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Equivalence of the Drinfeld–Sokolov Reduction to a Bi-Hamiltonian Reduction★

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Abstract. We show that the Drinfeld–Sokolov reduction is equivalent to a bi-Hamiltonian reduction, in the sense that these two reductions, although different, lead to the same reduced Poisson (more correctly, bi-Hamiltonian) structure. In order to do this, we heavily use the fact that they are both particular cases of a Marsden–Ratiu reduction.

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1. Introduction

The Drinfeld–Sokolov reduction [1] is a very important tool in the theory of soliton equations [2, 3], allowing the definition of generalized Korteweg–DeVries equations associated with an arbitrary Kac–Moody algebra. Recently, it has been attracting even more attention in relation with the definition of classical and quantum $\mathcal{W}$-algebras [4–7].

This Letter aims to compare the Drinfeld–Sokolov reduction with another reduction process, namely the bi-Hamiltonian reduction, which can be performed on any bi-Hamiltonian manifold. It is a particular case of a reduction process for Poisson manifolds by Marsden and Ratiu [8] and has been suggested in [9], where a lot of properties typical of soliton equations (Lax and zero-curvature representations, dressing transformations, τ-function) are shown to be natural consequences of the existence of the bi-Hamiltonian structure. This point of view has been developed further in [10] and [11].

Preliminary results about the comparison between Drinfeld–Sokolov reduction and bi-Hamiltonian reduction have been obtained in [12]. However, the main object of that paper was the technique of the transversal manifold, allowing to the performance of the bi-Hamiltonian reduction with a saving of computations. This technique is applied to the case where the bi-Hamiltonian manifold is the dual of (a loop-algebra over) a simple Lie algebra; the recipe for the construction of the transversal manifold turns out to coincide with the one of Drinfeld and Sokolov for the canonical form of their Lax operator, and this coincidence strongly suggests that

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the two reductions, although different, should lead to the same reduced structure. This Letter definitively settles the matter in the following way: the Drinfeld–Sokolov reduction is recalled in the form of a Marsden–Weinstein reduction, which is another particular case of the Marsden–Ratiu reduction. At this point, the comparison between Drinfeld–Sokolov reduction and bi-Hamiltonian reduction is nothing else but a comparison between two different Marsden–Ratiu reductions.

In detail, in Section 2, we recall the definition of Poisson and bi-Hamiltonian manifolds, and three different reductions: Marsden–Ratiu, Marsden–Weinstein, and the bi-Hamiltonian one. In Section 3, we specialize the bi-Hamiltonian reduction to the case of the dual of a simple Lie algebra, getting the particular bi-Hamiltonian reduction to be compared with Drinfeld–Sokolov. Section 4 is the core of the Letter: the Drinfeld–Sokolov reduction is presented as a Marsden–Weinstein reduction [1, p. 2015], and the equivalence to the bi-Hamiltonian reduction is proved in Theorem 4.4. Finally, in Section 5, the example of $\mathfrak{sl}(3)$ is discussed in order to point out the differences between the two reduction processes.

2. Three Different Reductions

In this section, we recall three different reductions which are strictly related: the Marsden–Ratiu reduction, the Marsden–Weinstein reduction, and the bi-Hamiltonian reduction. The first two are defined on a Poisson manifold, while the third needs a bi-Hamiltonian manifold in order to be performed. Recall that a manifold $\mathcal{M}$ is said to be *Poisson* if it has a *Poisson bracket*, i.e. a composition law $\{\cdot,\cdot\}$ on $C^\infty(\mathcal{M})$ fulfilling $\mathbb{R}$-bilinearity, antisymmetry, the Leibnitz rule, and the Jacobi identity. An important object on a Poisson manifold is the Poisson tensor, sending 1-forms into vector fields, defined by

$$\langle df, P\,dg\rangle = \{f, g\}. \tag{2.1}$$

A *bi-Hamiltonian manifold* is a manifold endowed with two Poisson brackets $\{\cdot,\cdot\}_0$ and $\{\cdot,\cdot\}_1$ which are compatible, in the sense that for every λ, $\{\cdot,\cdot\}_\lambda := \{\cdot,\cdot\}_1 + \lambda\{\cdot,\cdot\}_0$ is still a Poisson bracket. Henceforth, $\{\cdot,\cdot\}_\lambda$ will be referred to as the *Poisson pencil* of $\mathcal{M}$.

We begin with the Marsden–Ratiu theorem.

