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Perfect Fluid Spheres in General Relativity

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Abstract

Spherically symmetric perfect fluid distributions in general relativity have been investigated under the assumptions of (i) uniform expansion or contraction and (ii) the validity of an equation of state of the form $p = p(\rho)$ with nonuniform density. An exact solution which is equivalent to a solution found earlier by Wyman is obtained and it is shown that the solution is *unique*. The boundary conditions at the interface of fluid distribution and the exterior vacuum are discussed and as a consequence the following theorem is established: *Uniform expansion or contraction of a perfect fluid sphere obeying an equation of state with nonuniform density is not admitted by the field equations.* It is further shown that the Wyman metric is not suitable on physical grounds to represent a cosmological solution.

§(1): *Introduction*

Perfect fluid configurations in general relativity have been studied by several investigators due to possible applications to problems of gravitational collapse. Of particular interest are the fluid distributions with spherical symmetry which serve as models for the phenomenon of gravitational implosion or explosion of a massive star [1–7]. On account of the high symmetry involved in this case, the complicated nonlinear gravitational field equations become, with some additional assumptions, mathematically manageable. In this context *uniform* contraction or expansion is one of the frequent simplifying assumptions chosen by various authors [2–5, 8]. In this note we also investigate perfect fluid distributions under the following assumptions: (1) the hypersurface orthogonal to the

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t axis is spatially isotropic and (2) the fluid obeys an equation of state. In particular, we prove the following theorem:

"Uniform contraction or expansion of a perfect fluid sphere embedded in an empty space and obeying an equation of state of the form $p = p(\rho)$ with nonuniform density is not admitted by the field equations of general relativity." Mashhoon and Partovi [1] and Mansouri [4] have claimed to establish this theorem but their arguments and the proofs are not convincing.

In Section 2 we present the field equations and their first integrals obtained independently by Taub [3] and by Wyman [8]. The paper of Wyman may be considered pioneering in the sense that it contains many of the results claimed to have been derived by subsequent authors, e.g., Taub [3]. In Section 3 we establish the uniqueness of Wyman's solution. In Section 4 the problem of matching this solution with vacuum Schwarzschild space-time is discussed and the necessary boundary conditions have been obtained. It is further shown that the matching is not possible. In the last section we discuss the Wyman metric as a cosmological solution and establish that the Wyman metric is not suitable to represent a cosmological distribution.

§(2): *The Field Equations and Their First Integrals*

We choose the spherically symmetric metric in comoving coordinates (r, θ, φ, t) as follows:

$$ds^2 = e^{\nu}\, dt^2 - e^{\lambda}\, dr^2 - e^{\mu}\, d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2 \tag{1}$$

where the metric coefficients are functions of r and t only. The energy-momentum tensor for the perfect fluid distribution is given by

$$T_{ij} = (\rho + p)\, V_i V_j - p g_{ij}$$

where p and ρ denote the pressure and density of the fluid. The Einstein field equations for this problem have been set up by several workers and, therefore, we do not detail them here. The field equations in a convenient form are [2, 3]

$$2p' = -(p + \rho)\, \nu' \tag{2}$$

$$\dot{\rho} = -(p + \rho)\, (\dot{\mu} + \dot{\lambda}/2) \tag{3}$$

$$m' = 2\pi \rho e^{3\mu/2} \mu' \tag{4}$$

$$\dot{m} = -2\pi p e^{3\mu/2} \dot{\mu} \tag{5}$$

Here and in what follows an overhead dot and a prime denote, respectively, partial derivatives with respect to t and r. The function $m(r, t)$ introduced by

Misner and Sharp [6] is defined as

$$8m = \dot{\mu}^2 e^{3\mu/2-\nu} - \mu'^2 e^{3\mu/2-\lambda} + 4e^{\mu/2} \tag{6}$$

and is interpreted as the mass enclosed within the spherical volume of radius r at a given instant t. It is remarked that the set of equations (2)-(6) is equivalent to the Einstein field equations only when $\dot{\mu} \neq 0$. If $\dot{\mu} = 0$, the solution becomes static, a case not of our interest.

