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Type-N, Shear-Free, Perfect-Fluid Spacetimes with a Barotropic Equation of State

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We present the class of Petrov-type N, shear-free, perfect-fluid solutions of Einstein's field equations in which the fluid satisfies a barotropic equation of state $p = p(w)$ and $w + p \not\equiv 0$. All solutions are stationary and possess a three-parameter, abelian group of local isometries which act simply transitively on timelike hypersurfaces. Furthermore, there exists one Killing vector parallel to the vorticity vector and another parallel to the four-velocity. This class of solutions is identified as part of a larger class present in the literature.

1. INTRODUCTION

Recently we proved [1] the following: For any Petrov-type N, shear-free, perfect-fluid solution of Einsteins' field equations in which the fluid satisfies a barotropic equation of state $p = p(w)$ with $w + p \not\equiv 0$ (w is the energy density and p is the fluid pressure), the volume expansion is zero but the vorticity is necessarily nonzero.

This theorem is part of a growing body of results [1–4] which suggest that any shear-free perfect fluid in general relativity with $p = p(w)$ and $w + p \not\equiv 0$ has either vanishing vorticity or vanishing expansion. In this paper we prove, by directly integrating the field equations, the following:

Theorem. Any Petrov-type N, shear-free, perfect-fluid spacetime, in which the fluid satisfies a barotropic equation of state with $w + p \not\equiv 0$, is stationary and possesses a three-parameter, abelian group of local

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isometries, acting simply transitively on timelike hypersurfaces. Furthermore, there exists one Killing vector parallel to the vorticity vector and another parallel to the four-velocity.

Thus the class of solutions presented is a subclass of those found by Krasinski [5], although expressed in a different coordinate system, who considered flow-stationary, cylindrically symmetric, perfect fluids [$p = p(w)$] with rigid rotation, under the restriction that there exists a Killing vector parallel to the vorticity vector.

This paper presupposes a knowledge of the Newman–Penrose [6] formalism (abbreviated NP). All considerations will be local. The units and conventions are as in [1].

2. CONSTRUCTION OF COORDINATES AND INTEGRATION OF FIELD EQUATIONS

In [1] we showed that, subject to the assumptions indicated above, there exists a tetrad in which the NP Weyl and trace-free Ricci tensor components, together with the spin coefficients, satisfy

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, \qquad \Psi_4 \neq 0 \tag{1}$$

$$\Phi_{01} = \Phi_{12} = \Phi_{02} = 0, \quad \Phi_{00} = \Phi_{22} = 2\Phi_{11} = \tfrac{1}{4}(w+p) \tag{2}$$

$$\sigma = \lambda = \rho = \mu = \varepsilon = \gamma = \tau + \pi = 0 \tag{3a}$$

$$2\kappa = (\alpha+\beta)(3 + 9\dot{p} - S) \tag{3b}$$

$$2\tau = (\alpha+\beta)(S - 1 - 3\dot{p}) \tag{3c}$$

$$2\nu = (\alpha+\beta)(1 - 9\dot{p} + S) \tag{3d}$$

$$\Phi_{11} = \tfrac{1}{8}(w+p) = \kappa(\alpha - \beta) \tag{3e}$$

$$\Lambda = \tfrac{1}{24}(w - 3p) = \tau(\beta - \alpha) \tag{3f}$$

$$(\alpha+\beta)\, G(w) + 8(\alpha - \beta) = 0 \tag{3g}$$

$$\Psi_4 = 4(\alpha+\beta)(\alpha-\beta) \tag{3h}$$

$$\delta w = 3(w+p)(\alpha+\beta) \tag{3i}$$

where

$$9(w+p)(w+3p)\,\ddot{p} = (1+3\dot{p})(9\dot{p}w - 9\dot{p}p - w - 3p) \tag{3j}$$

