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RC^*-FIELDS

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It is stated that if a Boolean family W of valuation rings of a field F satisfies the block approximation property (BAP) and a global analog of the Hensel–Rychlick property (THR), in which case $\langle F, W\rangle$ is called an RC^*-field, then F is regularly closed with respect to the family W (Theorem 1). It is proved that every pair $\langle F, W\rangle$, where W is a weakly Boolean family of valuation rings of a field F, is embedded in the RC^*-field $\langle F_0, W_0\rangle$ in such a manner that $R_0 \mapsto R_0 \cap F$, $R_0 \in W_0$ is a continuous map, W_0 is homeomorphic over W to a given Boolean space, and R_0 is a superstructure of $R_0 \cap F$ for every $R_0 \in W_0$ (Theorem 2).

In the present paper we prove most of the statements announced in [1].

For definitions of the basic notions in valued field theory, we ask the reader to consult Chap. 4 in [2]. If R is a valuation ring of a field F, then $\mathfrak{m}(R)$ is the maximal ideal of R, v_R is the corresponding valuation, $F_R \rightleftharpoons R/\mathfrak{m}(R)$ is a residue class field, Γ_R is a value group, R^h is the Henselization of R, and $H_R(F) \rightleftharpoons q(R^h) \leq \widetilde{F}$ is the field of fractions of R^h; moreover, $\mathbb{H}_R(F) \rightleftharpoons \langle H_R(F), R^h\rangle$.

1. Let W be some family of pairwise incomparable (w.r.t. inclusion) valuation rings of a field F. A *canonical topology on* W is one defined via the subbase of subsets of the form $V_a \rightleftharpoons \{R \mid R \in W,\ a \in R\}$, $a \in F^* \rightleftharpoons F\backslash\{0\}$; if $A \subseteq F^*$ is a finite nonempty subset, then $V_A \rightleftharpoons \bigcap\limits_{a\in A} V_a$.

We call W a *weakly Boolean* family of valuation rings if W endowed with the canonical topology becomes a Boolean space, and V_A is closed-open for every finite A, $\varnothing \neq A \subseteq F^*$. A weakly Boolean family W is *Boolean* if all closed-open subsets of W are of the form V_a, $a \in F^*$. In [3], it was established that W is a Boolean family iff F is the field of fractions of the ring R_W defined by W as follows: $R_W \rightleftharpoons \cap\{R \mid R \in W\}$, R_W is a regular Prüfer ring, and $W = \{(R_W)_{\mathfrak{m}} \mid \mathfrak{m}$ is a maximal ideal of $R_W\}$. If W is a (weakly) Boolean family of valuation rings of F, F_0 is an algebraic extension of F, and $W_0 \rightleftharpoons \{R_0 \mid R_0$ is a valuation ring of a field F_0 and $R_0 \cap F \in W\}$, then W_0 is a (weakly) Boolean family (Theorems 1 and 2 in [3]).

Let W be a weakly Boolean family of valuation rings of a field F. We say that W possesses the *block approximation property* (BAP) if, for every partition $[V_0, \ldots, V_n]$ of W, i.e., for a family $V_0, \ldots, V_n$ of closed-open subsets of W such that $W = \bigcup\limits_{i\leq n} V_i$ and $V_i \cap V_j = \varnothing$ for $0 \leq i < j \leq n$, for all $a_0, \ldots, a_n \in F$ and $\varepsilon_0, \ldots, \varepsilon_n \in F^*(= F\backslash\{0\})$, there exists an element $a \in F$ for which $v_R(a - a_i) \geq v_R(\varepsilon_i)$ with all $R \in V_i$, $i \leq n$.

Proposition 1. If a Boolean family W has BAP, then W is Boolean.

Note that for every partition $[V_0, \ldots, V_n]$, there exists a finer partition $[V_{00}, \ldots, V_{0k_0}, \ldots, V_{n0}, \ldots, V_{nk_n}]$ such that for every V_{ij}, there is an element $\varepsilon_{ij} \neq 0$ satisfying $v_R(\varepsilon_{ij}) > 0$ for all $R \in V_{ij}$. This follows from the fact that a family of closed-open subsets of the form $V_\varepsilon^* \rightleftharpoons V_\varepsilon \backslash V_{\varepsilon^{-1}}$, $\varepsilon \in F^*$, covers W. Let V be closed-open and let $[V_0, \ldots, V_n]$ be a refinement of the partition $[V, W\backslash V]$ such that for some

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$\varepsilon_0, \ldots, \varepsilon_n \in F^*$, we have $v_R(\varepsilon_i) > 0$ with all $R \in V_i$, $i \le n$. Put $a_i \rightleftharpoons 1$ if $V_i \subseteq V$, and $a_i \rightleftharpoons 0$ if $V_i \subseteq W \backslash V$, $i \le n$; if $a \in F$ is such that $v_R(a - a_i) \ge v_R(\varepsilon_i)$ for $R \in V_i$, $i \le n$, then $V = V_{a^{-1}}$.

If W is known to be Boolean, then the block approximation property for W is reworded as follows: If $\alpha_0, \ldots, \alpha_n \in F^*$ determines the partition $V_{\alpha_0}, \ldots, V_{\alpha_n}$ of the space W (in this case we say that $[\alpha_0, \ldots, \alpha_n]$ is a partition), $a_0, \ldots, a_n \in F$, and $0 \ne \varepsilon \in R_W$, then there exists an element $a \in F$ such that $v_R(a - a_i) \ge v_R(\varepsilon)$ for any $R \in V_{\alpha_i}$, $i \le n$.

Proposition 2. If W is a Boolean family of valuation rings of a field F, then the pairs $\langle F, R_W \rangle$ such that W has BAP are expressed via a system of axioms in the signature $\sigma_f \cup \langle R \rangle$.

The proposition follows from the following facts:

(1) For every $a \in F$, there exists an $\alpha \in R_W$ such that $V_a = V_{\alpha^{-1}}$.

Since R_W is a Prüfer ring, there exists a $\beta \in F$ such that $aR_W + R_W = \beta R_W$. It is easy to see that $\alpha \rightleftharpoons \beta^{-1} \in R_W$ and $V_a = V_\beta = V_{\alpha^{-1}}$. The elements a and α are connected via relations $\alpha \in R_W$ and $\exists r_0, r_1 \in R_W$ $(a\alpha r_0 + \alpha r_1 = 1)$, which follow from the condition of $aR_W + R_W = \beta R_W$ being equivalent to $\alpha a R_W + \alpha R_W = R_W$.

(2) If $\alpha, \beta \in R_W$ and $\alpha\beta \ne 0$, then $V_{\alpha^{-1}} \cap V_{\beta^{-1}} = V_{(\alpha\beta)^{-1}}$.

Indeed, $R \in V_{\alpha^{-1}} \cap V_{\beta^{-1}} \Rightarrow \alpha, \beta \in R \backslash \mathfrak{m}(R) \Rightarrow \alpha\beta \in R \backslash \mathfrak{m}(R) \Rightarrow R \in V_{(\alpha\beta)^{-1}}$ $R \in V_{(\alpha\beta)^{-1}} \Rightarrow \alpha^{-1} = \beta(\alpha\beta)^{-1} \in R$, and $\beta^{-1} = \alpha(\alpha\beta)^{-1} \in R \Rightarrow R \in V_{\alpha^{-1}} \cap V_{\beta^{-1}}$.

(3) If $\alpha \in R_W$, then $V_{\alpha^{-1}} = \varnothing$ iff α belongs to the Jacobson radical $J(R_W)$ of the ring R_W.

Let $V_{\alpha^{-1}} = \varnothing$; if $\mathfrak{m}$ is a maximal ideal of R_W, then $(R_W)_\mathfrak{m} \in W$. We have $\alpha^{-1} \notin (R_W)_\mathfrak{m}$ and $\alpha \in \mathfrak{m}((R_W)_\mathfrak{m}) \cap R_W = \mathfrak{m}$, i.e., $\alpha \in \cap\{\mathfrak{m} | \mathfrak{m}$ is a maximal ideal of $R_W\} = J(R_W)$.

Conversely, let $\alpha \in J(R_W)$; then for $R \in W$, $\mathfrak{m}(R) \cap R_W$ is a maximal ideal of R_W, and $\mathfrak{m}(R) \cap R_W \supseteq J(R_W)$ holds; hence $\alpha \in \mathfrak{m}(R)$, and so $\alpha^{-1} \notin R$, i.e., $V_{\alpha^{-1}} = \varnothing$.

(4) If $\alpha \in R_W$, then $\alpha \in J(R_W)$ iff $R_W \models \forall x \exists y((1 + \alpha x)y = 1)$.

See, for instance, Proposition 1.9 in [4].

Let $0 \ne \alpha \in R_W$ and $R_\alpha \rightleftharpoons \cap\{R | R \in W, \alpha^{-1} \in R\}$.