Owing to the nonlinearity of the field equations it is very difficult to obtain a solution in its generality and, therefore, one has to make some simplifying assumptions to derive useful results. The assumptions are motivated either by physical considerations or by mathematical convenience. One such assumption used extensively is

$$\dot{\lambda} = \dot{\mu} \tag{7}$$

Under this assumption the shear of the 4-velocity vector vanishes and the expansion is uniform[2] [2, 9]. Further the three-dimensional space orthogonal to the t axis is spatially isotropic. In this note we also assume this condition and take

$$\lambda = \mu \tag{8}$$

the constant of integration being absorbed into the r scale. The metric (1) now becomes

$$ds^2 = e^{\nu}\, dt^2 - e^{\mu}\, (dr^2 + d\Omega^2) \tag{9}$$

Equation (2) in view of the integrability condition for the function m and equations (3)-(5) leads to

$$2\dot{\mu}' - \dot{\mu}\nu' = 0 \tag{10}$$

It is easily integrated to obtain

$$e^{-\nu/2}\dot{\mu} = L(t) \tag{11}$$

where L is an arbitrary function of its argument. Further from equation (3) and using equations (2) and (10) one gets

$$\dot{\rho}' + \tfrac{3}{2}\,\dot{\mu}\rho' = 0$$

This equation on integration yields

$$\rho' e^{3\mu/2} = -A(r)'/4\pi \tag{12}$$

[2] The converse result has been obtained by Nariai [9] assuming the solution to be regular at the center (see also Misra and Srivastava [1]). We will also consider the regularity conditions and hence the assumption (7) would be equivalent to the assumption of uniform expansion or contraction of perfect fluid distribution.

We have chosen the arbitrary function of integration as $-A(r)'/4\pi$ for later convenience. Now equation (4) in view of the above result is integrated to yield

$$3m = 4\pi\rho e^{3\mu/2} + A(r) + N(t) \tag{13}$$

where N is again an arbitrary function of time. We require the solution to be regular at the center of the symmetry. This condition determines $N(t)$. The regularity conditions in the neighborhood of the origin require

$$\left.\begin{array}{l} e^{\mu/2} \longrightarrow 0 \\ e^{\lambda/2} \longrightarrow e^{\mu/2}\mu'/2 \end{array}\right\} \quad \text{as } r \longrightarrow 0 \Longrightarrow m(0, t) = 0$$

Hence $N(t)$ is a constant. We absorb this constant into the function $A(r)$ such that

$$A(0) = 0 \tag{14}$$

Thus equation (13) becomes

$$3m = 4\pi\rho e^{3\mu/2} + A \tag{15}$$

It is to be noted that if one assumes $\rho = \rho(t)$ in addition to (7) equation (12) is satisfied identically and one obtains $A = 0$. Spherical distributions satisfying these conditions have been considered in detail by Thompson and Whitrow [2] and by Bondi [5]. We are interested in fluid distributions with $\rho \neq \rho(t)$ and hence we must have

$$A(r) \neq \text{const} \tag{16}$$

In this case equation (5) with the help of equations (12) and (15) becomes

$$\dot{\rho}'(p + \rho) = \dot{\rho}p' \tag{17}$$

This is an equation relating p and ρ and further integration can be made only if one assumes a suitable equation of state. We assume that

$$p = p(\rho) \quad \text{such that} \quad \frac{1}{p+\rho} = \frac{1}{\sigma}\frac{d\sigma}{d\rho}$$

$$\sigma = \sigma(\rho) \tag{18}$$

Perfect fluid distributions with the above equation of state have also been discussed by Taub [3], Mansouri [4], and Faulkes [10] in connection with the collapse of spherical balls, and by Wyman [8] for cosmological distributions. The results established in this section are due to Wyman [8] and Taub [3] and are presented for the sake of continuity and clarity. Integrating equation (17) we obtain