THEOREM 2.1 (Marsden–Ratiu). *Let $\mathcal{M}$ be a Poisson manifold; $\{\cdot,\cdot\}^{\mathcal{M}}$ (resp. $P^{\mathcal{M}}$) its Poisson bracket (resp. tensor); $\mathcal{S}$ a submanifold of $\mathcal{M}$; $i_{\mathcal{S}}: \mathcal{S} \hookrightarrow \mathcal{M}$ the canonical immersion of $\mathcal{S}$ in $\mathcal{M}$; D a distribution on $\mathcal{M}$ such that:*

(1) *$E = D \cap T\mathcal{S}$ is an integrable distribution of $\mathcal{S}$;*
(2) *the foliation induced by E on $\mathcal{S}$ is regular, so that $\mathcal{N} := \mathcal{S}/E$ is a differentiable manifold and $\pi: \mathcal{S} \to \mathcal{N}$ a submersion;*
(3) *if F, G are functions on $\mathcal{M}$ whose differentials vanish on D, then $d\{F, G\}^{\mathcal{M}}$ also vanishes on D;*
(4) *$P^{\mathcal{M}}(D^0) \subset T\mathcal{S} + D$, where D^0 is the annihilator of D.*

Then the triple $(\mathscr{M}, \mathscr{S}, D)$ *is Poisson-reducible, that is to say,* $\mathscr{N}$ *is a Poisson manifold whose Poisson bracket* $\{\cdot, \cdot\}^{\mathscr{N}}$ *(the so-called reduced Poisson bracket) is given by*

$$\{f, g\}^{\mathscr{N}} \circ \pi = \{F, G\}^{\mathscr{M}} \circ i_{\mathscr{S}}, \tag{2.2}$$

where F, G *are extensions of* $f \circ \pi$, $g \circ \pi$, *with differentials vanishing on* D.

A very interesting example of Marsden – Ratiu reduction is given by the Marsden–Weinstein reduction for Poisson manifolds [8, p. 165]. We shall need this example in Section 4 for the comparison between bi-Hamiltonian reduction and Drinfeld–Sokolov reduction.

EXAMPLE 2.2 (Marsden–Weinstein reduction). Let $\mathscr{M}$ be a Poisson manifold, G a Lie group and $\mathfrak{g}$ its Lie algebra; suppose that G acts on $\mathscr{M}$ by a Hamiltonian action ϕ, with momentum map $J: \mathscr{M} \to \mathfrak{g}^*$. This means that [13, p. 194], for every $\xi \in \mathfrak{g}$, the fundamental vector field X_ξ is a Hamiltonian vector field,

$$X_\xi = P\, \mathrm{d}H_\xi, \tag{2.3}$$

with Hamiltonian $H_\xi(m) = \langle J(m), \xi \rangle$. Suppose the momentum map J to be Ad*-equivariant, that is

$$J(\phi_n(u)) = \mathrm{Ad}_n^* J(u). \tag{2.4}$$

Let then $\mu \in \mathfrak{g}^*$ be a regular value of J, so that $\mathscr{S} = J^{-1}(\mu)$ is a submanifold of $\mathscr{M}$, and let D be the tangent distribution to the orbits of ϕ. Then the triple $(\mathscr{M}, \mathscr{S}, D)$ is Poisson-reducible. The quotient manifold turns out to be $\mathscr{N} = J^{-1}(\mu)/G_\mu$, where G_μ is the isotropy group of μ (the so-called small group).

We are finally ready to state the theorem about bi-Hamiltonian reduction. It allows us to reduce simultaneously both Poisson brackets (that is, the Poisson pencil) of a bi-Hamiltonian manifold. The submanifold $\mathscr{S}$ and the distribution D are pointed out by the bi-Hamiltonian structure itself, in the following way:

(1) $\mathscr{S}$ is any symplectic leaf of $\{\cdot, \cdot\}_0$ which is an embedded submanifold (recall that symplectic leaves generally are only immersed submanifold);
(2) Suppose $D = \{P_1\, \mathrm{d}k_0 \,|\, P_0\, \mathrm{d}k_0 = 0\} = P_1(\mathrm{Ker}\, P_0)$ to be a distribution of constant rank★; then D is integrable on account of the compatibility condition between $\{\cdot, \cdot\}_0$ and $\{\cdot, \cdot\}_1$ (see [9, p. 216]).

Then we can state

THEOREM 2.3. *Let* $(\mathscr{M}, \{\cdot, \cdot\}_0, \{\cdot, \cdot\}_1)$ *be a bi-Hamiltonian manifold;* $\mathscr{S}$ *and* D *as above;* $E = D \cap T\mathscr{S}$ *the distribution induced by* D *on* $\mathscr{S}$. *Suppose the foliation induced by* E *to be sufficiently regular, so that the quotient set* $\mathscr{N} = \mathscr{S}/E$ *is a differentiable manifold. Then the triple* $(\mathscr{M}, \mathscr{S}, D)$ *is Poisson-reducible w.r.t. every bracket* $\{\cdot, \cdot\}_\lambda$.