(5) If $a \in F$ and $0 \ne \varepsilon \in R_W$, then the fact that $v_R(a) \ge v_R(\varepsilon)$ for all $R \in V_{\alpha^{-1}}$ is equivalent to the relation $a\varepsilon^{-1} \in R_\alpha$.

We have $v_R(a) \ge v_R(\varepsilon) \Leftrightarrow v_R(a\varepsilon^{-1}) \ge 0 \Leftrightarrow a\varepsilon^{-1} \in R$.

(6) If $0 \ne \alpha \in R_W$ and $\alpha_* \in R_W$ is such that $V_{\alpha^{-1}} \cap V_{\alpha_*^{-1}} = \varnothing$ and $V_{\alpha^{-1}} \cup V_{\alpha_*^{-1}} = W$, then $R_\alpha = \{ab^{-1} | a, b \in R_W, b \ne 0, V_{b^{-1}} \cup V_{\alpha_*^{-1}} = W\}$.

Indeed, $V_{b^{-1}} \cup V_{\alpha_*^{-1}} = W$ implies that $V_{\alpha^{-1}} \subseteq V_{b^{-1}}$, and so $b^{-1} \in R$ for all $R \in V_{\alpha^{-1}}$; then we have $b^{-1} \in \cap\{R | R \in V_{\alpha^{-1}}\} = R_\alpha$, and $R_W \le R_\alpha$ implies $ab^{-1} \in R_\alpha$ for every $a \in R_W$.

Let $a \in R_\alpha$ and $\beta \in F$ be such that $aR_W + R_W = \beta R_W$; then it is obvious that $V_{\alpha^{-1}} \subseteq V_a \subseteq V_\beta$. Thus, $\beta^{-1} \in R_W$, and $a = r\beta$ for a suitable $r \in R_W$.

(7) Let $\alpha_0, \ldots, \alpha_n \in R_W \backslash \{0\}$. Then $\bigcup_{i \le n} V_{\alpha_i^{-1}} = W$ iff $(\alpha_0, \ldots, \alpha_n) = R_W$ [here $(\alpha_0, \ldots, \alpha_n)$ is the ideal of R_W generated by elements $\alpha_0, \ldots, \alpha_n$] and $(\alpha_0, \ldots, \alpha_n) = R_W \Leftrightarrow \exists \beta_0, \ldots, \beta_n \in R_W(\sum_{i \le n} \alpha_i \beta_i = 1)$.

If $(\alpha_0, \ldots, \alpha_n) \ne R_W$, then there exists a maximal ideal $\mathfrak{m} \ge (\alpha_0, \ldots, \alpha_n)$, in which case $\alpha_0, \ldots, \alpha_n \in R \rightleftharpoons (R_W)_\mathfrak{m} \in W$ and $R \in W \backslash \bigcup_{i \le n} V_{\alpha_i^{-1}}$. But if $(\alpha_0, \ldots, \alpha_n) = R_W$ and $R \in W$, then $1 \notin \mathfrak{m}(R)$ and $\alpha_i \notin \mathfrak{m}(R)$ for some $i \le n$, from which it follows that $R \in V_{\alpha_i^{-1}}$ and $W = \bigcup_{i \le n} V_{\alpha_i^{-1}}$.

Now, for every $n > 1$ we can write the BAP_n sentence in the signature $\sigma_f \cup \langle R \rangle$ to express the following property: For all $\alpha_0, \ldots, \alpha_n$, $a_0, \ldots, a_n \in F$ and $0 \ne \varepsilon \in R_W$, let $\beta_0, \ldots, \beta_n \in R_W \backslash \{0\}$ be such that $V_{\alpha_i} = V_{\beta_i^{-1}}$, $i \le n$, and $V_{\beta_i^{-1}} \cap V_{\beta_j^{-1}} = \varnothing$ for $0 \le i < j \le n$ [which is equivalent to the condition

$\beta_i\beta_j \in J(R_W)$, $0 \leq i < j \leq n$] and $\bigcup_{i\leq n} V_{\beta_i^{-1}} = W$ [which is equivalent to the condition $(\beta_0, \ldots, \beta_n) = R_W$]. Then there exists an $a \in F$ satisfying $(a - a_i)\varepsilon^{-1} \in R_{\beta_i}$, $i \leq n$.

The collection of axioms BAP_n, $n > 1$, does axiomatize BAP.

LEMMA 1. If W is a Boolean family of valuation rings of a field F such that $W \neq \{F\}$, then $J(R_W) \neq \{0\}$.

If $J(R_W) = \{0\}$, then R_W is a subdirect product of fields: $R_W \leq \prod_{R\in W} F_R$. Since R_W has no zero divisors, it follows that $|W| = 1$ and $R_W \simeq F_R$ for R such that $W = \{R\}$, which is possible only in the case $R = F$.

We call the family W of valuation rings *independent* if any two distinct valuation rings, R and R', in W are independent, i.e., $RR' = F$.

Proposition 3. For W a Boolean family of valuation rings of a field F, the following conditions are equivalent:

(a) $\langle F, W\rangle \in \mathrm{BAP}$;

(b) W is independent.

Let $\langle F, W\rangle \in \mathrm{BAP}$, assuming that there exist $R_0 \neq R_1 \in W$ such that $R \rightleftharpoons R_0R_1 \neq F$. Let $a_i \in F^*$, $i = 0, 1$, be such that $R_0 \in V_{a_0}$, $R_1 \in V_{a_1}$, $V_{a_0} \cap V_{a_1} = \varnothing$, and $V_{a_0} \cup V_{a_1} = W$, let R be a valuation ring, and let $\mathfrak{m}(R) = \mathfrak{m}(R_0) \cap \mathfrak{m}(R_1) \geq J(R_W)$. Take $0 \neq b_0 \in J(R_W)$, which we can do since $J(R_W) \neq \{0\}$ by Lemma 1. Let $b_1 \in F \backslash R_1$. By BAP for $\varepsilon \rightleftharpoons b_0^2$, we can find an element b for which, in particular, $v_{R_0}(b - b_0) \geq v_{R_0}(\varepsilon)$ and $v_{R_1}(b - b_1) \geq v_{R_1}(\varepsilon)$. The first condition implies that $b - b_0 \in J(R_W)$, $b \in J(R_W) \leq \mathfrak{m}(R) \leq \mathfrak{m}(R_1)$, and $v_{R_1}(b) > 0$; it follows from the second condition that $v_{R_1}(b) = v_{R_1}(b_1) < 0$. Contradiction. Thus, (a) implies (b).

In order to prove that (b)$\Rightarrow$(a), we need the following:

LEMMA 2. If W is an independent Boolean family and if $\alpha, \beta \in F$ are such that $V_\alpha \cap V_\beta = \varnothing$ and $V_\alpha \cup V_\beta = W$, then for every $0 \neq \varepsilon \in R_W$, there exists an element a for which $v_R(1 - a) \geq v_R(\varepsilon)$ if $R \in V_\alpha$, and $v_R(a) \geq v_R(\varepsilon)$ if $R \in V_\beta$.

There is no loss of generality in assuming that $V_\alpha \neq \varnothing$, $V_\beta \neq \varnothing$, and $\varepsilon \in J(R_W)$. Let $R_0 \in V_\alpha$; then for every $R_1 \in V_\beta$ the equality $R_0R_1 = F$ implies the existence of an a_{R_1} satisfying $v_{R_0}(1 - a_{R_1}) \geq v_{R_0}(\varepsilon)$ and $v_{R_1}(a_{R_1}) \geq v_{R_1}(\varepsilon)$. Let $V_{R_1} \rightleftharpoons V_{a_{R_1}\varepsilon^{-1}} \cap V_\beta$; then $R_1 \in V_{R_1} \subseteq V_\beta$ and $V_\beta = \bigcup_{R_1 \in V_\beta} V_{R_1}$. Since V_β is compact, there exist $R_1, \ldots, R_s \in V_\beta$ such that $V_\beta = \bigcup_{i=1}^{s} V_{R_i}$. If $a \rightleftharpoons \prod_{i=1}^{s} a_{R_i}$, then $v_R(a) \geq v_R(\varepsilon)$ for $R \in V_\beta$, and $v_{R_0}(1 - a) \geq v_{R_0}(\varepsilon)$ [the latter is implied by the following: $1 - a = (1 - a_{R_1}) + a_{R_1}(1 - a_{R_0}) + \ldots + a_{R_1} \cdot \ldots \cdot a_{R_{s-1}}(1 - a_{R_s})$; $v_{R_0}(1 - a_{R_i}) \geq v_{R_0}(\varepsilon)$ and $v_{R_0}(a_{R_i}) = 0$, $i = 1, \ldots, s$]. Thus, for every $R_0 \in V_\alpha$, there exists an $e_{R_0} \rightleftharpoons a \in F^*$ for which $v_{R_0}(1 - e_{R_0}) \geq v_{R_0}(\varepsilon)$ and $v_R(e_{R_0}) \geq v_R(\varepsilon)$ for all $R \in V_\beta$. That element e_{R_0} is such that for every $R_0' \in V_{R_0} \rightleftharpoons V_{(1-e_{R_0})\varepsilon^{-1}} \cap V_\alpha$, the condition that $v_{R_0'}(1 - e_{R_0}) \geq v_{R_0'}(\varepsilon)$ and $v_R(e_{R_0}) \geq v_R(\varepsilon)$ is satisfied also for all $R \in V_\beta$. Therefore, $R_0 \in V_{R_0} \subseteq V_\alpha$ and $V_\alpha = \bigcup_{R \in V_\alpha} V_R$. Since V_α is compact, there exist $R_0, \ldots, R_k \in V_\alpha$ such that $V_\alpha = \bigcup_{i\leq k} V_{R_i}$. Let $e \rightleftharpoons \sum_{i\leq k} e_{R_i} - \sum_{0\leq i<j\leq k} e_{R_i}e_{R_j} + \sum_{0\leq i<j<l\leq k} e_{R_i}e_{R_j}e_{R_l} + \ldots + (-1)^k e_{R_0} \cdot \ldots \cdot e_{R_k}$. It is not hard to verify (e.g., for the case $k = 1$) that $v_R(1 - e) \geq v_R(\varepsilon)$ if $R \in V_\alpha$, and $v_R(e) \geq v_R(\varepsilon)$ if $R \in V_\beta$.