$$x\rho_x = \sigma \tag{19}$$

$$\rho = \rho(x), \qquad x = B(r)/P(t) \tag{20}$$

where B and P are arbitrary functions of their arguments. Here and in what follows a subscript denotes the partial derivative. Making use of this result in equation (12) we obtain

$$Xe^{\mu/2} = C(r) \tag{21}$$

where X and C are given by

$$X = \sigma^{1/3}, \quad C = -(A'B/4\pi B')^{1/3} \tag{22}$$

Equation (18) in view of equations (19) and (22) leads to

$$p + \rho = X^4/3xX_x \tag{23}$$

For a physical system we must have $p \geqslant 0$ and $\rho \geqslant 0$. Hence

$$xX_x \geqslant 0 \tag{24}$$

Using equations (20) and (21) one obtains

$$\dot{\mu} = \frac{xX_x}{X^2}\frac{\dot{P}}{P}$$

Therefore $\dot{P}/P > 0$ or < 0 according as the system is expanding or contracting. For definiteness we consider the system to be contracting, and without any loss of generality choose the function $P(t)$ as

$$P = \exp(-t) \tag{25}$$

It is remarked that the results obtained hereafter also apply for an expanding system by the interchange of t to $-t$. Now equation (11) yields

$$e^{\nu/2} = -2xX_x/LX$$

We now derive an equation for e^{μ}, namely,

$$(e^{-\mu/2})'' = e^{-\mu/2} - Ae^{-\mu} \tag{26}$$

in the following manner. Eliminate ν from equation (6) with the help of equation (11) and obtain

$$e^{-\mu/2}\mu' = \pm(4e^{-\mu} - 8me^{-3\mu/2} + L^2)^{1/2}$$

Differentiate this equation with respect to r and eliminate m' using equations (4) and (15) and the result follows. Thus using equations (20) and (21) one obtains

$$x^2X_{xx}\left(\frac{B'}{B}\right)^2 + xX_x\left(\frac{B''}{B} - \frac{2B'}{B}\frac{C'}{C}\right) + X\left(\frac{2C'^2}{C^2} - \frac{C''}{C}\right) = X - X^2\frac{A}{C}$$

This equation is really remarkable in the sense that A, B, and C are functions of r and the other terms are functions of x. Hence because of the independence of variables x and r, the equation can be satisfied only if

$$\left(\frac{B'}{B}\right)^2 = \alpha \frac{A}{C} \tag{27}$$

$$\frac{B''}{B} - 2\frac{B'}{B}\frac{C'}{C} = \beta \frac{A}{C} \tag{28}$$

$$\frac{C''}{C} - 2\left(\frac{C'}{C}\right)^2 + 1 = -\gamma \frac{A}{C} \tag{29}$$

where α, β, and γ are constants. Thus one obtains

$$\alpha x^2 X_{xx} + \beta x X_x + \gamma X + X^2 = 0 \tag{30}$$

It may be noted that $\alpha = 0$ makes $B(r)$ a constant and vice versa. This result in view of equation (20) gives $\rho = \rho(t)$. Hence we must have

$$\alpha \neq 0 \quad \text{and} \quad B \neq \text{const} \tag{31}$$

Now equation (28) can be integrated with the help of equation (27) to yield

$$B'/B^{\epsilon} = dC^2, \quad \epsilon = \beta/\alpha \tag{32}$$

where d is an arbitrary constant not equal to zero. Thus equation (27) becomes

$$A = \eta\, d^2\, C^5 B^{2\epsilon - 2}, \quad \eta = 1/\alpha \tag{33}$$

It may be noted that in view of equations (14) and (16) one must have

$$\eta \neq 0 \tag{34}$$

Thus the metric is determined in terms of functions B, C, and X satisfying equations (32), (29), and (30), respectively. These equations contain several constants and it is difficult to find a general solution. Wyman, in the quest for solutions of these equations, could achieve his objective by *choosing* the constants conveniently and obtained only particular solutions. However, in the next section we show that the constants are determined uniquely from the field equations and the equation of state. Thus we will prove that the Wyman solution is unique and the general solution for the given equation of state.