★Marle [14] remarked that it is not necessary to suppose D to be of constant rank in order to get its integrability; it is sufficient to use a theorem by Sussmann [15].

Therefore, $\mathcal{N}$ is a bi-Hamiltonian manifold, whose Poisson pencil $\{\cdot,\cdot\}^\lambda_{\mathcal{N}}$ is given by

$$\{f, g\}^\lambda_{\mathcal{N}} \circ \pi = \{F, G\}^\lambda_{\mathcal{M}} \circ i_{\mathcal{S}}, \tag{2.5}$$

where F and G are functions on $\mathcal{M}$ extending $f \circ \pi$ and $g \circ \pi$, and constant on D.

The proof can be found in [9].

3. Bi-Hamiltonian Reduction on the Dual of a Simple Lie Algebra

In this section, we discuss, following [12], the particular bi-Hamiltonian reduction to be compared with the Drinfeld–Sokolov reduction. The bi-Hamiltonian manifold $\mathcal{M}$ will be the dual of (a loop-algebra over) a simple Lie algebra.

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and let $\mathcal{G} = C^\infty(S^1, \mathfrak{g})$ be the Lie algebra of C^∞-functions from the unit circumference to $\mathfrak{g}$. More generally, if $\mathfrak{l}$ is a subset of $\mathfrak{g}$, we shall denote by $\mathcal{L} = \{u \in \mathcal{G} \mid u(x) \in \mathfrak{l}\ \forall x \in S^1\}$ the 'loop-set' of $\mathfrak{l}$. For the basic notations regarding simple Lie algebras, we refer to [16], or to Sec. 5 of [1]. The appendix of [12] is a very short summary of what we shall need later on.

First of all, let us suppose that $\mathcal{G}^*$ can be identified with $\mathcal{G}$ by means of the bilinear form

$$(v, u)_{\mathcal{G}} = \int_{S^1} (v(x), u(x))_{\mathfrak{g}}\, \mathrm{d}x, \quad u, v \in \mathcal{G}, \tag{3.1}$$

induced by the Killing form $(\cdot,\cdot)_{\mathfrak{g}}$ of $\mathfrak{g}$. Let a be any element in $\mathcal{G}$; then it is well known [13] that $\mathcal{M} = \mathcal{G}^*$ can be endowed with the Poisson pencil

$$\{f, g\}_\lambda(u) = (\mathrm{d}f(u), \mathrm{d}g(u)_x + [\mathrm{d}g(u), u + \lambda a])_{\mathcal{G}}, \tag{3.2}$$

that is, with the following pair of compatible Poisson brackets:

$$\{f, g\}_1(u) = (\mathrm{d}f(u), \mathrm{d}g(u)_x + [\mathrm{d}g(u), u])_{\mathcal{G}}, \tag{3.3}$$

$$\{f, g\}_0(u) = (\mathrm{d}f(u), [\mathrm{d}g(u), a])_{\mathcal{G}}. \tag{3.4}$$

The corresponding Poisson tensors are

$$(P_1)_u v = v_x + [v, u], \tag{3.5}$$

$$(P_0)_u v = [v, a], \tag{3.6}$$

where $v \in T^*_u \mathcal{G}^* \simeq \mathcal{G}$.

Now we will apply the bi-Hamiltonian reduction theorem stated in the previous section to this particular bi-Hamiltonian manifold.

The first step is to choose a symplectic leaf $\mathcal{S}$ of $\{\cdot,\cdot\}_0$. Since $\operatorname{Ker} P_0 = \mathcal{G}_a$ (the isotropy algebra of a) and $\operatorname{Im} P_0 = (\operatorname{Ker} P_0)^\perp = \mathcal{G}_a^\perp$, where orthogonality is defined w.r.t. the bilinear form (3.1), one has that the symplectic leaves $\mathcal{S}$ of $\{\cdot,\cdot\}_0$ are affine subspaces modelled on $\mathcal{G}_a^\perp$: $\mathcal{S} = \mathcal{G}_a^\perp + b$, with b an arbitrary element of $\mathcal{G}$. In view of the comparison with the Drinfeld–Sokolov reduction, we choose

$$a \text{ in the center of } \mathfrak{n}_-, \quad a \neq 0, \tag{3.7}$$

$$b = \sum_{i=1}^{\operatorname{rank} \mathfrak{g}} E_i, \tag{3.8}$$

where the E_i are the vectors of the Weyl basis that generate $\mathfrak{n}_+$ (i.e. the elements associated with the simple positive roots).