Now, let $[\alpha_0, \ldots, \alpha_n]$ be a partition, $a_0, \ldots, a_n \in F$, and $0 \neq \varepsilon \in R_W$. For each a_i, we will find b_i such that $a_iR_W + R_W = b_iR_W$, $i \leq n$. Note that $c_i \rightleftharpoons b_i^{-1} \in R_W$ and $a_ic_i \in R_W$, $i \leq n$. Choose $\varepsilon' \neq 0$ in $J(R_W)$ and put $\varepsilon_0 \rightleftharpoons \varepsilon \cdot \varepsilon' \cdot \prod_{i\leq n} c_i$. By Lemma 2, we can find elements $e_0, \ldots, e_n (\in R_W)$ such that $v_R(1 - e_i) \geq v_R(\varepsilon_0)$

if $R \in V_{\alpha_i}$, and $v_R(e_i) \geq v_R(\varepsilon_0)$ if $R \in W \backslash V_{\alpha_i}$. Therefore, the element $a \rightleftharpoons \sum_{i \leq n} a_i e_i$ satisfies the condition $v_R(a - a_i) \geq v_R(\varepsilon)$ if $R \in V_{\alpha_i}$, $i \leq n$. Indeed, if $R \in V_{\alpha_i}$, then $a - a_i = a_i(e_i - 1) + \sum_{j \neq i} a_j e_j$. We have $v_R(a_i(e_i - 1)) = v_R(a_i) + v_R(e_i - 1) \geq v_R(a_i) + v_R(\varepsilon_0) \geq v_R(a_i) + v_R(c_i) + v_R(\varepsilon) = v_R(a_i c_i) + v_R(\varepsilon) \geq v_R(\varepsilon)$ since $v_R(a_i c_i) \geq 0$. Further, $v_R(a_j e_j) = v_R(a_j) + v_R(e_j) \geq v_R(a_j) + v_R(\varepsilon_0) \geq v_R(a_j) + v_R(c_j) + v_R(\varepsilon) = v_R(a_j c_j) + v_R(\varepsilon) \geq v_R(\varepsilon)$. Hence $v_R(a - a_i) \geq v_R(\varepsilon)$.

COROLLARY. If W is an independent Boolean family of valuation rings of the field F, $F_0 \geq F$ is an algebraic extension of F, and $W_0 \rightleftharpoons \{R_0 | R_0$ is a valuation ring of the field F_0, and $R_0 \cap F \in W\}$, then $\langle F_0, W_0 \rangle \in$ BAP.

Indeed, if W is independent, then so is W_0.

2. Let us recall a number of basic definitions from [5].

Let $F \leq F_0$ be a field extension. The field F_0 is a *1-extension* of F ($F \leq_1 F_0$) if, for every finite subset $C \subseteq F_0$, there exists an F-homomorphism φ of the ring $F[C]$ into F.

For W a Boolean family of valuation rings of the field F, F is said to be *regularly closed with respect to* W, and we write $F \in RC(W)$ or $\langle F, W \rangle \in RC$, if for every regular extension F_0 of F the fact that $H_R(F) \leq_1 H_R(F) F_0$ for all $R \in W$ implies that $F \leq_1 F_0$.

If $F \leq F_0$ and R_0 is a valuation ring of the field F_0, we say that F is *dense in the valued field* $\langle F_0, R_0 \rangle$ if, for any $a_0 \in F_0$ and $\gamma_0 \in \Gamma_{R_0}$, there exists an $a \in F$ such that $v_{R_0}(a - a_0) \geq \gamma_0$.

COROLLARY. If R is a valuation ring of the field F dense in $\mathbb{H}_R(F) = \langle H_R(F), R^h \rangle$ and if $R < R_0 < F$ and $R = R_0 \circ \bar{R}$ for a suitable valuation ring $\bar{R}$ of the field F_{R_0}, then $\bar{R}$ is Henselian.

Proposition 4. If $\langle F, W \rangle$ is an RC-field, W is independent, and $R \in W$, then F is dense in $\mathbb{H}_R(F) = \langle H_R(F), R^h \rangle$.

As in the proof of Theorem 3 in [5], we argue that the property of being dense is a consequence of the following property shared by RC- fields $\langle F, W \rangle$ possessing BAP.

If $g \in R_W[x]$ is a unitary polynomial and $a \in R_W$ is such that $g(a) \in J(R_W)$ and $g'(a)^{-1} \in R_W$, then for every $c \in J(R_W) \backslash \{0\}$ the polynomial $g(x) \cdot g(y) - c^2 \in R_W[x, y]$ has a root in F.

This property is proved in exactly the same way as (*) for RC_π-fields.

We proceed to formulate the THR property for $\langle F, R_W \rangle$ ($\langle F, W \rangle$).

Let F be the field of fractions of a regular Prüfer ring R_W. Then, for every absolutely irreducible polynomial $f \in R_W[x, \bar{y}]$ ($\bar{y} = y_0, \ldots, y_n$) unitary in x and for all a, $\bar{b}$, $0 \neq \varepsilon \in R_W$ such that

$$f'_x(a, \bar{b}) \neq 0, \quad f(a, \bar{b}) f'_x(a, \bar{b})^{-2} \in J(R_W),$$

there exist c, $\bar{d} \in R_W$ satisfying

$$f(c, \bar{d}) = 0; \quad (b_i - d_i)\varepsilon^{-1} \in J(R_W), \quad i \leq n;$$

$$(a - c)^{-1} f(a, \bar{b}) f'_x(a, \bar{b})^{-1}, \ (a - c) f(a, \bar{b})^{-1} f'_x(a, \bar{b}) \in R_W.$$

A pair $\langle F, W \rangle$ ($\langle F, R_W \rangle$) is called an *RC^*-field* if W is a Boolean family of valuation rings of the field F, and $\langle F, W \rangle$ satisfies BAP and THR.

Proposition 5. If $\langle F, W \rangle$ is an RC^*-field, then F is dense in $\mathbb{H}_R(F)$ for every $R \in W$.

Suppose that $g \in R_W[x]$ is a unitary polynomial for which there exists an $a \in R_W$ such that $g(a) \in J(R_W)$ and $g'(a)^{-1} \in R_W$. We show that for every $\varepsilon \in J(R_W) \backslash \{0\}$, we can find $\alpha_\varepsilon \in R_W$ for which $g(\alpha_\varepsilon)\varepsilon^{-1} \in R_W$.

Let $h(x, y) \rightleftharpoons g(x) + y$. The polynomial h is absolutely irreducible, $h(a, \varepsilon) = g(a) + \varepsilon \in J(R_W)$, and $h'_x(a, \varepsilon) = g'(a)$ is invertible in R_W. By THR, there exist c, $d \in R_W$ such that $h(c, d) = 0$ and $(\varepsilon - d)\varepsilon^{-1} \in J(R_W)$, and so we have $g(c) + d = 0$, $g(c) = -d$ and $g(c)\varepsilon^{-1} = -d\varepsilon^{-1} = (\varepsilon - d)\varepsilon^{-1} - 1 \in R_W$.

Now the conclusion of our proposition follows from this property and Proposition 7 in [6, Sec. 1], reasoning in the same way as in the proof of Theorem 3 in [5].

THEOREM 1. If $\langle F, R_W\rangle$ ($\langle F, W\rangle$) is an RC^*-field, then $\langle F, R_W\rangle$ is an RC-field.