§(3): *The Wyman Solution and Its Uniqueness*

As remarked in the previous section the metric is obtained explicitly provided the functions B, C, and X are determined. We note that the functions A, B, and C are not independent but are related via equation (22) as

$$4\pi C^3 = -A'B/B'$$

This equation, in view of equations (32) and (33), gives

$$\frac{5C'}{C} = (2\,d\delta\, C^2 B^{-\delta} - \xi B^{\delta}), \qquad \xi = 4\pi/d\eta, \quad \delta = 1 - \epsilon \tag{35}$$

Further equations (28) and (29) with the help of equations (27), (32), and (35) yield

$$\frac{5B''}{B} + (\delta - 5)\left(\frac{B'}{B}\right)^2 + 2\xi B^{\delta-1}B' = 0 \tag{36}$$

$$\xi^2 B^{2\delta} + 5\xi\delta B^{\delta-1}B' + (6\delta^2 - 25\gamma\eta)\,B^{-2}B'^2 = 25 \tag{37}$$

respectively. Now it is easy to integrate equation (36) and one obtains

$$B'B^{(\delta/5)-1} = -\xi B^{6\delta/5}/3\delta + k, \qquad \delta \neq 0$$

$$B'/B = -(2\xi \ln B)/5 + k, \qquad \delta = 0 \tag{38}$$

where k is an arbitrary constant. Let us consider the case $\delta \neq 0$. Equation (37) now yields

$$\xi k B^{4\delta/5}(\delta + 50\gamma\eta/3\delta) - 25\gamma\eta\xi^2 B^{-2\delta}/9\delta^2 + k^2 B^{-2\delta/5}(6\delta^2 - 25\gamma\eta) = 25$$

This turns out to be an algebraic equation for $B(r)$ leading to the conclusion that $B(r)$ is a constant. But we have shown earlier that constant B leads to $\rho = \rho(t)$. Alternatively, the above equation may be satisfied identically only with $\delta = 0$. Thus with this value of δ equation (37) reduces to

$$\xi^2 - \gamma\eta(2\xi \ln B - 5k)^2 = 25$$

Hence in view of above discussion we must have

$$\gamma\eta = 0 \quad \text{and} \quad \xi = 5u, \quad u = \pm 1 \tag{39}$$

Since $\eta \neq 0$ [equation (34)] we obtain $\gamma = 0$. Collecting our results together we thus get

$$\gamma = 0 = \delta \quad \text{and } \xi = 5u \tag{40}$$

Now equations (29), (30), and (32) can easily be integrated. From equation (29) and (35) we obtain

$$C = e^{-ur+l}$$

where l is an arbitrary constant. Without any loss of generality one may absorb this constant into the origin of r scale and hence

$$C = e^{-ur} \tag{41}$$

Equation (33) now yields

$$A = (c)^2\, e^{-5ur}/\alpha, \qquad c = -\frac{4\pi\alpha}{5} \tag{42}$$

It may be noted that A so determined does not satisfy the regularity condition at the center viz., equation (14). Hence we rescale the r coordinate as

$$e^{-ur} \longrightarrow \bar{r} \tag{43}$$

The functions A and C are now obtained as

$$A = c^2 \bar{r}^5 / \alpha, \qquad C = \bar{r} \tag{44}$$

Further from equation (32) we get

$$\ln B = \tfrac{1}{2} c\bar{r}^2 + f \tag{45}$$

where f is an arbitrary constant of integration. Also equation (30) is integrated to yield

$$xX_x = (\psi - 2\eta X^3/3)^{1/2} \tag{46}$$

ψ is again an arbitrary constant. Thus X is the Weierstrass elliptic function. The variable x in view of equations (20), (24) and (45) is given by

$$\ln x = \tfrac{1}{2} c\bar{r}^2 + f + t$$

Here again one may absorb f into the origin of the t scale and thus

$$\ln x = \tfrac{1}{2} c\bar{r}^2 + t \tag{47}$$

The metric (9) is thus obtained as

$$ds^2 = 4(xX_x/LX)^2 \, dt^2 - \frac{1}{X^2} (d\bar{r}^2 + \bar{r}^2 \, d\Omega^2) \tag{48}$$