$$G \equiv 4[3\dot{p} - 1 + 9\ddot{p}(w+p)/(1+3\dot{p})] \tag{3k}$$

$$S \equiv 2 - 3\dot{G}(w+p)/G \tag{3l}$$

and where κ, ν, τ, α, β are real and D and Δ derivatives of all quantities are zero. (We have chosen $K=0$.) In addition, $p=p(w)$ must satisfy the conditions[2]

$$\ddot{p}(w) \neq 0 \tag{4}$$

and

$$6w\dot{p}(w) \neq w + 3p(w) \tag{5}$$

In view of (3a), Cartan's first structure equations become

$$d\theta^1 = [(\alpha+\beta+\tau)\,\theta^1 - \nu\theta^2] \wedge [\theta^3+\theta^4] \tag{6a}$$

$$d\theta^2 = [\kappa\theta^1 - (\alpha+\beta-\tau)\,\theta^2] \wedge [\theta^3+\theta^4] \tag{6b}$$

$$d\theta^3 = (\alpha-\beta)\,\theta^3 \wedge \theta^4 \tag{6c}$$

where

$$\theta^1 = n_i\,dx^i, \qquad \theta^2 = l_i\,dx^i, \qquad \theta^3 = -\bar{m}_i\,dx^i = \bar{\theta}^4 \tag{6d}$$

It follows from the Frobenius theorem [7] that there exist coordinates u, v, z, and $\bar{z}$, where u and v are real and z complex, and five functions A_1, A_2, B_1, B_2, and E of these coordinates such that

$$\theta^1 = A_1\,du + A_2\,dv \tag{7a}$$

$$\theta^2 = B_1\,du + B_2\,dv \tag{7b}$$

$$\theta^3 = E\,dz \tag{7c}$$

A_1, A_2, B_1, and B_2 are real-valued functions, while E is complex-valued. With the tetrad θ^i given in (7), we have

$$\frac{\partial A_1}{\partial v} = \frac{\partial A_2}{\partial u} \tag{8a}$$

$$\tau+\alpha+\beta = \frac{1}{EJ}\left(B_1\frac{\partial A_2}{\partial z} - B_2\frac{\partial A_1}{\partial z}\right) \tag{8b}$$

$$\nu = \frac{1}{EJ}\left(A_1\frac{\partial A_2}{\partial z} - A_2\frac{\partial A_1}{\partial z}\right) \tag{8c}$$

$$\frac{\partial B_1}{\partial v} = \frac{\partial B_2}{\partial u} \tag{8d}$$

[2] The second condition was omitted in [1].

$$\kappa = \frac{1}{EJ}\left(B_1 \frac{\partial B_2}{\partial z} - B_2 \frac{\partial B_1}{\partial z}\right) \tag{8e}$$

$$\alpha + \beta - \tau = \frac{1}{EJ}\left(A_1 \frac{\partial B_2}{\partial z} - A_2 \frac{\partial B_1}{\partial z}\right) \tag{8f}$$

$$\frac{\partial E}{\partial u} = \frac{\partial E}{\partial v} = 0 \tag{8g}$$

$$\beta - \alpha = \frac{1}{E\bar{E}} \frac{\partial E}{\partial \bar{z}} \tag{8h}$$

where we require

$$J \equiv A_1 B_2 - A_2 B_1 \not\equiv 0 \tag{8i}$$

in order that the metric be nondegenerate. Equations (3b), (3d), (8c), and (8e) imply

$$(A_1 - B_1) \frac{\partial}{\partial z} (A_2 - B_2) - (A_2 - B_2) \frac{\partial}{\partial z} (A_1 - B_1) = 0 \tag{9}$$

We must now distinguish two separate cases:

I. $A_1 \not\equiv B_1$ and $A_2 \not\equiv B_2$

II. $A_1 \equiv B_1$ and $A_2 \not\equiv B_2$

The case when $A_2 \equiv B_2$ and $A_1 \not\equiv B_1$ is essentially equivalent to II.

Case I. $A_1 \not\equiv B_1$ and $A_2 \not\equiv B_2$.