Once we have chosen the symplectic leaf $\mathcal{S}$, we must study the distribution $E = P_1(\operatorname{Ker} P_0) \cap T\mathcal{S}$ of $\mathcal{S}$. Consequently, we fix a point $u \in \mathcal{S}$, i.e. of the form $u = b + s$ with $s \in \mathcal{G}_a^\perp$. Because of the definition of E, one has that $(P_1)_u v \in E$ if and only if $v \in \mathcal{G}_a$ and $v_x + [v, u] \in T_u\mathcal{S} = \mathcal{G}_a^\perp$. But $v_x + [v, u] = v_x + [v, b] + [v, s]$, and $[v, s] \in \mathcal{G}_a^\perp$ since $[\mathcal{G}_a, \mathcal{G}_a^\perp] \subset \mathcal{G}_a^\perp$. Hence, $(P_1)_u v \in E$ if and only if $v \in \mathcal{G}_a$ and $v_x + [v, b] \in \mathcal{G}_a^\perp$. In other words, $E = P_1 \mathcal{G}_{ab}$, where

$$\mathcal{G}_{ab} = \{v \in \mathcal{G}_a \mid v_x + [v, b] \in \mathcal{G}_a^\perp\}. \tag{3.9}$$

The final step of the reduction process is the computation of the reduced Poisson pencil on the quotient manifold $\mathcal{N} = \mathcal{S}/E$. An algorithm for simplifying this computation is described in [12]; it makes use of a submanifold of $\mathcal{S}$ which is transversal to E. This submanifold is strictly related with the canonical form of the matrix Lax operator of Drinfeld and Sokolov. We will not recall these results, because they play no role in the sequel.

4. Comparison Between Drinfeld–Sokolov Reduction and Bi-Hamiltonian Reduction

As we have already stated in the Introduction, this section is the core of the Letter, being devoted to show that Drinfeld–Sokolov reduction and bi-Hamiltonian reduction are equivalent, i.e. they lead to the same reduced Poisson pencil. This is not surprising, as in the case $\mathfrak{g} = \mathfrak{sl}(2)$ both reductions give rise to the Poisson pencil of KdV (see [9]).

In order to carry out this comparison, we first recall the Drinfeld–Sokolov reduction in a form very similar to the geometric interpretation given by Drinfeld and Sokolov themselves [1, p. 2015]. The only remarkable difference is that Drinfeld and Sokolov reduce only P_1 (in the notations of last section), while we shall extend their procedure to the whole pencil. From now on, we fix a matrix representation of $\mathfrak{g}$. Moreover, we systematically identify $\mathcal{G}^*$ with $\mathcal{G}$.

PROPOSITION 4.1. *Consider the group N_- having as its Lie algebra $\mathcal{N}_-$, and the action of N_- on $\mathcal{G}^*$ given by*

$$\phi_n : u \mapsto nun^{-1} + n_x n^{-1}, \quad u \in \mathcal{G}^* \simeq \mathcal{G},\ n \in N_-. \tag{4.1}$$

(a) *This action is Hamiltonian on $(\mathcal{G}^*, \{\cdot, \cdot\}_\lambda)$ for all λ, and admits as momentum map $J : \mathcal{G}^* \to \mathcal{N}_-^*$ the restriction to $\mathcal{N}_-$:*

$$J : u \in \mathcal{G}^* \mapsto u|_{\mathcal{N}_-} \in \mathcal{N}_-^*. \tag{4.2}$$

(b) *The momentum map J is* Ad*-*equivariant:*

$$J(\phi_n(u)) = \mathrm{Ad}_n^* J(u). \tag{4.3}$$

Proof. (a) Let $\xi \in \mathcal{N}_-$ and let X_ξ be the associated fundamental vector field. It is easy to show that

$$X_\xi(u) = \xi_x + [\xi, u]. \tag{4.4}$$

On the other hand, let us consider the function $H_\xi(u) = \langle J(u), \xi \rangle$: since J is the restriction to $\mathcal{N}_-$, and $\xi \in \mathcal{N}_-$, one has that $H_\xi(u) = \langle u, \xi \rangle$ and therefore $\mathrm{d}H_\xi(u) = \xi$. It follows that the Hamiltonian vector field with Hamiltonian H_ξ, w.r.t. the Poisson bracket $\{\cdot, \cdot\}_\lambda$, is

$$(P_\lambda)_u \, \mathrm{d}H_\xi(u) = (P_\lambda)_u \xi = \xi_x + [\xi, u + \lambda a] = \xi_x + [\xi, u], \tag{4.5}$$

where last equality is due to the fact that a belongs to the center of $\mathcal{N}_-$. From (4.4) and (4.5), we have the first part of the thesis.