The density of the fluid distribution is obtained by integrating equation (19) and one gets

$$\rho = (3/4\pi) \{c[X(\psi - 2\eta X^3/3)^{1/2} - \psi \ln x] + h\} \tag{49}$$

Here h is an arbitrary constant. The pressure of the distribution is now determined from equation (23). It is to be noted that the arbitrary function $L(t)$ may be determined using equations (6), (11), (15), and (49) and one gets

$$L = [8(h - \psi c t)]^{1/2} \tag{50}$$

Equation (48) is the solution obtained by Wyman by assuming the condition (40). But we have shown that equation (40) follows from the field equations themselves and need not be taken as a separate assumption. Hence we establish the following:

The uniformly expanding or contracting perfect fluid distributions having spherical symmetry and obeying an equation of state with nonuniform density are described uniquely by the Wyman metric.

It is to be pointed out that recently Mashhoon and Partovi [1] have established a similar theorem, namely, "The only solution for the spherically symmetric, shear-free motion of an uncharged perfect fluid obeying an equation of state is the Friedmann solution." In view of our result it is clear that this theorem is valid only when the density is independent of r.

It is interesting to remark that the two solutions reported by Wyman and related via the transformation of r scale follow according to our formalism in a natural way with $u = \pm 1$.

§(4): *The Boundary Conditions*

The solution given by (48) will describe the interior gravitational field of a spherical ball of perfect fluid provided it is matched smoothly at the boundary with the exterior field, namely, the Schwarzschild solution given in curvature coordinates (R, θ, φ, T) as follows:

$$ds^2 = (1 - 2M/R)\, dT^2 - (1 - 2M/R)^{-1}\, dR^2 - R^2\, d\Omega^2 \tag{51}$$

Here M is the mass of the sphere. This requires the metric coefficients and their first derivatives to be continuous. The matching problem has been dealt with in depth by several workers, e.g., Misner and Sharp [6] and Robson [7]. Accordingly, the following conditions must hold:

$$p(\bar{r}_0, t) = p_0 = O \tag{52}$$

and

$$m(\bar{r}_0, t) = m_0 = M \tag{53}$$

where $\bar{r}_0$ is a constant defining the boundary of the sphere. Here and in what follows a subscript $_0$ represents the boundary value of the quantity under consideration. Equation (23) in view of the boundary condition (52) leads to

$$\rho_0 = (X^4/3xX_x)_0 \tag{54}$$

Further from equation (15) and using equations (21), (44) and (54) we get

$$3m_0 = 4\pi(X/3xX_x)_0\, \bar{r}_0^3 + c^2\bar{r}_0^5/\alpha \tag{55}$$

Now combining this equation with the condition (53) one obtains

$$(4\pi/3)\,(X/xX_x)_0\, \bar{r}_0^3 = 3M - c^2\bar{r}_0^5/\alpha \tag{56}$$

Thus at the boundary (X/xX_x) is a constant. Therefore, in view of equation (46) X_0 is a constant. Further equation (54) leads to

$$\rho_0 = \text{const}$$

Now equation (50) becomes

$$-\psi c\,(\ln x)_0 + h = \text{const} \tag{57}$$

This equation must be satisfied identically otherwise it would lead to an algebraic equation for t. Hence we must have

$$\psi = 0$$

and this in view of equations (49) and (50) leads to

$$\begin{aligned} 8\pi\rho &= 6cxX_x X + 6h \\ 8\pi p &= -\frac{8\pi X^4}{15xX_x} - 6h \\ (p+\rho) &= \frac{X^4}{3xX_x} \end{aligned} \tag{58}$$

Now one must have $p > 0$, $\rho > 0$ throughout the distribution so that equation (58) requires $h < 0$. But we must also have $g_{00} > 0$ and hence in view of equations (48) and (50) $h > 0$, contrary to our previous conclusion. Thus $h = 0$ leading to $\dot{\mu} = 0$ makes the distribution *static.* The above discussion shows that the gravitational field of a perfect fluid distribution characterized by equation (48) does not match smoothly with a spherically symmetric exterior vacuum. Hence we conclude the following:

Uniform expansion or contraction of a perfect fluid sphere obeying an equation of state of the form $p = p(\rho)$ and embedded in an empty space is not admitted by the field equations of general relativity.