Integrating (9) yields

$$A_2 - B_2 = Q(u, v)[A_1 - B_1] \tag{10}$$

where $Q \not\equiv 0$ is a real arbitrary function of u and v. Next, using the transformation

$$u = f(\tilde{u}, \tilde{v}), \qquad v = g(\tilde{u}, \tilde{v}), \qquad z = \tilde{z} \tag{11}$$

where f and g are real arbitrary functions, which preserves the form of the θ^i given in (7), it is possible to achieve $Q = 1$ so that

$$A_1 - B_1 = A_2 - B_2 \tag{12}$$

Combining (12) with (8a) and (8d) yields

$$A_1 = B_1 + \xi(u+v, z, \bar{z}), \qquad A_2 = B_2 + \xi(u+v, z, \bar{z}) \tag{13}$$

where ξ is some real function of the indicated variables.

In this coordinate system, we have

$$D = \frac{1}{J}\left(B_2 \frac{\partial}{\partial u} - B_1 \frac{\partial}{\partial v}\right), \qquad \Delta = \frac{1}{J}\left(A_1 \frac{\partial}{\partial v} - A_2 \frac{\partial}{\partial u}\right) \tag{14}$$

Since w and all spin coefficients are real and have vanishing D and Δ derivatives, it follows by a straightforward calculation that A_1, A_2, B_1, and B_2 must be of the form

$$A_1 = X(x) \frac{\partial \mathscr{P}(u, v)}{\partial u} + \xi(u+v)\, T(x) \tag{15a}$$

$$A_2 = X(x) \frac{\partial \mathscr{P}(u, v)}{\partial v} + \xi(u+v)\, T(x) \tag{15b}$$

$$B_1 = X(x) \frac{\partial \mathscr{P}(u, v)}{\partial u} + \xi(u+v)[T(x) - 1] \tag{15c}$$

$$B_2 = X(x) \frac{\partial \mathscr{P}(u, v)}{\partial v} + \xi(u+v)[T(x) - 1] \tag{15d}$$

$$E = E(x) \tag{15e}$$

where $z = x + iy$ and where use has been made of (8). X, $\mathscr{P}$, ξ, T, and E are all real arbitrary functions of the indicated variables.

It can be shown readily that the (nonsingular) transformation

$$\begin{aligned} u + v &= R(\tilde{u} + \tilde{v}) \\ v &= Q(\tilde{u}, \tilde{v}) \\ z &= \tilde{z} \end{aligned} \tag{16}$$

preserves the form of θ^i, given in (7), and A_1, A_2, B_1, B_2 as given in (15). This remaining coordinate freedom can be used to set

$$\xi = \partial \mathscr{P}/\partial u = -\partial \mathscr{P}/\partial v = 1 \tag{17}$$

Consequently

$$A_1 = T + X \tag{18a}$$

$$A_2 = T - X \tag{18b}$$

$$B_1 = T + X - 1 \tag{18c}$$

$$B_2 = T - X - 1 \tag{18d}$$

We now show that there are no solutions to Einstein's field equations with this metric structure and subject to the indicated assumptions. In this coordinate system, the nonzero spin coefficients are given by

$$\kappa = (1/2E^{-1}X^{-1})[X'T - XT' - X'] \tag{19a}$$

$$\nu = (1/2E^{-1}X^{-1})[X'T - XT'] \tag{19b}$$

$$\tau = -X'E^{-1}X^{-1}/4 \tag{19c}$$

$$\alpha + \beta = (1/4E^{-1}X^{-1})[2X'T - 2XT' - X'] \tag{19d}$$

$$\beta - \alpha = E'E^{-2}/2 \tag{19e}$$

where ≪$'$≫ denotes differentiation with respect to x. Equations (19a), (19c), and (19d) imply

$$\kappa - \tau = \alpha + \beta \tag{20}$$

whereas (3b) and (3c) yield

$$\kappa + \tau = (\alpha + \beta)(1 + 3\dot{p}) \tag{21}$$

From (20) and (21), it follows that

$$\kappa/\tau = (2 + 3\dot{p})/(3\dot{p}) \tag{22}$$

since $\tau \not\equiv 0$ is required. However, from (3e) and (3f), we have

$$\frac{\kappa}{\tau} = -\frac{\Phi_{11}}{\Lambda} = -\frac{3(w+p)}{w-3p} \tag{23}$$

Comparing (22) with (23) immediately yields

$$\dot{p} = p/(2w) - \tfrac{1}{6} \tag{24}$$

A simple check shows that the differential equation (24) is incompatible with (3j), and thus we conclude that no solutions are possible for this case.