(b) First of all we observe that under the identification of $\mathcal{G}^*$ with $\mathcal{G}$, $\mathcal{N}_-^*$ is identified with $\mathcal{N}_+$, and J is nothing else but the projection on $\mathcal{N}_+$, henceforth denoted by $\pi_{\mathcal{N}_+}$. Therefore,

$$\begin{aligned} J(\phi_n(u)) &= \pi_{\mathcal{N}_+}(nun^{-1} + n_x n^{-1}) = \pi_{\mathcal{N}_+}(nun^{-1}) \\ &= \pi_{\mathcal{N}_+}[n\pi_{\mathcal{N}_+}(u)n^{-1} + n\pi_{\mathcal{B}_-}(u)n^{-1}] = \pi_{\mathcal{N}_+}[n\pi_{\mathcal{N}_+}(u)n^{-1}] \\ &= \mathrm{Ad}_n^* J(u), \end{aligned} \tag{4.6}$$

where the last equality follows from the fact that the coadjoint action of N_- on $\mathcal{N}_-^* \simeq \mathcal{N}_+$ is given by

$$\mathrm{Ad}_n^* v = \pi_{\mathcal{N}_+}(nvn^{-1}), \quad v \in \mathcal{N}_-^*. \tag{4.7}$$

Equation (4.6) shows that J is equivariant, and the proof is complete. □

Bearing in mind what we said in Section 2, we realize that the Drinfeld–Sokolov reduction can be seen as a Marsden–Weinstein reduction, conforming to the following scheme:

(1) we consider b as an element of $\mathcal{N}_-^* \simeq \mathcal{N}_+$;
(2) as J is identified with $\pi_{\mathcal{N}_+}$, then $J^{-1}(b)$ turns out to be $\mathcal{B}_- + b$;
(3) we remark that $\mathcal{B}_- + b$ is invariant under the action ϕ, that is to say, the isotropy group of b is the whole N_-; hence the quotient manifold is $(\mathcal{B}_- + b)/N_-$.

In Section 2, we saw that the Marsden–Weinstein reduction is an example of Marsden–Ratiu reduction, where one chooses $\mathcal{S} = J^{-1}(\mu)$, with μ in the dual of the symmetry algebra, and D to be the tangent distribution to the orbits of the symmetry group. Hence, we have led the comparison between bi-Hamiltonian reduction and Drinfeld–Sokolov reduction back to a comparison between two different Marsden–

Ratiu reductions on the Poisson manifold $(\mathscr{G}^*, \{\cdot, \cdot\}_\lambda)$:

$$\begin{array}{lllll} \text{Biham. red.:} & \mathscr{S}_1 = \mathscr{G}_a^\perp + b, & D_1 = P_1\mathscr{G}_a, & E_1 = P_1\mathscr{G}_{ab}, & \mathscr{N}_1 = \mathscr{S}_1/E_1. \\ \text{DS red.:} & \mathscr{S}_2 = \mathscr{B}_- + b, & D_2 = P_1\mathscr{N}_-, & E_2 = P_1\mathscr{N}_-, & \mathscr{N}_2 = \mathscr{S}_2/E_2. \end{array}$$

The fact that $D_2 = P_1\mathscr{N}_-$ follows from Equation (4.5). E_2 coincides with D_2 because of item (3) above.

We want to show that the two reductions are equivalent; this means that, if we call $\{\cdot, \cdot\}_\lambda^{(1)}$ (resp. $\{\cdot, \cdot\}_\lambda^{(2)}$) the reduced pencil on $\mathscr{N}_1$ (resp. $\mathscr{N}_2$), then the Poisson manifolds $(\mathscr{N}_1, \{\cdot, \cdot\}_\lambda^{(1)})$ and $(\mathscr{N}_2, \{\cdot, \cdot\}_\lambda^{(2)})$ are isomorphic for all λ. As a first step, we state

LEMMA 4.2.

(a) $\mathfrak{g}_a^\perp \subset \mathfrak{b}_-$;
(b) $\mathfrak{g}_a \cap \mathfrak{g}_b = 0$;
(c) $\mathfrak{g}_a^\perp + \mathrm{ad}_b(\mathfrak{n}_-) = \mathfrak{b}_-$.

Proof.

(a) $\mathfrak{g}_a \supset \mathfrak{n}_- \Rightarrow \mathfrak{g}_a^\perp \subset \mathfrak{n}_-^\perp = \mathfrak{b}_-$.
(b) Let $v \in \mathfrak{g}_a \cap \mathfrak{g}_b$; recall that $\mathfrak{g}_b \subset \mathfrak{n}_+$ [17, p. 1008]. Hence, v is nilpotent. But $v \in \mathfrak{g}_{a+b}$, which is a Cartan subalgebra, because $a+b$ is a regular element [17, Cor. 6.4]; then v is semisimple. But the only element which is nilpotent and semisimple is $v = 0$.
(c) $\mathfrak{g}_a^\perp + \mathrm{Im}(\mathrm{ad}_b) = \mathfrak{g}_a^\perp + \mathfrak{g}_b^\perp = (\mathfrak{g}_a \cap \mathfrak{g}_b)^\perp = 0^\perp = \mathfrak{g}$; it follows that $\mathfrak{g}_a^\perp + \mathrm{ad}_b(\mathfrak{n}_-) = \mathfrak{b}_-$. □

We remark that, on account of item (a) of the previous lemma, $\mathscr{S}_1 \subset \mathscr{S}_2$; moreover, $D_1 \supset D_2$ because a belongs to the center of $\mathscr{N}_-$. The following lemma will entail that $E_1 \subset E_2$.