Recently Mansouri [4] has claimed to have established this theorem. He considers the first integrals of the field equations obtained by Taub, presented here in Section 2, and argues that the functional dependence $p = p(x) = p\,[B(\bar{r})/P(t)]$ and the boundary condition $p_0 = 0$ would imply

$$x_0 = 0$$

Also, making use of the boundary condition $m_0 = M$ one obtains

$$\gamma\eta + \epsilon = 0 \qquad \text{and} \qquad 2\epsilon + 3 = 0 \tag{59}$$

which is not the same as equation (40) obtained here. Further we have derived the explicit functional dependence of p on x and have demonstrated that $x \neq 0$ at the boundary. His derivation, therefore, seems to be incorrect although the final conclusions are correct.

§(5): *Wyman Metric as a Cosmological Solution*

In this section we discuss the nature of contraction of a cosmological distribution represented by the Wyman metric. The pressure and the density of the distribution are given by equations (23) and (49), which in view of equations (46), (47), and (50) are expressed as follows:

$$\rho = \frac{3cX}{4\pi}\left(\psi - \frac{2\eta}{3}X^3\right)^{1/2} - \frac{3}{8\pi}c^2\bar{r}^2\psi + \frac{3L^2}{32\pi} \tag{60}$$

$$p = -\frac{(3c\psi/4\pi)X + (X^4/15)}{[\psi - (2\eta/3)X^3]^{1/2}} + \frac{3}{8\pi}c^2\bar{r}^2\psi - \frac{3L^2}{32\pi}$$

In order that $p(0, t) \geqslant 0$ one must have $c\psi < 0$. We can have now two cases: $c > 0$, $\psi < 0$ and $c < 0$, $\psi > 0$. The first case is unphysical because in view of equations (12), (44), and (48) it leads to the situation $\rho' > 0$. Thus we obtain

$$c < 0 \quad \text{and} \quad \psi > 0 \tag{61}$$

Equations (60) in view of equations (46), (47), and (50) lead to

$$\frac{d\rho}{dp} = \frac{3c\psi + 8\pi X^3/5}{c\psi - 4\pi X^3/15}$$

For a physical system we must also demand $d\rho/dp \geqslant 1$ and this will require $X^3 \leqslant -15c\psi/14\pi$. But from equations (46) and (48) one obtains

$$0 < X \leqslant \left(-\frac{15c\psi}{8\pi}\right)^{1/3}$$

Table I. Physical Parameters of the Distribution $c = -8\pi/15$, $\psi = 1$

X	ρ	p	ρ/p	$t = 0$ $\bar{r}/\bar{r}_0$	$\bar{r} = \bar{r}_0$ t
0.000	0.000	0.000	3.00	1.000	0.000
0.100	0.000	0.000	3.00	0.949	0.100
0.200	0.000	0.000	2.98	0.894	0.200
0.300	0.000	0.000	2.93	0.836	0.301
0.400	0.007	0.002	2.84	0.772	0.403
0.500	0.016	0.006	2.68	0.701	0.508
0.600	0.035	0.014	2.45	0.618	0.618
0.700	0.067	0.031	2.14	0.514	0.736
0.800	0.124	0.072	1.72	0.364	0.868
0.850	0.166	0.114	1.46	0.238	0.943
0.900	0.225	0.195	1.15	–	1.031
0.950	0.313	0.406	0.77	–	1.142
1.000	0.561	∞	0.00	–	1.402