Case II. $A_1 \equiv B_1$ and $A_2 \not\equiv B_2$.

For this case, (9) plus the fact that w and all spin coefficients are real and have vanishing D and Δ derivatives, implies the following functional forms

$$A_1 = B_1 = X(x)[\partial\mathscr{P}(u, v)/\partial u]$$
$$A_2 = \hat{R}(x)\,Q(v) + X(x)[\partial\mathscr{P}(u, v)/\partial v]$$
$$B_2 = \hat{T}(x)\,Q(v) + X(x)[\partial\mathscr{P}(u, v)/\partial v]$$
$$E = E(x)$$

where use has been made of (8). X, $\mathscr{P}$, $\hat{R}$, $\hat{T}$, Q, and E are all real arbitrary functions of the indicated variables. Next, using the coordinate freedom expressed by the form-preserving transformation

$$u = u(\tilde{u}, \tilde{v}) \qquad v = v(\tilde{v}) \qquad z = \tilde{z} \tag{25}$$

we may achieve

$$A_1 = B_1 = X(x) \tag{26a}$$
$$A_2 = \hat{R}(x) + X(x) \equiv R(x) \tag{26b}$$
$$B_2 = \hat{T}(x) + X(x) \equiv T(x) \tag{26c}$$
$$E = E(x) \tag{26d}$$

In this coordinate system the metric has the form

$$ds^2 = 2\theta^1\theta^2 - 2\theta^3\theta^4 = 2(X\,du + R\,dv)(X\,du + T\,dv) - 2E^2\,dz\,d\bar{z} \tag{27}$$

and the Einstein field equations (together with the assumptions of the theorem) yield

$$X = \exp\left\{-\int \frac{dp}{w+p}\right\} \tag{28a}$$
$$t = \exp\left\{\frac{1}{3}\int \frac{(6\dot{p}w + w - 3p)}{(w+3p)(w+p)}\,dw\right\} \tag{28b}$$
$$E = \exp\left\{\frac{1}{3}\int \frac{(6\dot{p}w - w - 3p)}{(w+3p)(w+p)}\,dw\right\} \tag{28c}$$
$$r = \frac{2X}{3}\int \frac{t}{X(w+p)}\,dw \tag{28d}$$
$$3^{1/2}k_0 x = \int \frac{[(1+3\dot{p})(w+3p-6\dot{p}w)]^{1/2}}{E(w+3p)(w+p)}\,dw \tag{28e}$$

together with

$$9(w+p)(w+3p)\,\ddot{p}=(1+3\dot{p})(9\dot{p}w-9p\dot{p}-w-3p) \tag{3j}$$

where

$$2r\equiv T+R,\qquad 2t\equiv T-R,\qquad k_0=\pm 1 \tag{28f}$$

and

$$(1+3\dot{p})(w+3p-6\dot{p}w)>0 \tag{28g}$$

Using (3), (8), (26), and (28), the nonzero NP spin coefficients and Weyl tensor component may be expressed as

$$\begin{aligned} \kappa&=\frac{3^{1/2}}{4k_0}(w+p)\,L_2 & \nu&=\frac{k_0L_1}{4\cdot 3^{1/2}}(w+9p-9\dot{p}w-9p\dot{p}) \\ \alpha+\beta&=\frac{k_0}{2\cdot 3^{1/2}}(w+3p)\,L_1 & \alpha-\beta&=\frac{k_0}{2\cdot 3^{1/2}L_2} \\ \tau=-\pi&=\frac{k_0}{4\cdot 3^{1/2}}(3p-w)\,L_2 & \Psi_4&=\frac{w+3p}{3(1+3\dot{p})} \end{aligned} \tag{29}$$

where

$$L_1\equiv[(1+3\dot{p})(w+3p-6\dot{p}w)]^{-1/2},\qquad L_2\equiv\left(\frac{1+3\dot{p}}{w+3p-6\dot{p}w}\right)^{1/2}$$