LEMMA 4.3. *$\mathscr{G}_{ab}$ is a subset of $\mathscr{N}_-$.*

Proof. Let $t \in \mathscr{G}_{ab}$, and let $\mathfrak{g} = \oplus_{i=-k}^k \mathfrak{g}_i$ be the canonical gradation of $\mathfrak{g}$. It is not difficult to show that $t = \Sigma_{-k}^k\, t_i$, with $t_i \in \mathscr{G}_a \cap \mathscr{G}_i$. Furthermore, $t_k = 0$, since t_k is proportional to the Cartan involute $\sigma(a)$ of a that does not belong to $\mathscr{G}_a$. As $t_x + [t, b] \in \mathscr{G}_a^\perp \subset \mathscr{B}_-$, one has that $t_{ix} + [t_{i-1}, b] = 0$ $\forall i > 0$. For $i = k$, we get

$$[t_{k-1}, b] = 0 \Rightarrow t_{k-1} \in \mathscr{G}_a \cap \mathscr{G}_b \Rightarrow t_{k-1} = 0$$

on account of Lemma 4.2(b). In the same way, for $i = k-1$, we have $t_{k-2} = 0$. Finally, we get $t_i = 0$ $\forall i \geqslant 0$, that is $\mathscr{G}_{ab} \subset \mathscr{N}_-$.

It is not difficlt to show that $\mathscr{G}_{ab}$ is a subalgebra (see [12]), but later on it will suffice to know that $\mathscr{G}_{ab} \subset \mathscr{N}_-$.

Now we are going to prove that the Poisson manifolds $(\mathscr{N}_1, \{\cdot, \cdot\}_\lambda^{(1)})$ and $(\mathscr{N}_2, \{\cdot, \cdot\}_\lambda^{(2)})$ are isomorphic. This amounts to saying that the bi-Hamiltonian manifolds $(\mathscr{N}_1, \{\cdot, \cdot\}_0^{(1)}, \{\cdot, \cdot\}_1^{(1)})$ and $(\mathscr{N}_2, \{\cdot, \cdot\}_0^{(2)}, \{\cdot, \cdot\}_1^{(2)})$ are isomorphic.

THEOREM 4.4. (a) *$\mathcal{N}_1$ and $\mathcal{N}_2$ are diffeomorphic.* (b) *The reduced brackets $\{\cdot,\cdot\}_\lambda^{(1)}$ and $\{\cdot,\cdot\}_\lambda^{(2)}$ coincide.*

Proof. (a) The leaves of the distribution E_2 are the orbits of the action ϕ of the group N_-. As $\mathcal{G}_{ab}\subset\mathcal{N}_-$, the intersections of such leaves with $\mathcal{S}_1$ are exactly the leaves of the distribution E_1. From this, we deduce that $\mathcal{N}_1$ injects in $\mathcal{N}_2$. In order to affirm that this injection is surjective, we have to show that *every* leaf of E_2 intersects $\mathcal{S}_1$, i.e. that

$$\forall u\in\mathcal{S}_2 \quad \exists n\in N_- \text{ s.t. } nun^{-1}+n_xn^{-1}\in\mathcal{G}_a^\perp+b. \tag{4.8}$$

This can be shown in the following way (see also Proposition 6.1 of [1], p. 2010).

Fix $u\in\mathcal{B}_-+b$, and look for an element $n\in N_-$ of the form $n=\exp(v)$, with $v\in\mathcal{N}_-$, and for an $s\in\mathcal{G}_a^\perp+b$ such that

$$\exp(v)u\exp(-v)+(\exp(v))_x\exp(-v)=s. \tag{4.9}$$

Decompose u, v, and s w.r.t. the canonical gradation:

$$u=\sum_{i\geq 0}u_i+b,\qquad v=\sum_{i\geq 1}v_i,\qquad s=\sum_{i\geq 0}s_i+b, \tag{4.10}$$

with u_i, v_i, and $s_i\in\mathcal{G}_{-i}$. If we take the $\mathcal{G}_{-i}$-projection of Equation (4.9), we get

$$[v_{i+1},b]+(\text{terms in } v_j \text{ with } j\leq i)=s_i,\quad i\geq 0. \tag{4.11}$$

Lemma 4.2(c) ensures that it is possible to determine v_{i+1} and s_i in such a way that Equation (4.11) is satisfied. Hence, the injection of $\mathcal{N}_1$ into $\mathcal{N}_2$ is surjective, and the two quotient manifolds are diffeomorphic.