Suppose that $\langle F, W\rangle$ satisfies BAP and THR and let $F_0 \geq F$ be a finitely generated regular extension of F such that $H_R(F) \leq_1 H_R(F)F_0$ for all $R \in W$.

Let $\bar{b}, a$ be an R_W-special system of generators of the field F_0 over F (see [5, p. 595]), let $f \in R_W[x, \bar{y}]$ be the corresponding absolutely irreducible polynomial, and let $0 \neq h \in R_W[\bar{y}]$. For every $R \in W$, the fact that $H_R(F) \leq_1 H_R(F)F_0$ implies that there exist a_R, $\bar{b}_R \in R^h$ such that $f(a_R, \bar{b}_R) = 0$, $f'_x(a_R, \bar{b}_R) \neq 0$, and $h(\bar{b}_R) \neq 0$. Since F is dense in $\mathbb{H}_R(F)$, there exist α_R, $\bar{\beta}_R \in R$ for which $f'_x(\alpha_R, \bar{\beta}_R) \neq 0$, $f(\alpha_R, \bar{\beta}_R) f'_x(\alpha_R, \bar{\beta}_R)^{-2} \in \mathfrak{m}(R)$, $h(\bar{\beta}_R) \neq 0$, $f(\alpha_R, \bar{\beta}_R) \neq 0$, and $f(\alpha_R, \bar{\beta}_R) h(\bar{\beta}_R)^{-1} \in \mathfrak{m}(R)$. Then $R \in V_R \rightleftharpoons (\cap V_{\bar{\beta}_R}) \cap V_{\alpha_R} \cap (V_{f(\alpha_R,\bar{\beta}_R)h(\bar{\beta}_R)^{-1}} \setminus V_{h(\bar{\beta}_R) f(\alpha_R,\bar{\beta}_R)^{-1}}) \cap (V_{f(\alpha_R,\bar{\beta}_R) f'_x(\alpha_R,\bar{\beta}_R)^{-2}} \setminus V_{f'_x(\alpha_R,\bar{\beta}_R)^2 f(\alpha_R,\bar{\beta}_R)^{-1}})$ and $W = \bigcup_{R \in W} V_R$. Let $0 \neq \varepsilon_R \in R_W$ be such that $v_{R'}(f(\alpha_R, \bar{\beta}_R)) \leq v_{R'}(\varepsilon_R)$ for all $R' \in V_R$. Such ε_R can well be chosen in view of the following: If $\eta \in F$ satisfies the condition that $f(\alpha_R, \bar{\beta}_R) R_W + R_W = \eta R_W$, then $\eta^{-1} \in R_W$ and $\eta^{-1} f(\alpha_R, \bar{\beta}_R) \in R_W$, and we put $\varepsilon_R \rightleftharpoons \eta^{-1} f(\alpha_R, \bar{\beta}_R)$. Since W is compact, there exist $R_0, \ldots, R_n \in W$ such that $W = \bigcup_{i \leq n} V_{R_i}$. Since W is Boolean, there exists a partition $[a_0, \ldots, a_n]$ with $V_{a_i} \subseteq V_{R_i}$, $i \leq n$. Let $\varepsilon \rightleftharpoons \prod_{i \leq n} \varepsilon_{R_i}$ and choose γ, $\bar{\delta} \in F$ such that $v_R(\gamma - \alpha_{R_i}) \geq v_R(\varepsilon)$ and $v_R(\bar{\delta} - \bar{\beta}_{R_i}) \geq v_R(\varepsilon)$ for $R \in V_{a_i}$, $i \leq n$; the choices are possible in view of BAP. Now it is easy to verify that $\gamma, \bar{\delta} \in R_W$, $f'_x(\gamma, \bar{\delta}) \neq 0$, and $f(\gamma, \bar{\delta}) f'_x(\gamma, \bar{\delta})^{-2} \in J(R_W)$. If, by THR, γ', $\bar{\delta}' \in R_W$ are chosen in such a way that $(\bar{\delta} - \bar{\delta}')\varepsilon^{-1} \in J(R_W)$ and $f(\gamma', \bar{\delta}') = 0$, we also have $h(\bar{\delta}') \neq 0$. Indeed, let $R \in V_{a_i}$; then $v_R(h(\bar{\beta}_{R_i})) < v_R(f(\alpha_{R_i}, \bar{\beta}_{R_i})) \leq v_R(\varepsilon_{R_i}) \leq v_R(\varepsilon)$. Also $v_R(h(\bar{\beta}_{R_i}) - h(\bar{\delta}')) \geq v_R(\bar{\beta}_{R_i} - \bar{\delta}') \geq v_R(\varepsilon)$. Finally, $v_R(h(\bar{\beta}_{R_i})) = v_R(h(\bar{\delta}'))$. Hence $F \leq_1 F_0$ and $\langle F, W\rangle \in RC$.

Remark. THR can obviously be expressed via a system of axioms in the signature $\sigma_f \cup \langle R\rangle$, so that the property of being an RC^*-field for a pair $\langle F, R_W\rangle$ is axiomatizable.

3. In this section we argue to show that the class of RC^*-fields is sufficiently broad.

Let $F \leq F_0$, let R_0 be a valuation ring of the field F_0, and let $R \rightleftharpoons R_0 \cap F$. We call R_0 a *superstructure* of R if $\mathbb{H}_R(F) \leq_1 \mathbb{H}_{R_0}(F_0)$ and there exists a decomposition $R_0 = R'_0 \circ \bar{R}$, where $R'_0 > R_0$ is a valuation ring of F_0, $\bar{R}$ is a valuation ring of $F_{R'_0}$, and $R'_0 \circ \bar{R} \rightleftharpoons \{a | a \in R'_0,\ a + \mathfrak{m}(R'_0) \in \bar{R}\}$, such that $F \leq R'_0$, $R (\simeq R + \mathfrak{m}(R'_0)/\mathfrak{m}(R'_0)) \leq \bar{R} \leq R^h$, and $\Gamma_{R'_0}$ is a divisible group.

THEOREM 2. If W is a weakly Boolean family of valuation rings of the field F and $\pi : X \to W$ is a continuous surjective map of Boolean spaces, then there exist a regular extension F_0 of the field F, a Boolean family W_0 of valuation rings of F, and a homeomorphism $\varepsilon : W_0 \xrightarrow{\sim} X$ such that $\langle F_0, W_0\rangle$ is an RC^*-field, and for every $R_0 \in W_0$, $\pi\varepsilon(R_0) = R_0 \cap F$ and R_0 is the superstructure of $R_0 \cap F$.

In [1], a proof of the theorem was sketched for the case where W is Boolean and π is a homeomorphism. Proposition 1 shows that the case where W is a weakly Boolean family can be proved following essentially the same line. We thus need the following:

Proposition 6. If W is a weakly Boolean family of valuation rings of a field F and if $\pi : X \to W$ is a continuous surjective map of Boolean spaces, then there exist a regular extension F_* of the field F, a weakly Boolean family W_* of valuation rings of F_*, and a homeomorphism $\varepsilon : W_* \xrightarrow{\sim} X$ such that for every $R_* \in W_*$, $\pi\varepsilon(R_*) = R_* \cap F$ and R_* is the superstructure of $R_* \cap F$.

In order to prove this, we need a number of auxiliary facts concerning maps of Boolean spaces. The facts laid out below are probably well known. It is difficult, however, to point out a convenient source, and so we cite them to make our treatment self-contained.

A surjective continuous map $\pi: X \to Y$ of Boolean spaces is called *two-sheeted* if there exists a closed-open subspace $V \subseteq X$ such that $\pi \upharpoonright V$ and $\pi \upharpoonright (X \backslash V)$ are one-to-one.

Every surjective continuous map $\pi: X \to Y$ of Boolean spaces proves to be an inverse limit of two-sheeted maps.

Let $\pi: X \to Y$ be a surjective continuous map of Boolean spaces and let $V \subseteq X$ be closed-open. Denote by $(Y)_V$ a subspace of the Boolean space $Y \times \{V, X\backslash V\}$ (the latter two-element space is endowed with discrete topology), consisting of elements of the form $\langle \eta, V\rangle$, $\eta \in \pi(V)$, and of the form $\langle \eta, X\backslash V\rangle$, $\eta \in \pi(X\backslash V)$. Now, $(Y)_V$ is closed and, hence, Boolean. The projection $\varepsilon_V : (Y)_V \to Y$ on the first coordinate is a two-sheeted map and $\pi_V : X \to (Y)_V$ is a continuous map defined as follows: $\pi_V(\xi) \rightleftharpoons \langle \pi(\xi), V\rangle$ if $\xi \in V$, and $\pi_V(\xi) \rightleftharpoons \langle \pi(\xi), X\backslash V\rangle$ if $\xi \in X\backslash V$.

Let a surjective continuous map $\pi: X \to Y$ of the Boolean spaces X and Y be defined.

