General Relativity and Gravitation, Vol. 22, No. 6, 1990

A Critical Density Cosmological Model with Varying Gravitational and Cosmological "Constants"

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Received June 23, 1989

A cosmological model in which the universe has its critical density and the gravitational and cosmological "constants" G and Λ are time-dependent is presented. The model may possibly solve the horizon and monopole problems. It predicts a perpetually expanding universe in which G increases and Λ decreases with time in a manner consistent with conservation of the energy-momentum tensor. The model also allows the calculation of various cosmological parameters.

1. INTRODUCTION

In an attempt to solve long-standing cosmological problems Özer and Taha [1] have recently proposed a new singularity-free cosmological model. In their work the universe has the critical density of the Einstein-de Sitter model and a time-dependent cosmological constant Λ, interpreted as a vacuum energy density, is present. Conservation of the matter energy-momentum tensor is replaced by conservation of the sum of this tensor and a "vacuum energy-momentum tensor." The latter arises from the Λ-term in the field equations. The model predicts an initially cold universe in which the entropy, horizon and monopole problems are resolved. It also gives sensible estimates of various cosmological parameters.

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0001-7701/90/0600-0655$06.00/0

The interpretation of the cosmological constant as a vacuum energy density is common but other interpretations also exist [2]. Of relevance is a recent work by Wilkins [3]. Investigating the distance dependence of gravity under very general conditions, Wilkins finds that the gravity field at a distance r from a point mass has two components: one going as r^{-2}, the other as r (Hookian field). The latter component is identifiable with the weak field limit of the Λ-term in Einstein's equations with a cosmological constant. Like G the constant Λ is a gravity coupling and both should therefore be treated on an equal footing. In particular there can be no reason for keeping either of them constant when the other is allowed to vary. This point of view was also stressed by Beesham [4]. We adopt it here and assume that both constants do in fact vary with time, but in a manner that conserves the energy-momentum tensor [4].

Another important assumption that we make is the critical density assumption of Özer and Taha [1]. With the present Hubble constant $H_p = 5 \times 10^{-11}\text{yr}^{-1}$, the present critical energy density $\rho_{cp} = 3H_p^2/8\pi G_p \approx 2 \times 10^{-47}(\text{GeV})^4$. The current energy density ρ of the universe, on the other hand, is between $10^{-46}(\text{GeV})^4$ and $10^{-48}(\text{GeV})^4$. Since ρ and ρ_c are time-dependent but apparently independent cosmic parameters the closeness in their present values is difficult to understand. For unless the universe actually does have the critical density, it is expected to diverge very rapidly from it. As there is no justification for conferring a special status upon the present epoch, one is inescapably led to surmise that $\rho = \rho_c$ for all times. We will assume that this is so.

With the foregoing assumptions we show that there are models in which G increases in a continuously expanding initially hot Robertson-Walker universe that has evolved from a finite minimum volume. The cosmological constant, on the other hand, decreases with time. The initial singularity in this scenario may be avoided and the horizon and monopole (but not the entropy) problems may possibly be solved. Other predictions include estimates of the present values of the scale factor, the cosmological constant, the deacceleration parameter and estimates of the matter density and age of the universe.

This paper is organised as follows. In Section 2 we present the model. Its implications for the early and matter and radiation universes are explored in Sections 3 and 4 respectively. We wind up the article in Section 5 with some concluding remarks.

2. THE MODEL

In a Robertson-Walker universe

$$(d\tau)^2 = (dt)^2 - R^2(t)\left[\frac{(dr)^2}{1-kr^2} + r^2\{(d\theta)^2 + \sin^2\theta(d\phi)^2\}\right], \qquad (1)$$

Einstein field equations with time-dependent cosmological and gravitational "constants" [5][1]

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R^{\lambda}_{\lambda} = 8\pi G(t)T_{\mu\nu} + \Lambda(t)g_{\mu\nu} \qquad (2)$$

and the perfect-fluid energy-momentum tensor

$$T_{\mu\nu} = -pg_{\mu\nu} + (p+\rho)U_\mu U_\nu \qquad (3)$$

yield the two independent equations

$$3\ddot{R} = -4\pi GR\left(3p + \rho - \frac{\Lambda}{4\pi G}\right), \qquad (4)$$

$$3\dot{R}^2 = \ 8\pi GR^2\left(\rho + \frac{\Lambda}{8\pi G}\right) - 3k. \qquad (5)$$

Elimination of $\ddot{R}$ between (4) and the differentiated form of eq. (5) gives

$$3(p+\rho)\dot{R} = -\left(\frac{\dot{G}}{G}\rho + \dot{\rho} + \frac{\dot{\Lambda}}{8\pi G}\right)R. \qquad (6)$$

On the other hand, the vanishing of the covariant divergence of the Einstein tensor in eq. (2) and the usual energy-momentum conservation relation $T^{\mu\nu}_{;\nu} = 0$, lead to [4]

$$\rho\dot{G} = -\frac{\dot{\Lambda}}{8\pi} \qquad (7)$$

so that eq. (6) becomes

$$3(p+\rho) = -R\frac{d\rho}{dR}. \qquad (8)$$

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[1] Our inertial coordinate system metric $\eta_{\mu\nu}$ has diagonal elements $1, -1, -1, -1$.