(b) Let f and g be two functions on $\mathcal{N}_1\simeq\mathcal{N}_2$. We have to compare the brackets $\{f,g\}_\lambda^{(1)}$ and $\{f,g\}_\lambda^{(2)}$, defined in the following way:

$$\{f,g\}_\lambda^{(k)}\circ\pi_k=\{F_k,G_k\}_\lambda\circ i_k,\quad k=1,2, \tag{4.12}$$

where $\pi_k\colon\mathcal{S}_k\to\mathcal{N}_k$ and $i_k\colon\mathcal{S}_k\to\mathcal{G}$ are, respectively, the canonical projections and injections, F_k, G_k are extensions of $f\circ\pi_k$ and $g\circ\pi_k$, which are constant on the leaves of D_k, and $\{\cdot,\cdot\}_\lambda$ is the bracket (3.2). We observe that, for what we proved in part (a), $\pi_2|_{\mathcal{S}_1}=\pi_1$.

Suppose now F to be an extension of $f\circ\pi_1$, which is constant along D_1. Then F is constant along D_2 and, therefore, it is not hard to see that F is an extension of $f\circ\pi_2$, too. Obviously, the same is true for g and G. Consequently, F and G can be used to compute both $\{f,g\}_\lambda^{(1)}$ and $\{f,g\}_\lambda^{(2)}$:

$$\{f,g\}_\lambda^{(1)}\circ\pi_1=\{F,G\}^\lambda\circ i_1, \tag{4.13}$$

$$\{f,g\}_\lambda^{(2)}\circ\pi_2=\{F,G\}^\lambda\circ i_2. \tag{4.14}$$

If we evaluate (4.14) on a point $u\in\mathcal{S}_1$, and we recall that $\pi_2|_{\mathcal{S}_1}=\pi_1$ and $i_2|_{\mathcal{S}_1}=i_1$, we immediately have that $\{f,g\}_\lambda^{(2)}$ also satisfies (4.13). But $\{f,g\}_\lambda^{(1)}$ is uniquely determined by Equation (4.13) and, therefore, we can conclude that $\{f,g\}_\lambda^{(1)}=\{f,g\}_\lambda^{(2)}$. □

In this way we proved that the result of the two reductions is the same. As a concluding remark, we observe that the Drinfeld–Sokolov reduction is based on the existence of a symmetry group, while the bi-Hamiltonian reduction, being defined on any bi-Hamiltonian manifold, makes use only of the information given by the two Poisson brackets.

5. Example: $\mathfrak{g}=\mathfrak{sl}(3)$

The purpose of this section is to show an example where the differences between the Drinfeld–Sokolov and the bi-Hamiltonian reduction processes can be explicitly seen. As in the case $\mathfrak{g}=\mathfrak{sl}(2)$ (corresponding to KdV) the two processes coincide, we are forced to consider the case $\mathfrak{g}=\mathfrak{sl}(3)$ (leading to the Boussinesq hierarchy).

The elements a and b are

$$a=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad b=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.1}$$

and, therefore, in the notations of the previous sections,

$$\mathscr{G}_a^{\perp}=\operatorname{Im} P_0 : \begin{pmatrix} v_1 & 0 & 0 \\ v_2 & 0 & 0 \\ v_3 & v_4 & -v_1 \end{pmatrix}, \tag{5.2}$$

where the v_k are C^∞-functions from S^1 to $\mathbb{R}$. It follows that the elements s_1 of $\mathscr{S}_1=\mathscr{G}_a^{\perp}+b$ have the form

$$s_1=\begin{pmatrix} p_1 & 1 & 0 \\ p_2 & 0 & 1 \\ q_1 & q_2 & -p_1 \end{pmatrix}. \tag{5.3}$$

The choice for the notations in (5.3) is motivated by the fact that p_1, p_2, q_1, q_2 are 'canonical coordinates' on the symplectic leaf $\mathscr{S}_1$, but we will not prove this assertion. The elements s_2 of $\mathscr{S}_2=\mathscr{B}_-+b$ are given by

$$s_2=\begin{pmatrix} p_1+r & 1 & 0 \\ p_2 & -2r & 1 \\ q_1 & q_2 & -p_1+r \end{pmatrix}, \tag{5.4}$$

so that $\mathscr{S}_1$ is the submanifold of $\mathscr{S}_2$ corresponding to $r=0$. In order to determine the distributions $D_1=P_1\mathscr{G}_a^{\perp}$, $D_2=E_2=P_1\mathscr{N}_-$, and $E_1=P_1\mathscr{G}_{ab}$, let us consider the generic element $t \in \operatorname{Ker} P_0 = \mathscr{G}_a$:

$$t = \begin{pmatrix} a & 0 & 0 \\ b & -2a & 0 \\ c & d & a \end{pmatrix}. \tag{5.5}$$