The velocity, acceleration, and vorticity vectors may be written as, respectively,

$$u=\frac{1}{2^{1/2}X}\frac{\partial}{\partial u} \tag{30}$$

$$\dot{u}=-\frac{3^{1/2}k_0}{2E}\,\dot{p}(w+3p)\,L_1\frac{\partial}{\partial x} \tag{31}$$

and

$$\omega=-\frac{k_0}{2\cdot 3^{1/2}E}(w+3p)\,L_1\frac{\partial}{\partial y} \tag{32}$$

Thus once a solution for p, $p=p(w)$, is determined from (3j), the remaining field equations are essentially reduced to quadrature. It is possible to reduce the order of the differential equation (3j) by considering the following change of variables

$$q \equiv 1 + 3pe^{-h}, \qquad e^h \equiv w \tag{33}$$

If we let

$$\eta = \eta(q) \equiv (dq/dh) + q \tag{34}$$

then (3j) becomes

$$q(2+q)(q-\eta)(d\eta/dq) = 4\eta + \eta^2(q-4) \tag{35}$$

which is an Abel equation of the second kind. Thus far we have found only the singular solutions $\eta \equiv 0$ and $\eta = 1 + (q/2)$, to (35), which are both unacceptable since, interestingly enough, they correspond to $\ddot{p} = 0$ and $6w\dot{p} = w + 3p$, respectively.

3. PROPERTIES OF THE SOLUTIONS

The metric (27) admits only the Killing vectors $\partial/\partial u$, $\partial/\partial v$, and $\partial/\partial y$, and hence the spacetime possesses a maximal three-parameter, abelian (Bianchi type I) group of local isometries, acting simply transitively on timelike hypersurfaces T_3 (temporally homogeneous [8]). Since $\partial/\partial u$ is timelike and not hypersurface-orthogonal, the spacetime is stationary. The solutions (27), (28) constitute the type-N subclass of the class of solutions obtained by Krasinski [5], although expressed in a different coordinate system, who determined all flow-stationary (there exists a timelike Killing vector, which is not hypersurface-orthogonal, collinear with the fluid four-velocity), cylindrically symmetric solutions to Einstein's field equations for a rigidly rotating, isentropic perfect fluid in which there exists a Killing vector collinear with the vorticity vector. In fact, his solutions appear to be the only ones known in which the spacetime is stationary and cylindrically symmetric and the perfect fluid is rigidly rotating with nonconstant pressure [4, 8].

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REFERENCES

1. Carminati, J. (1987). *J. Math. Phys.*, **28**, 1848.
2. Collins, C. B. (1986). *Can. J. Phys.*, **64**, 191.
3. Lang, J. M., and Collins, C. B. (1987). "Observationally Homogeneous, Shear-Free Perfect Fluids," preprint (University of Waterloo, Ontario, Canada).
4. Collins, C. B. (1987). "Homogeneous and Hypersurface-Homogeneous, Shear-Free Perfect Fluids in General Relativity," preprint (University of Waterloo, Ontario, Canada).
5. Krasinski, A. (1978). *Rep. Math. Phys.*, **14**, 225.
6. Newman, E. T., and Penrose, R. (1962). *J. Math. Phys.*, **3**, 566.
7. Flanders, H. (1963). *Differential Forms with Applications to the Physical Sciences* (Academic Press, New York).
8. Kramer, D., Stephani, H., MacCallum, M., and Herlt, E. (1980). *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, Cambridge, England).