As mentioned and justified in the introduction we follow [1] in setting

$$\rho = \rho_c \equiv \frac{3\dot{R}^2}{8\pi G R^2} \tag{9}$$

for all cosmic time t. Consequently eq. (5) gives

$$\Lambda = \frac{3k}{R^2} \tag{10}$$

and eq. (7) reduces to

$$\rho \frac{dG}{dR} = \frac{3k}{4\pi R^3}. \tag{11}$$

Thus depending on whether Λ is positive, negative or vanishing one has $k = 1$, -1 or zero, corresponding respectively to an increasing, decreasing or a constant G in an expanding universe. For $\Lambda \leq 0$ eq. (4) implies that $\ddot{R} < 0$ as long as $(3p + \rho)$ is positive. Hence for $k = -1$ or zero, R must pass through zero at some finite time in the past ($t = 0$, say).

Equations (8) and (11), together with some equation of state $p = p(\rho)$, determine ρ and G in terms of R. Equation (9) will then produce the explicit dependence of R on t. These aspects are treated in the next two sections.

3. THE EARLY UNIVERSE

Substitution of the equation of state for the radiation-dominated early universe

$$p = \frac{1}{3}\rho \tag{12}$$

in eq. (8) leads to

$$\rho R^4 = \text{constant}. \tag{13}$$

As noted before if $\Lambda \leq 0$ the scale factor R must pass through zero at some finite time in the past. The density, according to eq. (13), was then infinite. But if $\Lambda > 0$ the existence of this singularity is not compelling. We will in fact make the physical assumption that the initial density was maximum but finite so that the universe has expanded from a finite minimum volume with a minimum scale factor at some initial instant $t = 0$. The necessary conditions for the existence of this minimum in an expanding universe are $\dot{R} = 0$ at $t = 0$ and $\dot{R} > 0$ for $t > 0$. With this assumption $\Lambda > 0$, $k = 1$ and eq. (10) is

$$\Lambda = 3/R^2. \tag{14}$$

Equation (11) can now be immediately integrated and yields (the subscript o indicates values of the parameters at $t = 0$)

$$G = \alpha(R^2 - R_o{}^2) \tag{15}$$

where $\alpha = 3/8\pi\rho_o R_o{}^4$. Combining (15) with eqs. (9) and (13) we find

$$R^2 = R_o{}^2 + t^2, \tag{16}$$

and hence

$$G = \alpha t^2. \tag{17}$$

Thus at $t = 0$, $G = 0$, $\dot{R}^2/G = 1/\alpha R_o{}^2$ and $\ddot{R} = 1/R_0$. One possible interpretation is that the universe came into being just prior to the onset of gravity at $t = 0$ as a result of a vacuum fluctuation propelled by the repulsive effect of the positive cosmological constant [see eq. (4)]. Universes in which the initial vacuum was in a state of tension under the action of a stretching or antigravity force have recently been discussed [6] in association with inflationary universe models.

Equation (16) coincides with the result obtained by Özer and Taha [1] although essential differences between our model and theirs occur. As shown by them the time-dependence (16) solves the horizon and monopole problems. In particular global causality is established at $t = 2.3R_o$. However we cannot mimic [1] in relating R_o to the Planck length, because in our case the Planck length is not a fundamental constant but rather evolves with time.

It follows from the preceding equations that

$$\rho = \frac{3t^2}{8\pi G(R_o{}^2 + t^2)^2}. \tag{18}$$

Hence assuming that the radiation temperature T is related to ρ by[2]

$$\rho = \frac{\pi^2}{15} T^4, \tag{19}$$

we have

$$T = \left(\frac{45}{8\pi^3 G}\right)^{1/4} \left\{\frac{t^2}{(R_o{}^2 + t^2)^2}\right\}^{1/4}, \tag{20}$$

[2] We use units such that $\hbar = c = k_B = 1$.

with T maximum at $t = 0$ (initially hot universe):

$$T_{\max} = T_o = \left(\frac{45}{8\pi^3 \alpha R_o{}^4}\right)^{1/4} \tag{21}$$

The expressions (18) and (20) are of the same form as eqs. (3.3) and (3.5) of [1] except that G here is time-dependent. As in the standard model, $T \propto 1/R$ throughout the pure radiation era. On the other hand [1] predicts $T = 0$ at $t = 0$ (cold beginning scenario), $T = T_{\max}$ at $t = R_o$ and $RT \propto t^{1/2}$.

