3A^2] \end{aligned} \tag{C29}$$

with $v_{(\mu}\Pi_{\nu\rho)} \equiv v_\mu\Pi_{\nu\rho} + v_\nu\Pi_{\rho\mu} + v_\rho\Pi_{\mu\nu}$, etc., and $A_{(\mu}v_{\nu)} = A_\mu v_\nu + A_\nu v_\mu$, etc., and

$$\Pi_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu, \qquad \omega = (u\cdot\hat{u}'), \qquad \rho = -(z - \hat{z}')\cdot\hat{u}' \tag{C30}$$

$$\tilde{\nu}^\alpha = -\hat{u}'^\alpha + (z^\alpha - \hat{z}'^\alpha)/\rho + u^\alpha\{-\omega + [(z - \hat{z}')\cdot u]/\rho\} = \nu^\alpha + (\nu\cdot u)u^\alpha \tag{C31}$$

$$v^\alpha = \hat{u}'^\alpha + \omega u^\alpha \tag{C32}$$

$$A^\alpha = \tilde{\nu}^\alpha + v^\alpha, \qquad A = (A\cdot A)^{1/2} \tag{C33}$$

$\hat{u}'$ being the tangent to $L'$ at $\hat{z}'$, retarded point associated with $z$.

The derivatives of $P^\alpha$ are deduced from (C21), using

$$\underset{(0)}{D_\alpha}\rho = \tilde{\nu}_\alpha \equiv \nu_\alpha + (\nu\cdot u)u^\alpha \tag{C34}$$

$$\underset{(0)}{D'_\alpha}\rho = -\nu_\alpha \tag{C35}$$

$$\underset{(0)}{D_\alpha}\nu_\beta = [\Pi_{\alpha\beta} + \hat{u}'_\beta v_\alpha - \nu_\alpha \tilde{\nu}_\beta]/\rho \tag{C36}$$

$$\underset{(0)}{D'_\alpha}\nu_\beta = [-\eta_{\alpha\beta} - \hat{u}'_\alpha \hat{u}'_\beta + \nu_\alpha \nu_\beta]/\rho \tag{C37}$$

The Laurent series expansion of $(u^\alpha P'_\alpha)$ is

$$(u \cdot P') = S'/\epsilon + R'^0 + \xi^\mu R'^1_\mu + \mathcal{O}(\epsilon^2) \tag{C38}$$

$$S' = [\xi^2 (B - A)]/4\rho^2 \tag{C39}$$

$$R'^0 = (\ln A)/\rho \tag{C40}$$

$$R'^1_\mu = (B_\mu - A_\mu)\frac{\ln A}{2\rho^2} + \frac{A_\mu}{4\rho^2}\left(1 + \frac{2B}{A} - \frac{2}{A^2}\right) + \frac{B_\mu}{4\rho^2} \tag{C41}$$

where

$$B_\mu = v_\mu - \nu_\mu, \qquad B = (B \cdot B)^{1/2}. \tag{C42}$$

From (C21) and (C38) the contributions of $h^{\alpha\beta}_{\times}$ to the second-order equations of motion can be calculated [equation (123) in text].

## *Appendix D*

The notations of Section 7 for a generic configuration $(z_a, u_a, z_{a'}, u_{a'})$ are

$$h^\alpha_{aa'} = z^\alpha_a - z^\alpha_{a'} - \tau_a u^\alpha_a + \tau_{a'} u^\alpha_{a'}, \qquad (h_{aa'} \cdot u_a) = (h_{aa'} \cdot u_{a'}) = 0 \tag{D1}$$

$$t^\alpha_{a'} = u^\alpha_{a'} - k u^\alpha_a, \qquad (t_{a'} \cdot u_a) = 0 \tag{D2}$$

$$k = -(u_a \cdot u_{a'}) \tag{D3}$$

$$\Lambda^2 = k^2 - 1 \tag{D4}$$

$$r_a = (h^2_{aa'} + \Lambda^2 {\tau_a}^2)^{1/2} \tag{D5}$$

$$\tau_a = \Lambda^{-2}(z_a - z_{a'}) \cdot (u_a - k u_{a'}) \tag{D6}$$

As a consequence of (140) we obtain the basic formula for the replacement of $z^\alpha - \hat{z}^{\alpha'}$ when passing from a light cone connected to a generic configuration:

$$z^\alpha_a - \hat{z}^\alpha_{a'} \longrightarrow h^\alpha_{aa'} + \tau_a u^\alpha_a + (r_a - k\tau_a) u^\alpha_{a'} \tag{D7}$$

from which one deduces easily $(a = 1, a' = 2)$

$$\rho \longrightarrow r_a \tag{D8}$$

$$\omega \longrightarrow -k \tag{D9}$$

$$v^{\alpha} \longrightarrow t^{\alpha}_{a'} \tag{D10}$$

$$A \longrightarrow r_a^{-1}(kr_a - \Lambda^2\tau_a) \equiv r_a^{-1} p_a \tag{D11}$$

$$A^{\alpha} \longrightarrow r_a^{-1} [h^{\alpha}_{aa'} + (r_a - k\tau_a) t^{\alpha}_{a'}] \tag{D12}$$

From (D8)-(D12), $\underset{(1)}{\xi^{\alpha}_a}$ [equation (142)] and $\underset{(2)}{\xi^{*\alpha}_a}$ [equation (147)] can be deduced from (129).

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