It follows that, at the points of $\mathscr{S}_1$,

$$D_1 : \begin{pmatrix} a_x - b & 3a & 0 \\ b_x + p_1 b - 3p_2 a - c & -2a_x + b - d & -3a \\ c_x + 2p_1 c + p_2 d - q_2 b & d_x + c + 3q_2 a + p_1 d & a_x + d \end{pmatrix}, \tag{5.6}$$

and that $P_1(t) \in T\mathscr{S}_1 \Leftrightarrow$

$$t = \begin{pmatrix} 0 & 0 & 0 \\ d & 0 & 0 \\ c & d & 0 \end{pmatrix}. \tag{5.7}$$

In the notations of Section 3, (5.7) is the form of the generic element of $\mathscr{G}_{ab}$. One can check that $\mathscr{G}_{ab} \subsetneq \mathscr{N}_- \subsetneq \mathscr{G}_a$ (see Lemma 4.3). This obviously implies that $E_1 \subsetneq D_2 \subsetneq D_1$ at the points of $\mathscr{S}_1$. The distribution E_1 is given by

$$E_1 : \begin{pmatrix} -d & 0 & 0 \\ d_x + p_1 d - c & 0 & 0 \\ c_x + 2p_1 c + p_2 d - q_2 d & d_x + c + p_1 d & d \end{pmatrix}, \tag{5.8}$$

or

$$\begin{aligned} \dot{p}_1 &= -d, \\ \dot{p}_2 &= d_x + p_1 d - c, \\ \dot{q}_1 &= c_x + 2p_1 c + p_2 d - q_2 d, \\ \dot{q}_2 &= d_x + c + p_1 d. \end{aligned} \tag{5.9}$$

A rather long calculation shows that a possible choice for the coordinates (u_1, u_2) on the quotient manifold $\mathscr{N}_1 = \mathscr{S}_1 / E_1$ is

$$\begin{aligned} u_1 &= q_1 - p_1 q_2 + p_1 p_2 + p_1 p_{1x} + p_{2x} + p_{1xx}, \\ u_2 &= q_2 + p_2 + 2p_{1x} + p_1^2. \end{aligned} \tag{5.10}$$

This is also the form of the projection π_1. Another way to compute π_1 is to observe that the leaves of E_1 are the orbits of the action (4.1) of the group G_{ab} whose Lie algebra is $\mathscr{G}_{ab}$. Thus, π_1 can be obtained by writing $s' = gsg^{-1} + g_x g^{-1}$, with $s', s \in \mathscr{S}_1$ and $g \in G_{ab}$, and eliminating g.

Now we turn to the Drinfeld–Sokolov reduction. If

$$n=\begin{pmatrix}0&0&0\\ b&0&0\\ c&d&0\end{pmatrix}\in\mathcal{N}_-,\tag{5.11}$$

then the distribution $D_2=P_1\mathcal{N}_-$ at the generic point $s_2\in\mathcal{S}_2$ is given by

$$D_2:\begin{pmatrix}-b&0&0\\ b_x+(p_1+3r)b-c&b-d&0\\ c_x+p_1c+p_2d-q_2b&d_x+c+(p_1-3r)d&d\end{pmatrix}.\tag{5.12}$$

After long computations, one can find (a possible form for) the projection π_2:

$$\begin{aligned}u_1&=q_1-p_1q_2+p_1p_2+p_1p_{1x}+p_{2x}+p_{1xx}-p_2r-q_2r+p_{1x}r+\\&\quad+2p_1^2r-2r^3+3p_1r_x+r_{xx}+3rr_x,\\ u_2&=q_2+p_2+2p_{1x}+p_1^2+3r^2.\end{aligned}\tag{5.13}$$

Comparing (5.10) and (5.13), we see that $\pi_2|_{\mathcal{S}_1}=\pi_1$.

This example shows very well that the submanifolds and the distributions taking part in the Drinfeld–Sokolov and bi-Hamiltonian reductions are really different. Nevertheless, we proved in Theorem 4.4 that the quotient manifold is the same Poisson (more correctly, bi-Hamiltonian) manifold.

A detailed study of the Boussinesq hierarchy from the bi-Hamiltonian point of view will be the object of a forthcoming paper.

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