For $t \gg R_o$ eqs. (18) and (20) become

$$\rho = \frac{3}{8\pi G t^2} \tag{22}$$

and

$$T = \left(\frac{45}{8\pi^3 G}\right)^{1/4} t^{-1/2}, \tag{23}$$

to be compared to the standard model results

$$\rho_{\text{SM}} = \frac{3}{8\pi G (2t)^2}, \tag{24}$$

and

$$T_{\text{SM}} = \left(\frac{45}{8\pi^3 G}\right)^{1/4} (2t)^{-1/2} \tag{25}$$

Since $G = \alpha t^2$ in our case but is constant in the standard model the thermal histories of the two models differ.

4. THE MATTER AND RADIATION ERA

For the matter and radiation era that follows in the wake of the pure radiation epoch, $\rho = \rho_m + \rho_\gamma$ where ρ_m and ρ_γ are the matter and electromagnetic radiation densities respectively. Assuming that matter does not contribute to pressure and that its conversion to radiation may be neglected, one has the equation of state

$$p = \frac{1}{3}(\rho - E_{mp} R^{-3}) \tag{26}$$

where $E_{mp} = \rho_{mp} R_p{}^3$ is the present total rest-mass energy of the universe (the subscript p indicates present-day parameters). Then eq. (8) integrates to

$$\rho = E_{mp} R^{-3} + R_p E_{\gamma p} R^{-4} \tag{27}$$

where $E_{\gamma p} = \rho_{\gamma p} {R_p}^3$ is the present radiation energy.

Next substitute eq. (27) into eq. (11) (with $k = 1$) to get

$$G = G_p + \frac{3R_p}{4\pi E_{mp}} \left(x - \xi \ln \left\{ \frac{\xi + x}{\xi + 1} \right\} - 1 \right), \tag{28}$$

where $x = R/R_p$ and $\xi = E_{\gamma p}/E_{mp} = \rho_{\gamma p}/\rho_{mp}$. From this equation and eqs. (27) and (9) we deduce that $R \to \sqrt{2}$ as $t \to \infty$. Clearly the model predicts a closed perpetually expanding universe.

An expression for the age of the universe t_p may be obtained upon inserting eqs. (27) and (28) in eq. (9) and integrating the result. The expression is

$$t_p = \sqrt{3/2}\, {R_p}^{3/2} \int_0^1 dx\, x[\xi\lambda + (\lambda + 3R_p\xi)x + 3R_p x^2 - 3R_p\xi(x + \xi)\ln(x + \xi)]^{-1/2}, \tag{29}$$

where

$$\lambda = 4\pi E_{mp} G_p + 3R_p \xi \ln(\xi + 1) - 3R_p. \tag{30}$$

Letting $\rho = \rho_c$ and using eq. (27) and its time derivative one deduces the relations

$$E_{mp} = \frac{3}{2\pi G_p(1 + \xi)} \left(\frac{H_p}{2{\delta_p}^3} \right)^{1/2}, \tag{31}$$

and

$$\frac{1}{2}(1 + \xi)^{-1} = 1 - q_p - \frac{\delta_p}{2H_p}, \tag{32}$$

where H_p and q_p are the present values of Hubble's parameter $H \equiv \dot{R}/R$ and the deacceleration parameter $q = -R\ddot{R}/\dot{R}^2$ respectively, and

$$\delta_p \equiv \frac{\dot{G}_p}{G_p} = \frac{2}{H_p {R_p}^2}. \tag{33}$$

For the values $H_p = 5 \times 10^{-11}\mathrm{yr}^{-1}$ and $\delta_p = 10^{-11}\mathrm{yr}^{-1}$ [7] eq. (33) gives

$$R_p = 3 \times 10^{42}\,(\mathrm{GeV})^{-1}, \tag{34}$$

leading to

$$\Lambda_p = 3 \times 10^{-85}\,(\mathrm{GeV})^2, \tag{35}$$

which is well within the upper limit of $10^{-82}\,(\mathrm{GeV})^2$ placed on $|\Lambda_p|$ from verifications of Hubble's law for distant galaxies [8]. Assuming that the density of the present microwave background radiation is less than one-hundredth the density of galactic matter [5] we deduce from eq. (32) that