Suppose that $B(X)$ refers to a family of all closed-open subsets of X. Denote by $|B(X)|$ the cardinality of the set $B(X)$ and assume that $\{V_\alpha | \alpha < |B(X)|\} = B(X)$ is some well-ordering of $B(X)$.

For every ordinal $\alpha < |B(X)|$, a Boolean space Y_α and surjective continuous maps $\pi_\alpha : X \to Y_\alpha$ and $\varepsilon_{\alpha,\beta}: Y_\alpha \to Y_\beta$, $\beta \leq \alpha$, are defined as follows:

$Y_0 \rightleftharpoons Y$, $\pi_0 \rightleftharpoons \pi$, and $\varepsilon_{0,0} \rightleftharpoons \mathrm{id}_Y$.

Let Y_α, $\pi_\alpha : X \to Y_\alpha$, and $\varepsilon_{\alpha,\beta} : Y_\alpha \to Y_\beta$, $\beta \leq \alpha$, be defined. We then put $Y_{\alpha+1} \rightleftharpoons (Y_\alpha)_{V_\alpha}$, $\pi_{\alpha+1} \rightleftharpoons (\pi_\alpha)_{V_\alpha}$, and $\varepsilon_{\alpha+1,\alpha+1} \rightleftharpoons \mathrm{id}_{Y_{\alpha+1}}$, $\varepsilon_{\alpha+1,\alpha} \rightleftharpoons \varepsilon_{V_\alpha} : Y_{\alpha+1} = (Y_\alpha)_{V_\alpha} \to Y_\alpha$, letting $\varepsilon_{\alpha+1,\beta} \rightleftharpoons \varepsilon_{\alpha,\beta} \cdot \varepsilon_{\alpha+1,\alpha}$, $\beta \leq \alpha$.

Suppose that Y_α, π_α, and $\varepsilon_{\alpha,\beta}$ are defined for all $\beta \leq \alpha < \gamma < |B(X)|$ and let γ be a limit ordinal.

By induction we can assume that for $\delta \leq \beta \leq \alpha < \gamma$, we have $\pi_\beta = e_{\alpha,\beta}\pi_\alpha$, $\varepsilon_{\alpha,\delta} = \varepsilon_{\beta,\delta} \cdot \varepsilon_{\alpha,\beta}$, and $\varepsilon_{\alpha,\alpha} = \mathrm{id}_{Y_\alpha}$. Then $\{Y_\beta | \varepsilon_{\delta,\beta}, \delta \leq \beta < \alpha\}$ forms an inverse spectrum, and for limit $\alpha < \gamma$ Y_α, is isomorphic, by assumption, to $\varprojlim\{Y_\beta | \beta < \alpha\}$ (with projections $\varepsilon_{\alpha,\beta} : Y_\alpha \to Y_\beta$, $\beta < \alpha$). Hence, we can assume that $Y_\gamma \rightleftharpoons \varprojlim\{Y_\alpha | \alpha < \gamma\}$, $\varepsilon_{\gamma,\alpha} : Y_\gamma \to Y_\alpha$ are the corresponding projections, and that $\pi_\gamma : X \to Y_\gamma$ is uniquely determined from $\varepsilon_{\gamma,\alpha}\pi_\gamma = \pi_\alpha$, $\alpha < \gamma$.

It is not hard to verify that X is isomorphic to $\varprojlim\{Y_\alpha | \alpha < |B(X)|\}$.

We turn to the case where $\pi: X \to W$ is two-sheeted. Let V be a closed-open subset of X such that $\pi \upharpoonright V$ and $\pi \upharpoonright (X\backslash V)$ are one-to-one. Let $F_1 \rightleftharpoons F(t)$ be a field of rational functions in the variable t over F and let $F_0 \rightleftharpoons F(t^{n^{-1}}, n > 1)$. Suppose that R_0' is the valuation ring of F_1 such that $F \leq R_0'$ and $\mathfrak{m}(R_0') = tR_0'$, and that R_1' is the valuation ring of F_1 such that $F \leq R_1'$ and $\mathfrak{m}(R_1') = t^{-1}R_1'$. These are the conditions by which R_0' and R_1' are defined uniquely. There exist valuation rings R_0 and R_1 of F_0 which are uniquely defined by $R_i \cap F_1 = R_i'$, $i = 0, 1$. Let $W_V \rightleftharpoons \{R_0 \circ R | R \in \pi(V)\}$ and $W_{X\backslash V} \rightleftharpoons \{R_1 \circ R | R \in \pi(X\backslash V)\}$; then W_V, being a space, is homeomorphic to V, and so to $\pi(V)$; $W_{X\backslash V}$ is homeomorphic to $X\backslash V$, and so to $\pi(X\backslash V)$; and W_V, $W_{X\backslash V}$ are weakly Boolean families of valuation rings for F_0.

LEMMA 3. If W_0 and W_1 are weakly Boolean families of valuation rings of a field F, $W \rightleftharpoons W_0 \cap W_1$ is Hausdorff, and W_0 and W_1 are closed in W, then W is also a weakly Boolean family of valuation rings of F.

The family W is obviously compact. It suffices to show that V_A is closed in W for every finite $A \subseteq F$. Let $R \in W\backslash V_A$; then $R \in W_i$ with $i = 0$ or 1. Since W_i is weakly Boolean, there exists a finite $B \subseteq F$ such that $R \in V_B$ and $V_B \cap W_i \subseteq W_i\backslash V_A$. In addition, if $R \in W_{1-i}$, there exists a finite $C \subseteq F$ such that $R \in V_C$ and

$V_C \cap W_{1-i} \subseteq W_{1-i} \backslash V_A$. Then $R \in V_B \cap V_C \subseteq W \backslash V_A$. But if $R \notin W_{1-i}$, then $R \in V_B \cap (W \backslash W_{1-i}) \subseteq W \backslash V_A$ and $V_B \cap (W \backslash W_{1-i})$ is open.

It is not hard to verify that W_V and $W_{X \backslash V}$ satisfy the conditions of the lemma, so that $W_0 \rightleftharpoons W_V \cup W_{X \backslash V}$ is a weakly Boolean family of valuation rings of the field F_0, and there exists a homeomorphism $\varepsilon \colon W_0 \to X$ such that $\pi\varepsilon(R) = R \cap F$ holds for every $R \in W_0$. Moreover, R is the superstructure of $R \cap F$ since $R = R_0 \circ (R \cap F)$ or $R = R_1 \circ (R \cap F)$, and Γ_{R_0}, Γ_{R_1} are divisible groups.

LEMMA 4. Let $\langle F_i, W_i \rangle$, $i \in I$, be the family of fields with weakly Boolean families of valuation rings for which $\{F_i | i \in I\}$ is directed by inclusion, and for all i, $j \in I$, if $F_i \leq F_j$, then the map of W_j onto W_i is defined by the rule $R \mapsto R \cap F_i$, $R \in W_j$. Therefore, the field $F_* \rightleftharpoons \bigcup_{i \in I} F_i$ contains a weakly Boolean family of valuation rings W_* such that $R \cap F_i \in W_i$ holds for every $R \in W_*$, $i \in I$, and for every $R' \in W_i$, there exists an $R \in W_*$ for which $R' = R \cap F_i$.

Put $W_* \rightleftharpoons \{R | R \text{ is the valuation ring of } F_*, \text{ and } R \cap F_i \in W_i \text{ for all } i \in I\}$. The conditions of the lemma imply that $\{W_i | \pi_{i,j} \colon W_j \to W_i,\ F_i \leq F_j\ (\pi_{ij}(R) \rightleftharpoons R \cap F_i,\ R \in W_j)\}$ is an inverse spectrum of Boolean spaces with continuous maps onto. If $X \rightleftharpoons \varprojlim W_i$, then X is nonempty and, moreover, the projections $\pi_i \colon X \to W_i$ are onto. For each $\xi \in X$, let $R_\xi \rightleftharpoons \cup\{\pi_i(\xi) | i \in I\}$. It is easy to see that $R_\xi \in W_*$. Conversely, if $R \in W_*$, then the family $R_i \rightleftharpoons R \cap F_i$, $i \in I$, satisfies the condition that $\pi_{ij}(R_j) = R_j \cap F_i = R_i$ for all i, $j \in I$ such that $F_i \leq F_j$. The family R_i, $i \in I$, uniquely defines the point $\xi \in X$ for which $R_i = \pi_i(\xi)$, $i \in I$.

Thus, there does exist a one-to-one correspondence between X and W_*, which, as is easy to check, is a homeomorphism of these spaces.

For W_* in the conclusion of the lemma, we write $\varprojlim W_i$.