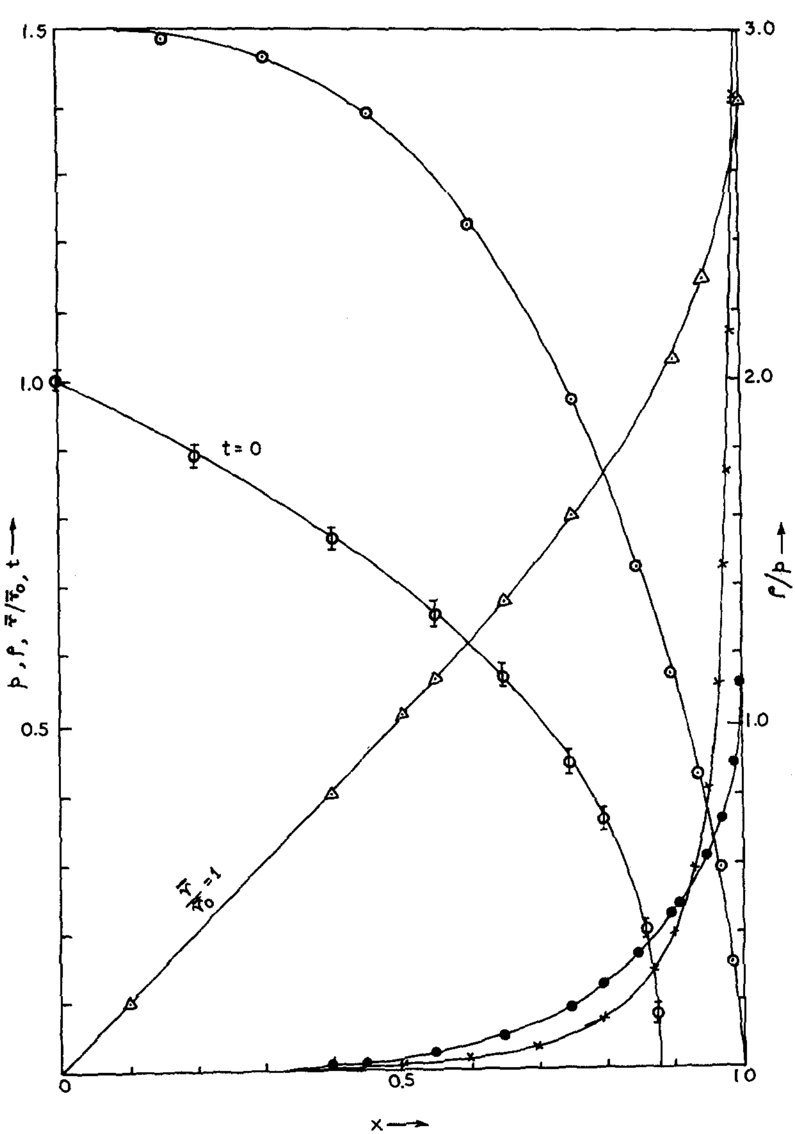

Fig. 1. Variation of various physical parameters with X for a model for typical values of $c = -8\pi/15$ and $\psi = 1$. ●, Variation of density of the fluid distribution; ×, variation of pressure of the fluid distribution; ⊙, variation of (ρ/p) of the fluid distribution; △, variation of t for $\bar{r} = \bar{r}_0$; Φ, variation of r in the units of r_0 at $t = 0$.

Thus during the contraction we find that in the interval $-15c\psi/8\pi \geqslant X^3 > -15c\psi/14\pi$ the system becomes unphysical.[3] The behavior of the model is illustrated in Table I and Figure 1 for a typical value of the parameters $c = -8\pi/15$ and $\psi = 1$. It is assumed that initially the pressure and density vanish at some finite value of $\bar{r}$, say, $\bar{r} = \bar{r}_0$. One finds that initially the density is greater than the pressure throughout the distribution but as the contraction proceeds there develops a region surrounding the center where pressure overtakes density. At $X = 1$ the pressure diverges off but the density remains finite. Thus we conclude that the Wyman metric is not suitable to represent a cosmological distribution.

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Note Added In Proof

Wyman in a private communication has informed us that he was aware of the uniqueness of his solution in 1946 but he did not provide the proof.

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