$$0.4 \leq q_p < 0.405 \tag{36}$$

where $q_p = 0.4$ when $\xi = 0$ (matter universe) and $q_p = 0.405$ when $\xi = 0.01$. Insertion of the preceding values of the relevant parameters together with $\xi = 0.01$ and $G_p = 3 \times 10^{-102}\mathrm{yr}^2$ in eq. (31) yields the estimate

$$\rho_{mp} = 2 \times 10^{-47}\,(\mathrm{GeV})^4. \tag{37}$$

The quoted and derived values of the various parameters permit a numerical calculation of the integral (29) for the age of the universe. With $\xi = 0.01$ we obtain

$$t_p = 13.8 \times 10^9\mathrm{yr}. \tag{38}$$

Variations in ξ induce little changes in t_p; e.g. for $\xi = 0$, $t_p = 13.9 \times 10^9$yr. The value given for t_p in eq. (38) is consistent with recent estimates of the age of the oldest starts [9]. Dirac's variable G theory [10] gives $t_p = 6.7 \times 10^9$yr only.

Finally, in the limit $R \to \infty$ eq. (28) reduces to

$$GM = R \tag{39}$$

where $M = \frac{4}{3}\pi\rho_{mp}{R_p}^3$. This equation was recently invoked [11] in support of an argument for an increasing G.

5. CONCLUDING REMARKS

We have investigated a cosmological model in which the cosmological and gravitational constants vary with time and the density is equal to the critical density. When the universe is required to have expanded from a finite minimum volume, the critical density assumption and conservation of the energy-momentum tensor dictate that G increases in a perpetually expanding universe. In most variable G cosmologies [5,12] G is a decreasing function of time. But the possibility of an increasing G has also been suggested [11]. The cosmological constant, on the other hand, is depleted as the universe expands.

For the early universe, the model indicates that the horizon and monopole problems may be resolved without abandoning conservation of the energy-momentum tensor. For the radiation and matter era the model makes several predictions. With mild assumptions and frequently quoted values for the Hubble and the rate of change of G parameters we estimated the present values of the scale factor, the cosmological constant and the deacceleration parameter. Estimates of the matter density and age of the universe are also obtained. The results fall within current theoretical and experimental bounds.

ACKNOWLEDGEMENT

I thank M. O. Taha for a discussion and Abdel-Mahmoud M. Abdel-Rahman for the verification of a computation.

REFERENCES

1. Özer, M., and Taha, M. O. (1986). *Phys. Lett.*, **171B**, 363; (1987). *Nucl. Phys. B*, **287**, 776.
2. Bergmann, P. G. (1968). *Int. J. Theor. Phys.*, **1**, 25. Wagoner, R. V. (1970). *Phys. Rev. D*, **1**, 3209. Linde, A. D. (1974). *JETP Lett.*, **19**, 183. DerSarkissian, M. (1985). *Nuovo Cimento*, **B88**, 29. Abdel-Rahman, A-M. M. (1988). *Nuovo Cimento*, **B102**, 225.
3. Wilkins, D. (1986). *Am. J. Phys.*, **54**, 726.
4. Beesham, A. (1986). *Int. J. Theor. Phys.*, **25**, 1295; (1986).*Nuovo Cimento*, **B96**, 17. Lau, Y.-K. (1985). *Australian J. Phys.*, **38**, 547.
5. Weinberg, S. (1971). *Gravitation and Cosmology* (Wiley, New York).
6. Grøn, Ø. (1986). *Am. J. Phys.*, **54**, 46.
7. Hellings, R. W., Adams, P. J., Anderson, J. D., Keesey, M. S., Lau, E. L., Standish, E. M., Canuto, V. M., and Goldman, I. (1983). *Phys. Rev. Lett.*, **51**, 1609. Van Flandern, T. C. (1981). *Astrophys. J.*, **248**, 813. Shapiro, I. I., Smith, W. B., Ash, M. E., Ingalls, R. P., and Pettengill, G. H. (1971). *Phys. Rev. Lett.*, **26**, 27. Rowan-Robinson, M. (1981). *Cosmology*, (Oxford University Press, Oxford), p. 124.
8. Hawking, S. W. (1984). *Phys. Lett.*, **134B**, 403.
9. Butcher, H. R. (1987). *Nature*, **328**, 127. Lachièze-Rey, M. (1988). *La Recherche*, **195**, 100.
10. Dirac, P. A. M. (1937). *Nature*, **139**, 323.
11. Levitt, L. S. (1980). *Lett. Nuovo Cimento*, **29**, 23. Chow, T. L. (1981). *Lett. Nuovo Cimento*, **31**, 119. Milne, E. A. (1935) *Relativity, Gravitation and World Structure*, (Oxford University Press, Oxford), p. 292.
12. Norman, E. B. (1986). *Am. J. Phys.*, **54**, 317.