We are now in a position to prove the proposition. For $\pi \colon X \to W$, we will find an inverse spectrum $\{Y_\alpha | \alpha < |B(X)|\}$ of Boolean spaces such that $Y_{\alpha+1} \to Y_\alpha$ is two-sheeted with all $\alpha < |B(X)|$, and $Y_\alpha \simeq \varprojlim_{\beta < \alpha} Y_\beta$ with all limit $\alpha < |B(X)|$. Define the sequence $\langle F_\alpha, W_\alpha \rangle$, $\alpha < |B(X)|$, of fields with weakly Boolean families W_α so that $F_0 \rightleftharpoons F$ and $W_0 \rightleftharpoons W$. If $\langle F_\alpha, W_\alpha \rangle$ is already defined, let $F_{\alpha+1} \rightleftharpoons F_\alpha(t_\alpha^{n^{-1}}, n > 1)$. Assume that $W_{\alpha+1}$ is obtained from W_α, and $\pi_{\alpha+1}(V_\alpha)$ [a closed-open subset of $Y_{\alpha+1} = \pi_{\alpha+1}(X)$], as was done in the case where $\pi \colon X \to W$ is two-sheeted, considered above. If $\alpha \leq |B(X)|$ is limit and $\langle F_\beta, W_\beta \rangle$ are defined for all $\beta < \alpha$, we put $F_\alpha \rightleftharpoons \bigcup_{\beta < \alpha} F_\beta$ and $W_\alpha \rightleftharpoons \varprojlim_{\beta < \alpha} W_\beta$, in which case $F_* \rightleftharpoons F_{|B(X)|}$ and $W_* \rightleftharpoons W_{|B(X)|}$ are, respectively, the required extension of F and the required weakly Boolean family of valuation rings of F_*.

Indeed, for every $\alpha < |B(X)|$ there exists a natural homeomorphism $\varepsilon_\alpha \colon W_\alpha \to Y_\alpha$ which induces the homeomorphism

$$\varepsilon \colon W_* = W_{|B(X)|} \xrightarrow{\sim} \varprojlim_{\alpha < |B(X)|} W_\alpha \simeq \varprojlim_{\alpha < |B(X)|} Y_\alpha \simeq X,$$

from which we can readily verify that $\pi\varepsilon(R_*) = R_* \cap F$ holds for every $R_* \in W_*$, and R_* is the superstructure of $R_* \cap F$.

To conclude this section, we point out yet another result related to the above considerations.

LEMMA 5. If $F \leq F_0$ is a field extension, W_0 is a weakly Boolean family of valuation rings of F_0, and $W \rightleftharpoons \{R_0 \cap F | R_0 \in W_0\}$ is Hausdorff, then W is also weakly Boolean.

Since W is an image of the compact space under the continuous mapping, W is compact, in which case we need only establish that $V_A^F \subseteq W$, where A is a finite subset of F closed in W. Further, $V_A^{F_0} \rightleftharpoons \{R_0 | R_0 \in W_0,$

$A \subseteq R_0\}$ is closed-open in W_0, and $V_A^F = \pi(V_A^{F_0})$, where $\pi(R_0) = R_0 \cap F$, $R_0 \in W_0$; consequently, V_A^F, being an image of the closed subset, is also closed.

Remark. If, in the conditions of the lemma, W_0 is Boolean, we cannot generally state that W is also. This is illustrated by the following example. Let W be a weakly Boolean family of valuation rings of the field F and let $F_0 \rightleftharpoons F(t)$ be the field of rational functions in t over F. Suppose $W_0 \rightleftharpoons \{R_t | R \in W\}$, where the valuation ring R_t of F_0 is uniquely determined from R, using the conditions that $R \leq R_t$, $t \in R_t \backslash \mathfrak{m}(R_t)$, and $t + \mathfrak{m}(R_t)$ is transcendental over $F_R = R/\mathfrak{m}(R) \leq R_t/\mathfrak{m}(R_t)$. Then W_0 is weakly Boolean and homeomorphic to W under the map $R_t \mapsto R_t \cap F (= R)$, $R \in W$. But if the polynomial $x^2 - t$ is taken as $f_a(x)$, the reasons for W_0 to be Boolean are provided by the corollary to Proposition 2 in [7].

4. Using Theorem 2, we show that Theorem 1 admits reversion in some cases.

A valuation ring R is called *distinguished* if at least one of the following conditions is met:

(0) the field F_R is not separably closed;

(n) the formula

$$\Phi_n \rightleftharpoons \forall x \exists y \forall z (x > 0 \to 0 < y \leq x \wedge (n+1)z \neq y),\ n > 0,$$

is satisfied in Γ_R.

COROLLARY 1. If R is a distinguished valuation ring of F, and $R \leq R' \leq F$, then R' also has this property, and so does $\bar{R}$ provided $R < R'$ and $R = R' \circ \bar{R}$.

COROLLARY 2. If R is distinguished, then $H_R(F)$ is not separably closed.

LEMMA 6. If R_0 and R_1 are distinguished valuation rings of a field F and if R_0 is Henselian, then R_0 and R_1 are comparable with respect to inclusion.

Assume on the contrary that $R_0 \not\leq R_1$ and $R_1 \not\leq R_0$; then $R \rightleftharpoons R_0 R_1 > R_0, R_1$. If $R = F$, then R_0 and R_1 are independent, and since R_0 is Henselian, $H_{R_1}(F)$ should be separably closed (see, e.g., the corollary to Prop. 4 in [6, Sec. 3]), which it is not because R_1 is distinguished. If $R \neq F$, then $R_0 = R \circ \bar{R}_0$ and $R_1 = R \circ \bar{R}_1$ for suitable nontrivial valuation rings $\bar{R}_0$ and $\bar{R}_1$ of the field F_R. But then $\bar{R}_0$ and $\bar{R}_1$ are distinguished and independent rings, and $\bar{R}_0$ is Henselian, an impossibility.

We call a Boolean family W of valuation rings of a field F *distinguished* if, for every elementary extension $\langle F_1, R_1 \rangle \succeq \langle F, R_W \rangle$ and for every $R \in W_{R_1} (= \{(R_1)_\mathfrak{m}\ | \mathfrak{m}$ is a maximal ideal in $R_1\})$, the conditions $R' \rightleftharpoons R^h F$ and $R^h = R' \circ \bar{R}$ for a suitable valuation ring $\bar{R}$ of $F_{R'}$ imply that $\bar{R} \cap \widetilde{F} \cap F_{R'}$ is a distinguished ring.

COROLLARY. If W is distinguished, $R \in W_{R_1}$, $R < R' < F_1$, $F \leq R'$, and $R = R' \circ \bar{R}$, then $H_{\bar{R}}(F_{R'})$ is not separably closed.

Remark. If W is finite, then it is distinguished iff every R in W is.

Nonrigid sufficient conditions for a family W to be distinguished are given below.

THEOREM 3. If $\langle F, W \rangle$ is an RC-field and W is distinguished, then $\langle F, W \rangle$ is an RC^*-field.

By Theorem 2, there exists an RC^*-field $\langle F_0, W_0 \rangle$ such that F_0 is a regular extension of F, the map $R_0 \mapsto R_0 \cap F$, $R_0 \in W_0$, is the homeomorphism of W_0 and W, and R_0 is the superstructure of $R_0 \cap F$ for all $R_0 \in W_0$. Then $\mathbb{H}_{R_0 \cap F}(F) \leq_1 \mathbb{H}_{R_0}(F_0)$ for all $R_0 \in W_0$, and the fact that $\langle F, W \rangle$ is an RC-field implies that $F \leq_1 F_0$. Therefore, there exists an ultrapower $F_1 \rightleftharpoons F^I/\mathcal{D}$ of F for which one can find an F-embedding $\varphi : F_0 \to F_1$. In what follows we identify F_0 with $\varphi(F_0)$, i.e., assume that $F_0 \leq F_1$. Let $R_1 \rightleftharpoons R_W^I/\mathcal{D}$; then $\langle F, R_W \rangle \preceq \langle F_1, R_1 \rangle$. We argue that $R_{W_0} = R_1 \cap F_0$, for which it suffices to show that for every $R \in W_{R_1}$, the ring $R_0 \rightleftharpoons R \cap F_0$ is in W_0.

We show that $H_{R_0}(F_0)$ is not separably closed. We have $H_{R_0}(F_0) \leq H_R(F_1)$. Let F'_0 be an algebraic closure of F_0 in $H_R(F_1)$ $[H_{R_0}(F_0) \leq F'_0]$ and let $R' \rightleftharpoons R^h F$ and $R'_0 \rightleftharpoons R' \cap F'_0$. We have $R^h = R' \circ \bar{R}$ and $R_0 = R'_0 \circ \bar{R}_0$ for suitable valuation rings $\bar{R}$ and $\bar{R}_0$ of the fields $F_{R'}$ and $F_{R'_0}$, respectively. Suppose that F' is an algebraic closure of the field F $(\leq R')$ in $F_{R'}$. By the definition of being distinguished for W, the ring $\bar{R} \cap F'$ is also distinguished, and so F' is not separably closed. Further, $F \leq F_{R'_0} \leq F'$ and $F' \cap F_{R'_0}$ is algebraically closed in $F_{R'_0}$, but it is not separably closed, and hence also $F_{R'_0}$ is not. If $R''_0 \rightleftharpoons R^h_0 F$ and $R^h_0 = R''_0 \circ \bar{R}'_0$, then $F \leq F_{R''_0} \leq F_{R'_0}$ and $F_{R''_0}$ is not separably closed, hence also $H_{R_0}(F_0) = q(R^h_0) = q(R''_0)$ is not.

In view of the corollary to Proposition 4 in [5], there exists an $R'_0 \in W_0$ such that $R' \rightleftharpoons R_0 R'_0 \neq F_0$. If $F \leq R'$, then $R' > R_0, R'_0$ since $R_0 \cap F$, $R'_0 \cap F \in W$; consequently, $R_0 \geq F$ and $R'_0 \geq F$. Let $R_0 = R' \circ \bar{R}_0$ and $R'_0 = R' \circ \bar{R}'_0$; then $\bar{R}_0$ and $\bar{R}'_0$ are nontrivial independent valuation rings of $F_{R'}$. But $\bar{R}'_0$ is Henselian, and hence $H_{\bar{R}_0}(F_{R'})$ should be separably closed, which contradicts the definition of being distinguished for W. If $F \not\leq R'$, let $R \rightleftharpoons R'F$. We have $R = R'_0 F = R_0 F$ (since $R'_0 \leq R'$, $R'_0 \leq R'_0 F$, and $R'_0 F \not\leq R'$, it follows that $R' \leq R'_0 F$; $R' \leq R_0 F$ is obtained similarly). The fact that R'_0 is the superstructure of $R'_0 \cap F \in W$ implies that there exists a decomposition $R'_0 = R'' \circ \bar{R}''_0$ such that $F \leq R''$ and $R'_0 \cap F \leq \bar{R}''_0 \leq (R'_0 \cap F)^h$. Since $F \leq R''$, we have $R = R'_0 F \leq R''$, but $F_{R''} = q(\bar{R}''_0)$ is an algebraic extension of F; consequently, $R'' = R$ and $F_R = F_{R''}$ is an algebraic extension of F also. The family W and, hence, the ring $\bar{R}_0 \cap \widetilde{F} \cap F_R = \bar{R}_0 \cap F_R = \bar{R}_0$ are distinguished by definition, and $\bar{R}_0$ is defined via the relation $R_0 = R \circ \bar{R}_0$. If $R'_0 = R \circ \bar{R}'_0$, then $\bar{R}'_0$ is distinguished because R'_0 is. Thus, $\bar{R}_0$ and $\bar{R}'_0$ are distinguished valuation rings of F; moreover, $\bar{R}'_0$ is Henselian [since $R > R'_0$ and F_0 is dense in $\mathbb{H}_{R'_0}(F_0)$]. Then we have $\bar{R}_0 \leq \bar{R}'_0$ or $\bar{R}'_0 \leq \bar{R}_0$ by Lemma 6, from which it follows that $\bar{R}_0 \cap F \leq \bar{R}'_0 \cap F$ or $\bar{R}'_0 \cap F \leq \bar{R}_0 \cap F$. But $\bar{R}_0 \cap F = R_0 \cap F \in W$ and $\bar{R}'_0 \cap F = R'_0 \cap F \in W$, and so $\bar{R}_0 \cap F = \bar{R}'_0 \cap F$, $\bar{R}_0 = \bar{R}'_0$, but $R_0 = R'_0$.

We have thus established that $R_{W_0} = R_1 \cap F_0$, from which it is easy to infer the following:

If $R' \in W_{R_1} = \{(R_1)_{\mathfrak{m}} | \mathfrak{m}$ is a maximal ideal in $R_1\}$, $R \rightleftharpoons R' \cap F$, and R_0 is that unique valuation ring in W_0 for which $R_0 \cap F = R$, then $R' \cap F_0 = R_0$.

Now we show that $\langle F, R_W \rangle \in RC^*$. Let $R' \neq R'' \in W$ and let $R'_1 \rightleftharpoons R'^I/\mathcal{D}$ and $R''_1 \rightleftharpoons R''^I/\mathcal{D} \in W_{R_1}$. Then $\langle F, R', R'' \rangle \preceq \langle F_1, R'_1, R''_1 \rangle$. Since $R'_0 \rightleftharpoons R'_1 \cap F_0 \neq R''_0 \rightleftharpoons R''_1 \cap F_0$, and W_0 is independent, it follows that $F_0 = R'_0 R''_0$, $F \leq R'_0 R''_0 \leq R'_1 R''_1$, and the fact that $\langle F, R', R'' \rangle \preceq \langle F_1, R'_1, R''_1 \rangle$ is an elementary embedding implies that $F \leq R'R''$, i.e., $F = R'R''$. Hence, the rings R' and R'' are independent, and so $\langle F, W \rangle$ satisfies BAP.

Let $f \in R_W[x, \bar{y}]$ be an absolutely irreducible polynomial unitary in x. Suppose that a, $\bar{b}$, $0 \neq \varepsilon \in R_W$ are such that

$$f'_x(a, \bar{b}) \neq 0, \quad f(a, \bar{b}) f'_x(a, \bar{b})^{-2} \in J(R_W).$$

Since $J(R_W) = J(R_{W_0}) \cap F$ (which is easily checked) and $\langle F_0, W_0 \rangle \in RC^*$, there exist $c, \bar{d} \in R_{W_0}$ satisfying $f(c, \bar{d}) = 0$, $(b_i - d_i)\varepsilon^{-1} \in J(R_{W_0})$, $i \leq n$, and $(a-c)^{-1} f(a, \bar{b}) f'_x(a, \bar{b})^{-1}$, $(a-c) f(a, \bar{b})^{-1} f'_x(a, \bar{b}) \in R_{W_0}$.

It is easy to verify that $J(R_1) \cap F_0 = J(R_{W_0})$, from which we can see that c, $\bar{d} \in R_1$, $(b_i - d_i)\varepsilon^{-1} \in J(R_1)$, $i \leq n$, and $(a-c)^{-1} f(a, \bar{b}) f'_x(a, \bar{b})^{-1}$, $(a-c) f(a, \bar{b})^{-1} f'_x(a, \bar{b}) \in R_1$.

Since $\langle F, R_W \rangle \preceq \langle F_1, R_1 \rangle$, there exist c', $\bar{d}' \in R_W$ such that $f(c', \bar{d}') = 0$, $f'_x(c', \bar{d}') \neq 0$, $(b_i - d'_i)\varepsilon^{-1} \in J(R_W)$, $i \leq n$, and $(a - c')^{-1} f(a, \bar{b}) f'_x(a, \bar{b})^{-1}$, $(a - c') f(a, \bar{b})^{-1} f'_x(a, \bar{b}) \in R_W$.

Consequently, $\langle F, W \rangle$ satisfies THR, proving that $\langle F, W \rangle$ is an RC^*-field.

Remark. In Proposition 2 [1], a stronger statement is formulated, which, however, is still not proved.

What we can prove is a part of the statement concerning BAP.

Proposition 7. Let W be a Boolean family of valuation rings of a field F, suppose that $\langle F, W\rangle \in RC$, and assume that the rings in W are all distinguished. Then W is independent.

Assume the contrary. Let $R_0 \neq R_1 \in W$, $R \rightleftharpoons R_0R_1 \neq F$, and $F_0 \rightleftharpoons H_R(F)$; R'_0 and R'_1 are the valuation rings of F_0 such that $R_0 \leq R'_0 \leq R_0^h$, $R_1 \leq R'_1 \leq R_1^h$, and $R'_0R'_1 = R^h$. Using Zorn's lemma, we can find a maximal algebraic extension F_1 of F_0 such that in F_1, there exist valuation rings R_0^* and R_1^* satisfying the conditions $R'_0 \leq R_0^* \leq R_0^h$ and $R'_1 \leq R_1^* \leq R_1^h$. Note that $R_0^*R_1^*$ dominates R^h ($R_0^*R_1^* \cap F_0 = R^h$), and hence $R_0^*R_1^* \neq F_1$, i.e., R_0^* and R_1^* are dependent.

We argue that F_1 is regularly closed with respect to the family $\{R_0^*, R_1^*\}$. Let W_1 be the family of all valuation rings R^* of the field F_1 such that $R^* \cap F \in W$. By Proposition 4 in [5], F_1 is then regularly closed with respect to W_1, and R_0^*, $R_1^* \in W_1$. We can show that for every $R^* \in W_1$, there exists either an F-embedding $H_{R_0^*}(F_1)$ in $H_{R^*}(F_1)$ or an F-embedding $H_{R_1^*}(F_1)$ in $H_{R^*}(F_1)$. Hence F_1 will be regularly closed with respect to $\{R_0^*, R_1^*\}$. Let $R^* \in W_1$. If R^* is independent of R_0^* and R_1^*, then by the corollary to Proposition 4 in [5], $H_{R^*}(F_1)$ is separably closed, i.e., it is a separable closure of F_1, and hence $H_{R_0^*}(F_1)$, $H_{R_1^*}(F_1) \leq H_{R^*}(F_1)$. Let R^* be such that $R' \rightleftharpoons R_0^*R^* \neq F_1$. Since R_0^*, $R^* \in W_1$, we have $R_0^* \not\leq R^*$ and $R^* \not\leq R_0^*$; hence, there exist representations $R_0^* = R' \circ \bar{R}_0$ and $R^* = R' \circ \bar{R}$ for suitable nontrivial independent valuation rings $\bar{R}_0$ and $\bar{R}$ of the field $F_{R'}$. In view of the maximality of F_1, it is not hard to show that $\bar{R}_0$ is Henselian, from which it will follow that $H_{\bar{R}}(F_{R'})$ is separably closed, and so we can assume that $H_{R_0^*}(F_1) \leq H_{R^*}(F_1)$. Similarly we argue for the case where $R_1^*R^* \neq F_1$ [and so $H_{R_1^*}(F_1) \leq H_{R^*}(F_1)$].

We have thus proved that $\langle F_1, \{R_0^*, R_1^*\}\rangle \in RC$. But $(R_0^*)^h = R_0^h$, $(R_1^*)^h = R_1^h$, and so the rings R_0^*, R_1^* and the family $\{R_0^*, R_1^*\}$ are distinguished. By Theorem 3, $\langle F, \{R_0^*, R_1^*\}\rangle$ is an RC^*-field and R_0^* and R_1^* should be distinguished, which they are not by construction. This is a contradiction, which proves the proposition.

A Boolean family W is called a *family of the first kind* if there exists a unitary polynomial $f \in R_W[x]$ such that for every $R \in W$, its reduction $\bar{f} \in F_R[x]$ is a separable polynomial without roots in F_R.

A Boolean family W is called a *family of the second kind* if there exists an $n > 0$ such that for every $R \in W$ we have $\Gamma_R \models \Phi_n$, and for every $a \in \mathfrak{m}(R)$, there exist a $b \in \mathfrak{m}(R)\backslash\{0\}$ and a neighborhood $W' \subseteq W$ of the ring R such that for every $R' \in W'$ we have $b \in \mathfrak{m}(R')$, $v_{R'}(b) \leq v_{R'}(a)$, and $v_{R'}(b)$ is not divisible by $(n+1)$ in $\Gamma_{R'}$.

Proposition 8. If, for every R in the Boolean family W, there exists a closed-open neighborhood of the first (second) kind, then W is distinguished.

It suffices to prove the proposition for the case where W is itself a family of the first (second) kind.

Suppose that W is a family of the first kind and $f \in R_W[x]$ is a unitary polynomial such that for every $R \in W$, its reduction $\bar{f} \in F_R[x]$ is a separable polynomial without roots in F_R. This condition is equivalent to stating that the following elementary sentence is valid on R_W:

$$\forall a \exists b (f(a) \cdot b = 1).$$

Hence, if $\langle F, R_W\rangle \preceq \langle F_1, R_1\rangle$, then

$$R_1 \models \forall a \exists b (f(a) \cdot b = 1),$$

and so for every $R \in W_{R_1}$, the reduction $\bar{f} \in F_R[x]$ has no roots in F_R. If $R' \rightleftharpoons R^hF$ and $R^h = R' \circ \bar{R}$, then $F_{\bar{R}} = F_R \geq F_{\bar{R}\cap\widetilde{F}\cap F_{R'}} \geq F_{R\cap F}$. The polynomial $\bar{f}$ is in $F_{R\cap F}[x]$ and has no roots in $F_{\bar{R}}$. Consequently, $F_{\bar{R}\cap\widetilde{F}\cap F_{R'}}$ is not separably closed and $\bar{R} \cap \widetilde{F} \cap F_{R'}$ is a distinguished ring.

Let W be a family of the second kind and let $n > 0$ be the number satisfying the definition. Suppose that $\langle F, R_W\rangle \preceq \langle F_1, R_1\rangle$, $R \in W_{R_1}$, $R' \rightleftharpoons R^h F$, and $R^h = R' \circ \bar{R}$ for a suitable valuation ring $\bar{R}$ of the field $F_{R'}$. Let F_0 be an algebraic closure of F in $F_{R'}$ and let $R_0 \rightleftharpoons \bar{R} \cap F_0$. We need to verify that $\Gamma_{R_0} \models \Phi_n$. Since $\langle F, R\cap F\rangle \leq \langle F_0, R_0\rangle$ and F_0 is the algebraic closure of F, it follows that for every $a_0 \in \mathfrak{m}(R_0)$, there exists an $a \in \mathfrak{m}(R\cap F)$ such that $(0 <)v_{R_0}(a) \leq v_{R_0}(a_0)$. Since $R \cap F \in W$, there exist a neighborhood (of the form $V^F_{\alpha^{-1}}$, $\alpha \in R_W$) of the ring $R\cap F$ and an element $b \in \mathfrak{m}(R\cap F)\backslash\{0\}$ such that for every $R' \in V_{\alpha^{-1}}$ we have $b \in \mathfrak{m}(R')$, $v_{R'}(b) \leq v_{R'}(a)$, and $v_{R'}(b)$ is not divisible by $(n+1)$ in $\Gamma_{R'}$.

Let $R_\alpha = \cap\{R' | R' \in W, \alpha^{-1} \in R'\}$; then the conditions formulated for a and b above can be represented as follows:

$$b \in J(R_\alpha)\backslash\{0\}, \;\; ab^{-1} \in R_\alpha,$$

and

$$\forall c \in R_\alpha(bc^{-(n+1)} \in R_\alpha \rightarrow bc^{-(n+1)} \in J(R_\alpha)).$$

These are the elementary conditions imposed on α, a, and b; consequently, they also hold in $\langle F_1, R_1\rangle$. In particular, every $R' \in V^{F_1}_{\alpha^{-1}}$ satisfies the following: $b \in \mathfrak{m}(R')$, $ab^{-1} \in R'$ [i.e., $v_{R'}(b) \leq v_{R'}(a)$], and $v_{R'}(b)$ is not divisible by $(n+1)$ in $\Gamma_{R'}$. We have $R \in V^{F_1}_{\alpha^{-1}}$ because $R \cap F \in V^F_{\alpha^{-1}}$, and so $0 < v_R(b) \leq v_R(a)$ and $v_R(b)$ is not divisible by $(n+1)$ in $\Gamma_R = \Gamma_{R^h}$. Further, $\Gamma_{\bar{R}}$ is isomorphic to a convex subgroup $\Gamma'_{\bar{R}}$ of Γ_{R^h}. Consequently, $\Gamma'_{\bar{R}}$ is a pure subgroup in Γ_{R^h} and $v_R(b) \in \Gamma'_{\bar{R}}$ is not divisible by $(n+1)$ in $\Gamma'_{\bar{R}}$, and so in $\Gamma_{\bar{R}}$. Therefore, $v_R(b) \in \Gamma_{R_0} \leq \Gamma_{\bar{R}}$ is not divisible by $(n+1)$ in Γ_{R_0}. Moreover, $v_R(b) = v_{R_0}(b) \leq v_{R_0}(a) \leq v_{R_0}(a_0)$, and since a_0 is an arbitrary element in $\mathfrak{m}(R_0)\backslash\{0\}$ (see above), it follows that $\Gamma_{R_0} \models \Phi_n$ and R_0 is a distinguished ring.

COROLLARY. If, for a Boolean family $W \neq \varnothing$ of valuation rings of the field F, there exist $\pi \in F$ and $n > 0$ such that $R \in W$ is a (π, n)-valuation ring, i.e., $W \subseteq W_{\pi,n}$ (see [1]), then W is distinguished.

Under the conditions of the corollary, W is a family of the second kind.

REFERENCES

1. Yu. L. Ershov, "Relatively regularly closed fields," *Dokl. Ross. Akad. Nauk*, **332**, No. 3, 286-288 (1993).
2. Yu. L. Ershov, *Decidability Problems and Constructive Models* [in Russian], Nauka, Moscow (1980).
3. Yu. L. Ershov, "Boolean families of valuation rings," *Algebra Logika*, **31**, No. 2, 119-131 (1992).
4. M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison–Wesley, Reading (1969).
5. Yu. L. Ershov, "Relative regular closeness and π-valuations," *Algebra Logika*, **31**, No. 6, 592-623 (1992).
6. Yu. L. Ershov, "Multiply valued fields," *Usp. Mat. Nauk*, **37**, No. 3, 55-93 (1982).
7. Yu. L. Ershov, "Boolean families of valuation rings. II," *Algebra Logika*, **32**, No. 3, 261-266 (1993).

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